“TOPOLOGICAL CHIRAL HOMOLOGY” AND CONFIGURATION SPACES OF SPHERES

by

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Abstract. — We compute the rational homology of all spaces of finite configurations on spheres. Our tool is a bar spectral sequence which can be viewed as coming from the notion of “topological chiral homology”, though we give a self-contained construction of the spectral sequence.

1. Result

For a topological manifold M, let $C_n(M)$ denote the space of n unordered distinct points in M. The purpose of this short note is to give a proof of the following.

Theorem 1.1. — *Suppose $d \geq 2$ is even. Then*

$$\widetilde{H}_*(C_n(S^d);\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ in degree } 2d-1 & n \geq 3 \\ \mathbb{Q} \text{ in degree } d & n = 1 \\ 0 & n = 0, 2. \end{cases}$$

Suppose $d \geq 3$ is odd. Then

$$\widetilde{H}_*(C_n(S^d);\mathbb{Q}) = \begin{cases} \mathbb{Q} \text{ in degree } d & n \geq 1 \\ 0 & n = 0. \end{cases}$$

This theorem is not new (for example, it may be quickly deduced from [**8**, Theorem 18]), but the method of proof we offer is different and, we feel, somewhat interesting.

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2. A "multiplicative" decomposition of configuration spaces

Let $C_n(M, X)$ denote the space of n unordered points in a topological manifold M, labeled by the space X, and let

$$C(M, X) := \coprod_{n \geq 0} C_n(M, X).$$

If the manifold M has dimension d, fix an embedding $e : \mathbb{R}^d \hookrightarrow M$ and let $S(t) \subset M$ denote the image under e of the sphere of radius t centred at the origin of $\mathbb{R}^d$. Define

$$B_n := \{(t_0, \ldots, t_n; c) \in \mathbb{R}^{n+1}_{>0} \times C(M, X) \,|\, t_0 < \cdots < t_n \text{ and } c \cap S(t_i) = \emptyset \text{ for all } i\}.$$

Let $d_i : B_n \to B_{n-1}$ be the map that forgets t_i, and $\epsilon : B_0 \to C(M, X)$ be the map that forgets t_0; with this structure $B_\bullet$ is a semi-simplicial topological space, augmented over $C(M, X)$.

Proposition 2.1. — *The map $|\epsilon| : |B_\bullet| \to C(M, X)$ is a weak homotopy equivalence.*

Proof. — The augmented semi-simplicial space $\epsilon : B_\bullet \to C(M, X)$ is a "topological flag complex" in the sense of [**3**, Definition 6.1]. Furthermore, it satisfies the conditions of [**3**, Theorem 6.2]: conditions i) and ii) are clear, and for condition iii) we observe that for any configuration c, and any finite collection of elements of the set

$$\{t \in \mathbb{R}_{>0} \mid c \cap S(t) = \emptyset\},$$

which is the set of vertices over the configuration c, there exists another element of that set which is larger than them all, as the set is infinite (because the configuration c is finite). Theorem 6.2 of [**3**] then implies that the augmentation map induces a weak homotopy equivalence on geometric realisation. □

The space $C((0, 1) \times S^{d-1}, X)$ is an H-space via stacking cylinders end-to-end. If we write $\mathring{M}$ for $M \setminus e(\overline{D}_1)$ then the spaces $C(\mathring{M}, X)$ and $C(\mathbb{R}^d, X)$ are right and left H-modules over $C((0, 1) \times S^{d-1}, X)$ respectively. Thus, fixing a field $\mathbb{F}$,

(i) $A := H_*(C((0, 1) \times S^{d-1}, X); \mathbb{F})$ is a ring,
(ii) $H_*(C(\mathring{M}, X); \mathbb{F})$ is a right A-module,
(iii) $D := H_*(C(\mathbb{R}^d, X); \mathbb{F})$ is a left A-module.

Proposition 2.2. — *There is a spectral sequence*

$$E^2_{s,*} := \mathrm{Tor}^s_A(H_*(C(\mathring{M}, X); \mathbb{F}), D) \Longrightarrow H_*(C(M, X); \mathbb{F}). \tag{2.1}$$

Proof. — Let us write $D_t \subset \mathbb{R}^d$ for the open ball of radius t, and $\overline{D}_t$ for its closure. There is a map

$$\varphi : B_n \longrightarrow C(\mathring{M}, X) \times (C((0, 1) \times S^{d-1}, X))^n \times C(\mathbb{R}^d, X)$$

given by the canonical identifications of

(i) $M \setminus e(\overline{D}_{t_n})$ with $\mathring{M}$,
(ii) $D_{t_{i+1}} \setminus \overline{D}_{t_i}$ with $(0, 1) \times S^{d-1}$,
(iii) D_{t_0} with $\mathbb{R}^d$.

