NON-TRIVIALITY OF TORSION UNIVERSAL CHARACTERISTIC CLASSES OF 3-MANIFOLD BUNDLES

by

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Abstract. — J. Ebert [**3**] has shown that there are no non-trivial rational universal characteristic classes of oriented 3-manifold bundles. We show that there do exist non-trivial *torsion* universal characteristic classes of oriented 3-manifold bundles.

Given a smooth fiber bundle $F \to E \xrightarrow{\pi} B$ having vertical tangent bundle T^vE, the Becker–Gottlieb pretransfer is a stable map $\Sigma^\infty B_+ \to \mathbf{Th}(-T^vE)$, and the transfer is a stable map $\mathrm{trf} : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ obtained from the pretransfer by composing with the inclusion $\omega : \mathbf{Th}(-T^vE) \to \mathbf{Th}(-T^vE \oplus T^vE) \simeq \Sigma^\infty E_+$. If the fiber F is oriented and d-dimensional, there is a universal pretransfer

$$\Sigma^\infty B_+ \to \mathbf{Th}(-T^vE) \to \mathbf{Th}(-\gamma_d \to BSO(d)) =: \mathbf{MTSO}(d).$$

These were studied by Madsen and Tillmann [**5**] in the case $d = 2$ where they showed, among other things, that the adjoint maps

$$\alpha_g : B\mathrm{Diff}^+(\Sigma_g) \longrightarrow \Omega^\infty\mathbf{MTSO}(2)$$

are non-trivial in the range of Harer stability [**4**].

For any dimension d, we consider $H^*(\Omega^\infty\mathbf{MTSO}(d))$ to be the ring of *universal characteristic classes* of oriented d-manifold bundles, in the sense that for any such bundle $M^d \to E \to B$ there is a map $\alpha_E : B \to \Omega^\infty\mathbf{MTSO}(d)$ adjoint to the universal pretransfer, and we pull back classes to obtain characteristic classes of $E \to B$. In [**3**], Ebert has shown that for any oriented 3-manifold M, the adjoint to the universal pretransfer for the universal smooth fibre bundle with fibre M,

$$\alpha_M : B\mathrm{Diff}^+(M) \longrightarrow \Omega^\infty\mathbf{MTSO}(3),$$

is trivial on rational cohomology, although $\Omega^\infty\mathbf{MTSO}(3)$ has plenty of rational cohomology. It is *not* true that all such maps are nullhomotopic: an argument suggested by the author which appears in [**3**] shows that the composite

$$BSO(4) \longrightarrow B\mathrm{Diff}^+(S^3) \xrightarrow{\alpha_{S^3}} \Omega^\infty\mathbf{MTSO}(3)$$

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is not nullhomotopic, although it is trivial in all ordinary homology theories. Ebert posed the question [**3**, Question 6.0.9] of whether there exists a manifold M having

$$\Sigma^\infty B\mathrm{Diff}^+(M)_+ \longrightarrow \mathbf{MTSO}(3)$$

non-trivial on $\mathbb{F}_p$-homology for some prime. The purpose of this note is to answer a variant of this question: is there a connected orientable 3-manifold M with the map α_M non-trivial on $\mathbb{F}_p$-homology for some prime?

We show that the example $M = T^3 = S^1 \times S^1 \times S^1$ at the prime 3 gives an affirmative answer to this question. This manifold has an orientation-preserving action of the cyclic group C_3 of order three — by cycling the factors — which gives a smooth fibre bundle

$$M \longrightarrow E \xrightarrow{\pi} BC_3.$$

We will study the composition

$$(*) \qquad BC_3 \longrightarrow B\mathrm{Diff}^+(M) \xrightarrow{\alpha_M} \Omega_0^\infty \mathbf{MTSO}(3) \xrightarrow{\omega} Q_0(BSO(3)_+) \longrightarrow Q_0(S^0),$$

and its effect on $\mathbb{F}_3$-homology.

Theorem A. — *The composition (∗) in $\mathbb{F}_3$-homology sends the generator e_4 of $H_4(BC_3;\mathbb{F}_3)$, dual to the square of the Euler class of the canonical complex representation, to the non-trivial class $Q^1([1]) * [-3] \in H_4(Q_0(S^0);\mathbb{F}_3)$. In particular, $H_*(\alpha_M;\mathbb{F}_3)$ is non-trivial.*

We will make use of standard facts about Dyer–Lashof operations and the homology of infinite loop spaces: our reference is [**2**], and we will use its notation. We require an auxiliary lemma on the Becker–Gottlieb transfer of the universal bundle $EC_3 \to BC_3$.

