Algebraic independence of topological Pontrjagin classes

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(joint work with Søren Galatius)

The spaces $B\mathrm{O}(d)$ classifying d-dimensional vector bundles carry two well-known kinds of rational characteristic classes. For $d = 2n$ there is the Euler class $e \in H^{2n}(B\mathrm{O}(2n); \mathbb{Q}^{w_1})$, in cohomology with twisted coefficients corresponding to the determinant line: its square yields an untwisted cohomology class $e^2 \in H^{4n}(B\mathrm{O}(2n); \mathbb{Q})$. On the other hand, for all d there are the Pontrjagin classes $p_i \in H^{4i}(B\mathrm{O}(d); \mathbb{Q})$. By their construction these satisfy some elementary relations:

$$(\dagger) \qquad \begin{aligned} p_i &= 0 \text{ for } 2i > d, \\ p_n &= e^2 \text{ for } d = 2n. \end{aligned}$$

Furthermore, the Euler and Pontrjagin classes give a complete description of the cohomology of $B\mathrm{O}(d)$, and these relations are the only ones satisfied: we have

$$H^*(B\mathrm{O}(d); \mathbb{Q}) = \begin{cases} \mathbb{Q}[p_1, p_2, \ldots, p_n] & d = 2n+1 \\ \mathbb{Q}[p_1, p_2, \ldots, p_n, e^2]/(p_n - e^2) & d = 2n. \end{cases}$$

There are analogous spaces $B\mathrm{Top}(d)$ classifying fibre bundles with fibre the euclidean space $\mathbb{R}^d$, and structure group $\mathrm{Top}(d)$, the group of homeomorphisms of $\mathbb{R}^d$ fixing the origin. Neglecting the fibrewise linear structure of a vector bundle gives maps $B\mathrm{O}(d) \to B\mathrm{Top}(d)$, and it follows from work of Sullivan and Kirby–Siebenmann that the map of stabilisations $B\mathrm{O} \to B\mathrm{Top}$ is a rational homotopy equivalence. We therefore have

$$H^*(BTop; \mathbb{Q}) \xrightarrow{\sim} H^*(B\mathrm{O}; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

This means that there are uniquely defined rational cohomology classes on $B\mathrm{Top}$ which restrict to the usual Pontrjagin classes on $B\mathrm{O}$: these are the topological Pontrjagin classes. These can be pulled back to $B\mathrm{Top}(d)$ for any d, and it is also easy to construct the (squared) Euler class on $B\mathrm{Top}(2n)$.

As their construction is rather indirect it is by no means clear whether the relations ($\dagger$) should be expected to hold on $B\mathrm{Top}(d)$. Reis and Weiss [1, 2, 3] developed a refined strategy to show that these relations do hold, but in breakthrough recent work Weiss [4] has shown that in fact they do not! For example, his results imply that $p_{n+i} \neq 0 \in H^{4(n+i)}(B\mathrm{Top}(2n); \mathbb{Q})$ for fixed i and all large enough n.

In my talk I presented the following result.

Theorem A. *For all $2n \geq 6$ the map*

$$\mathbb{Q}[e^2, p_1, p_2, p_3, \ldots] \longrightarrow H^*(B\mathrm{Top}(2n); \mathbb{Q})$$

is injective.

It is easy to deduce an analogous statement for $B\mathrm{Top}(2n+1)$. What this means is that not only do the relations (†) not hold, but no universal relations hold among the Euler and Pontrjagin classes for euclidean bundles of dimension $d \geq 6$.

In the form I have presented it this result relies on the theorem of Kupers [5] that $B\mathrm{Top}(2n)$ has finitely-generated cohomology groups for $n \geq 6$ (which in turn relies on the ideas of Weiss' [4]). However, if one is happy to work with rationalised integral cohomology, rather than rational cohomology, then this ingredient can be avoided, and in fact the argument then goes through equally well in dimension $2n = 4$, giving:

Theorem B. *The map*

$$\mathbb{Q}[e^2, p_1, p_2, p_3, \ldots] \longrightarrow H^*(B\mathrm{Top}(4); \mathbb{Z}) \otimes \mathbb{Q}$$

is injective.

I spent most of the talk outlining the proof of these results. The general idea is as follows, in the case $2n \geq 6$. If the maps in question were not injective then there would be non-zero rational polynomial Ξ in the Euler and Pontrjagin classes (or better, Hirzebruch L-classes) which vanishes. Such a counterexample would already exist over $\mathbb{Z}[\frac{1}{S}]$ for some large S, because Ξ has finitely-many rational coefficients and each Hirzebruch L-class is defined after inverting finitely-many primes. One can then try to obtain a contradiction by constructing non-linear representations $\phi : \mathbb{Z}/p \to \mathrm{Top}(2n)$ for infinitely-many primes $p > S$ satisfying $\phi^*\Xi \neq 0 \in H^{4i}(B\mathbb{Z}/p; \mathbb{Z}[\frac{1}{S}]) = \mathbb{Z}/p$. This what we do. We produce such ϕ's by constructing fake $(2n-1)$-dimensional lens spaces using Wall realisation and the calculation of the surgery obstruction groups $L^s_{2n}(\mathbb{Z}[\mathbb{Z}/p])$, then taking the non-linear $\mathbb{Z}/p$-representations given by the open cone on their universal covers. The crucial step is then to determine the Hirzebruch L-classes of these non-linear representations, which makes use of localisation in equivariant cohomology, as well as the Family Signature Theorem [6].

References

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