The product of φ with the map to $\mathbb{R}^{n+1}_{>0}$ given by $(t_0, \ldots, t_n; c) \mapsto (t_0, t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1})$ is a homeomorphism, and so φ is a homotopy equivalence. Furthermore, it is clear that φ gives an identification of semi-simplicial objects in the homotopy category of spaces, where

$$C(\mathring{M}, X) \times (C((0,1) \times S^{d-1}, X))^{\bullet} \times C(\mathbb{R}^d, X)$$

is such a semi-simplicial object via the H-space and H-module structure maps (and the simplicial identities hold by the homotopy associativity of these maps).

If we filter $|B_\bullet|$ by skeleta $|B_\bullet|^{(k)}$, then we have identifications

$$|B_\bullet|^{(k)}/|B_\bullet|^{(k-1)} \cong \Delta^k \times B_k/\partial\Delta^k \times B_k \cong S^k \wedge (B_k)_+.$$

The spectral sequence for this filtration has

$$E^1_{s,t} = H_{s+t}(|B_\bullet|^{(s)}, |B_\bullet|^{(s-1)}; \mathbb{F}) \cong H_t(B_s; \mathbb{F})$$

and, following [**9**, §5], we see that under this identification the differential $d^1 : E^1_{s,t} \to E^1_{s-1,t}$ is given by $\sum_{i=0}^{s}(-1)^i (d_i)_*$, the alternating sum of the maps induced on homology by the face maps. By the identification φ and the Künneth theorem we have

$$\begin{aligned} H_*(B_s; \mathbb{F}) &\cong H_*(C(\mathring{M}, X); \mathbb{F}) \otimes H_*(C((0,1) \times S^{d-1}, X); \mathbb{F})^{\otimes s} \otimes H_*(C(\mathbb{R}^d, X); \mathbb{F}) \\ &\cong H_*(C(\mathring{M}, X); \mathbb{F}) \otimes A^{\otimes s} \otimes D \end{aligned}$$

and by inspection of the differential d^1, the chain complex $(E^1_{*,*}, d^1)$ agrees with the bar complex $B(H_*(C(\mathring{M}, X); \mathbb{F}), A, D)$. Thus the E^2 page has the description claimed. □

Everything in sight has an extra grading: for any manifold N, there is a canonical decomposition $H_*(C(N, X)) = \bigoplus_{n \geq 0} H_*(C_n(N, X))$. We call this the *multiplicity grading*, and write the grading of an element as (h, m) where h is the homological grading and m is the multiplicity grading.

Remark 2.3. — The notion of topological chiral homology, c.f. [**4**, §5.3.2], [**1**], [**7**], roughly speaking associates to a framed E_n-algebra A (in topological spaces) and an n-manifold N a space $\int_N A$. The association $N \mapsto \int_M A$ is covariant, and sends disjoint union to cartesian product.

In particular, for an n-manifold N with boundary the space $\int_{[0,1]\times\partial N} A$ is an A_∞-algebra, as for each configuration of m little 1-cubes there is an embedding $\coprod^m [0,1] \times \partial N \to [0,1] \times \partial N$ to which $\int_{\_} A$ can be applied.

It can be shown that $\int_N A$ is a $\int_{[0,1]\times\partial N} A$-module (right or left, as $\int_{[0,1]\times\partial N} A$ has a canonical antiinvolution). Furthermore, if $\partial N = \partial N'$ then there is a natural equivalence

$$\left(\textstyle\int_N A\right) \otimes^{\mathbb{L}}_{\int_{[0,1]\times\partial N} A} \left(\textstyle\int_{N'} A\right) \longrightarrow \textstyle\int_{N \cup_{\partial N} N'} A,$$

from the (derived) tensor product of these two A_∞-modules. This gives a bar spectral sequence

$$\mathrm{Tor}^s_{H_*(\int_{[0,1]\times\partial N} A)}\left(H_*(\textstyle\int_N A), H_*(\int_{N'} A)\right) \Longrightarrow H_*(\textstyle\int_{N\cup_{\partial N} N'} A).$$