Lemma 0.1. — *The transfer map* trf : $BC_3 \longrightarrow Q_3(EC_{3+}) \simeq Q_3(S^0)$ *sends* $e_4 \in H_4(BC_3;\mathbb{F}_3)$ *to* $-Q^1([1]) \in H_4(Q_3(S^0);\mathbb{F}_3)$.

Proof. — Recall that $Q^1([1]) = -Q_4([1]) = -\theta_*(e_4 \otimes [1]^3)$ where θ_* is the action on homology of the operad action map $EC_3 \times_{C_3} (Q_1(S^0))^3 \to E\Sigma_3 \times_{\Sigma_3} (Q_1(S^0))^3 \xrightarrow{\theta_3} Q_3(S^0)$. Restricting to $BC_3 \times \{1\}$ this map is precisely the transfer map, so $\theta_*(e_4 \otimes [-1]^3) = \mathrm{trf}_*(e_4)$. □

Of course, the transfer map sends $e_2 \in H_2(BC_3;\mathbb{F}_3)$ (dual to the Euler class) to 0 in $H_2(Q_3(S^0);\mathbb{F}_3)$, as this group is zero.

Proof of Theorem A. — First note that this map coincides with the universal Euler characteristic,

$$BC_3 \xrightarrow{\mathrm{trf}_\pi} Q_0(EC_3 \times_{C_3} T_+) \longrightarrow Q_0(S^0),$$

where the first map is the transfer for the smooth fibre bundle π, and the second map collapses $EC_3 \times_{C_3} T$ to a point.

To compute the action of trf_π, we use the method of Brumfiel–Madsen [**1**] to reduce it to the computation of the transfer of a finite-sheeted cover. To do so we require a C_3-invariant vector field on T, or equivalently a $C_3 \ltimes \mathbb{Z}^3$-invariant vector field on $\mathbb{R}^3$. We take the vector field on $\mathbb{R}^3$ given by the gradient of the Morse function

$$(x_1, x_2, x_3) \mapsto \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi x_3).$$

On T this has 8 zeroes, on which C_3 acts by fixing two and having the remaining six fit into two transitive orbits. The two free orbits have indices ± 1, as do the two non-free orbits. Thus the map above coincides with

$$BC_3 \longrightarrow Q_1(BC_{3+}) \times Q_{-1}(BC_{3+}) \times Q_3(EC_{3+}) \times Q_{-3}(EC_{3+}) \longrightarrow Q_0(S^0)$$

where the first map is $\mathrm{Id} \times \chi \circ \mathrm{Id} \times \mathrm{trf} \times \chi \circ \mathrm{trf}$ and the second is the collapse map on each factor followed by the loop product map. By Lemma 0.1 this composition sends e_4 to

$$e_4 \otimes [-1] \otimes [3] \otimes [-3] + e_2 \otimes \chi(e_2) \otimes [3] \otimes [-3] + [1] \otimes \chi(e_4) \otimes [3] \otimes [-3]$$
$$-[1] \otimes [-1] \otimes Q^1[1] \otimes [-3] - [1] \otimes [-1] \otimes [3] \otimes \chi(Q^1[1])$$

but the top row maps to 0 in the homology of $Q_0(S^0)$. Thus we must understand the image of

$$-Q^1[1] \otimes [-3] - [3] \otimes \chi(Q^1[1])$$

under $Q_3(S^0) \times Q_{-3}(S^0) \to Q_0(S^0)$. We have the standard formula

$$\chi(Q^1[1]) = Q^1(\chi[1]) = Q^1[-1] = (Q^1[1]) * [-6]$$

and hence this element maps to $Q^1([1]) * [-3]$, as required. □

References

[1] G. Brumfiel and I. Madsen. Evaluation of the transfer and the universal surgery classes. *Invent. Math.*, 32(2):133–169, 1976.

[2] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 533.

[3] Johannes Ebert. A vanishing theorem for characteristic classes of odd-dimensional manifold bundles. arXiv:0902.4719. To appear in Crelle's Journal, 2009.

[4] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.

[5] Ib Madsen and Ulrike Tillmann. The stable mapping class group and $Q(\mathbb{CP}^\infty_+)$. *Invent. Math.*, 145(3):509–544, 2001.

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