ERRATUM TO: ON THE COHOMOLOGY OF TORELLI GROUPS

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Abstract. We resolve several mistakes surrounding the application of the results of the paper to the classical case $2n = 2$.

An overstatement. The statement "and is a monomorphism in degree $N + 1$, for all large enough g" in Theorem 4.1 of [KRW20] is not justified by the given proof, and should be removed. The corresponding statement should then be removed from Theorem B.

This means that in Theorem 8.1 only the calculation of $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}}$ can be obtained by employing Johnson's theorem that $H^1(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 3$. However, Theorem 8.1 can be rescued and even strengthened by applying the recent theorem of Minahan [Min23] that $H^2(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})$ is finite-dimensional for $g \geq 51$: using this, the equality

$$\begin{aligned} H^3(B\mathrm{Tor}(W_g, D^2); \mathbb{Q})^{\mathrm{alg}} = V_1 + V_{2,1} + 3V_{1^3} + 2V_{2^2,1} + 3V_{2,1^3} + V_{3,2,1^2} \\ + 2V_{2^3,1} + V_{3,2^3} + 4V_{1^5} + 2V_{2^2,1^3} + V_{3^2,1^3} \\ + 2V_{2,1^5} + V_{2^3,1^3} + 2V_{1^7} + V_{2^2,1^5} + V_{1^9} \end{aligned}$$

holds for all $g \gg 0$.

An expansion. Erik Lindell has pointed out that the last paragraph of the proof of Theorem 4.1 of [KRW20] is too brief. There we apply Proposition 2.16 with $B = H^i(B\mathrm{Tor}(W_g, D^{2n}); \mathbb{Q})$ and $i \leq N$, but have only assumed that these are finite-dimensional for $i < N$ and the statement of Proposition 2.16 asks for B to be a finite-dimensional G-representation. Nonetheless the conclusion is valid, by the following discussion.

Consider the setting of Proposition 2.16 but with B an arbitrary G-representation, and let $B^{\mathrm{alg}} \leq B$ denote its maximal algebraic subrepresentation, i.e. the union of its algebraic subrepresentations. The induced map $[K \otimes B^{\mathrm{alg}}]^G \to [K \otimes B]^G$ is then an isomorphism. As A is assumed to have finite length and $\phi^{\mathsf{Br}_{2g}} : i_*(A) \to [K \otimes B^{\mathrm{alg}}]^G$ is assumed to be an isomorphism, it follows that $[H(g)^{\otimes S} \otimes B^{\mathrm{alg}}]^G$ is finite-dimensional for every finite set S, and hence that $\mathrm{Hom}_G(V_\lambda, B^{\mathrm{alg}})$ is finite-dimensional for each irreducible algebraic G-representation V_λ. The evaluation map

$$\bigoplus_{\substack{\text{irreducible algebraic} \\ G\text{-representations } V_\lambda}} V_\lambda \otimes \mathrm{Hom}_G(V_\lambda, B^{\mathrm{alg}}) \longrightarrow B^{\mathrm{alg}}$$

is tautologically surjective, and there are finitely-many irreducibles, so B^{alg} is in fact finite-dimensional. One may now proceed with the proof of Proposition 2.16 as written.

A typo. On pp. 75-76 of [KRW20] (p. 52 of the arXiv version) we mistranscribed the computer-calculated Poincaré series for $H^*(B\mathrm{Tor}^+(W_g, *); \mathbb{Q})^{\mathrm{alg}}$ and $H^*(B\mathrm{Tor}^+(W_g); \mathbb{Q})^{\mathrm{alg}}$. In both cases the term $2s_{\langle 2^3, 1^3 \rangle}$ should instead be $s_{\langle 2^3, 1^3 \rangle}$.

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This now makes Remark 8.2 irrelevant: there is nothing to explain, as our expression now agrees with Sakasai's computation in [Sak05] (with the V_1 term present). Using Minahan's theorem as described above, this calculation completely describes $H^3(B\mathrm{Tor}(W_g,*);\mathbb{Q})^{\mathrm{alg}}$ and $H^3(B\mathrm{Tor}(W_g);\mathbb{Q})^{\mathrm{alg}}$.

Relation to Sakasai's result. On pp. 76-77 of [KRW20] (pp. 52-53 of the arXiv version) we described how to settle the ambiguity in Sakasai's paper [Sak05], but the argument given is fallacious. Even assuming Minahan's theorem, so that our calculations in degree 3 are valid, the image of the composition

$$\Lambda^3(V_{1^3}) \xrightarrow{\tau^*} H^3(B\mathrm{Tor}^+(W_g);\mathbb{Q}) \longrightarrow H^3(B\mathrm{Tor}(W_g,D^2);\mathbb{Q})$$

after applying $[-\otimes V_1]^{\mathrm{Sp}_{2g}(\mathbb{Z})}$ is *not* the subspace of those elements which can be represented by trivalent graphs with one leg, three internal vertices, and no loops as claimed, but is instead something more complicated: see [RW23, Section 3.4].

Nonetheless the conclusion is correct, as follows. With the correction indicated above our expression for $H^3(B\mathrm{Tor}(W_g);\mathbb{Q})^{\mathrm{alg}}$ agrees with Sakasai's expression for $\tau^*(\Lambda^3(V_{1^3}))$ with the V_1-term present, so showing that it should be present in Sakasai's paper is equivalent to showing that the $V_1 \leq H^3(B\mathrm{Tor}(W_g);\mathbb{Q})^{\mathrm{alg}}$ lies in the subspace spanned by products of degree-1 cohomology classes. It follows from Minahan's theorem and Theorem A, the discussion after it, and Section 5.2 of [RW23] that in fact all of $H^3(B\mathrm{Tor}(W_g);\mathbb{Q})^{\mathrm{alg}}$ is spanned by products of degree-1 classes (in the language of that paper, this is equivalent to the fact that $\mathcal{G}\mathrm{raph}_g(S)$ is spanned by trivalent graphs with all labels equal to 1, for any finite set S). Thus indeed the V_1-term should be present in Sakasai's result, and therefore $\kappa_{e^3} - (2-2g)e^2 \neq 0 \in H^4(B\mathrm{Tor}(W_g,*);\mathbb{Q})$ holds.

The argument given for Corollary 8.3 is correct, again invoking Minahan's theorem.

References

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