If we take $A = C(\mathbb{R}^d, X)$ to be the free E_d-algebra on a space X, then the topological chiral homology $\int_N A$ may be shown to be homotopy equivalent to $C(N, X)$, so taking $N = M \setminus \mathrm{int}(D^d)$ and $N' = D^d$ we obtain a spectral sequence

$$\mathrm{Tor}^s_{H_*(C([0,1]\times\partial S^{d-1}, X)}\Big(H_*(C(\mathring{M}, X)), H_*(C(\mathbb{R}^d, X))\Big) \Longrightarrow H_*(C(M, X))$$

which agrees with ours.

3. The structure of A and D in characteristic zero

Let $X = *$ and $\mathbb{F} = \mathbb{Q}$. In this section we wish to give a generators and relations description of the ring A and the left A-module D, and construct an explicit resolution (which will have length 1) of D as an A-module.

For a smooth manifold with boundary M, a choice of boundary component $\mathcal{E}$ gives a *stabilisation map*

$$s_{\mathcal{E}} : C_n(M) \longrightarrow C_{n+1}(M).$$

Let τM^+ denote the fibrewise one-point compactification of the tangent bundle of M, and $\Gamma_n(M)$ denote the space of sections of this bundle which are compactly supported in the interior of M, and which have degree n. There is an "electric charge", or "scanning", map

$$\mathcal{S} : C_n(M) \longrightarrow \Gamma_n(M),$$

cf. [**5**]. We shall need the following result. We state it for integral homology, though we only need it for rational homology.

Proposition 3.1. — *The map $\mathcal{S}$ induces an injection on integral homology, and an isomorphism on integral homology in degrees $2* \leq n$.*

Proof. — This is obtained by combining the main results of [**5**] and [**6**]. □

Remark 3.2. — In the following, for a set S we write $\mathbb{Q}[S]$ for the free commutative $\mathbb{Q}$-algebra on the set S, $\mathbb{Q}\langle S\rangle$ for the free noncommutative $\mathbb{Q}$-algebra on the set S, and $\mathbb{Q}\{S\}$ for the free $\mathbb{Q}$-vector space on the set S.

3.1. The disc: $C(\mathbb{R}^d)$. — We write $[n] \in H_0(C_n(\mathbb{R}^d); \mathbb{Q})$ for the class of any configuration of n points; these satisfy $s_{\mathcal{E}*}([n]) = [n+1]$, and $[n]$ has bidegree $(0, n)$. We also write $\tau \in H_{d-1}(C_2(\mathbb{R}^d); \mathbb{Q})$ for the image of the fundamental class under the map

$$S^{d-1} \longrightarrow C_2(\mathbb{R}^d) \tag{3.1}$$

which sends x to the configuration $\{0, x\}$, which has bidegree $(d-1, 2)$.

Proposition 3.3. — *The class $\tau^2 \in H_{2(d-1)}(C_4(\mathbb{R}^d); \mathbb{Q})$ is zero, τ and $[1]$ commute, and the induced map*

$$\phi : \begin{cases} \mathbb{Q}[[1]] & d \text{ odd} \\ \mathbb{Q}[[1], \tau]/(\tau^2) & d \text{ even} \end{cases} \longrightarrow H_*(C(\mathbb{R}^d)) = D$$

is an isomorphism.

Proof. — The scanning map in this case is

$$\mathcal{S} : C_n(\mathbb{R}^d) \longrightarrow \Omega^d_n S^d.$$

By a theorem of Serre, $\Omega^d_n S^d$ has trivial rational homotopy groups if d is odd, so also has trivial rational homology, and has a single nontrivial rational homotopy group $\pi_{d-1}(\Omega^d_n S^d) \otimes \mathbb{Q} \cong \mathbb{Q}$ if d is even. It is a simple calculation that in this case it also has a single nontrivial rational homology group in degree $(d-1)$, and we claim that as long as $n \geq 2$ the class $\mathcal{S}_*(\tau \cdot [n-2])$ is a generator. By the homotopy commutativity of the diagram

$$\begin{array}{ccc} C_2(\mathbb{R}^d) & \xrightarrow{\mathcal{S}} & \Omega^d_2 S^d \\ \big\downarrow s_{\mathcal{E}}^{\circ n-2} & & \simeq \big\downarrow - \cdot \mathcal{S}([n-2]) \\ C_n(\mathbb{R}^d) & \xrightarrow{\mathcal{S}} & \Omega^d_n S^d, \end{array}$$

and the injectivity of $\mathcal{S}_*$, it suffices to prove that $\tau \in H_{d-1}(C_2(\mathbb{R}^d);\mathbb{Q})$ is nontrivial. But $C_2(\mathbb{R}^d)$ is homeomorphic to $\mathbb{RP}^{d-1}$, an orientable manifold, and the map (3.1) has degree ± 2, so τ is nothing but $(\pm)$ twice the fundamental class of $\mathbb{RP}^{d-1}$, hence nontrivial. (For d odd, $H_{d-1}(\mathbb{RP}^{d-1};\mathbb{Q}) = 0$ so the class τ is zero.)

It is clear that τ and $[1]$ commute, by geometric considerations (the multiplication on $C(\mathbb{R}^d)$ extends to an E_d-algebra structure). The class $\mathcal{S}_*(\tau^2)$ lies in a group which is zero (as $\Omega^d_4 S^d$ has trivial rational homology in degree $2(d-1)$, by the above discussion), and $\mathcal{S}_*$ is injective so $\tau^2 = 0$ and we obtain an induced map ϕ as in the statement of the proposition.

This map is clearly an isomorphism in multiplicity grading 0 or 1, and it remains to show that $\phi_n : \mathbb{Q}\{[n], \tau \cdot [n-2]\} \to H_*(C_n(\mathbb{R}^d);\mathbb{Q})$ is an isomorphism for $n \geq 2$. But $\mathcal{S}_* \circ \phi_n$ is an isomorphism, and $\mathcal{S}_*$ is injective, so ϕ_n is an isomorphism too, as required. □

It is convenient to note, as we did in the proof, that the class τ is defined for all d, but is zero if d is odd.

3.2. The cylinder: $C((0,1) \times S^{d-1})$**.** — We perform an analysis similar to the above. There is a map

$$S^{d-1} \longrightarrow C_1((0,1) \times S^{d-1}) \qquad (3.2)$$

sending x to the one-point configuration $(\frac{1}{2}, x)$, and we let $\Delta \in H_{d-1}(C_1((0,1) \times S^{d-1});\mathbb{Q})$ be the image of the fundamental class, so $\deg(\Delta) = (d-1, 1)$. By identifying $\mathbb{R}^d$ with $(0,1) \times D^{d-1}_-$, the lower half of the cylinder, we obtain an inclusion

$$C(\mathbb{R}^d) \longrightarrow C((0,1) \times S^{d-1})$$

and we write $[n]$ and τ for the images of the elements defined in the previous section.

Proposition 3.4. — *In the ring $A = H_*(C((0,1) \times S^{d-1});\mathbb{Q})$ the relations*

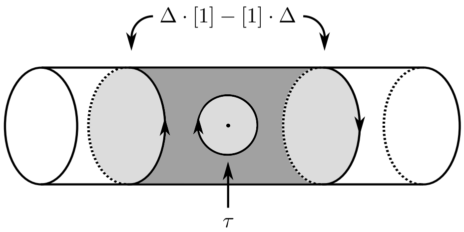
$$\tau \textit{ is central} \qquad [1] \cdot \Delta = \Delta \cdot [1] - \tau$$

hold. The induced map

$$\phi : \begin{cases} \mathbb{Q}[[1], \Delta] & d \text{ odd} \\ \mathbb{Q}\langle [1], \tau, \Delta\rangle/(\tau^2, [[1], \Delta] = -\tau, \tau \text{ central}) & d \text{ even} \end{cases} \longrightarrow H_*(C((0,1)\times S^{d-1}); \mathbb{Q}) = A$$

is an isomorphism.

Proof. — First consider the proposed relation $[1] \cdot \Delta = \Delta \cdot [1] - \tau$ in $H_{d-1}(C_2((0,1) \times S^{d-1}); \mathbb{Q})$. This follows from the homology between the cycles $\Delta \cdot [1] - [1] \cdot \Delta$ and τ, which may be seen in the following diagram of cycles in the configuration space of two points on the cylinder.



The fact that τ is central follows from similar geometric considerations (clearly τ and $[1]$ commute, then one sees from a similar figure that $[\Delta, \tau]$ is homologous to a cycle which is supported in a disc, but by the previous section $H_{2(d-1)}(C_3(\mathbb{R}^d); \mathbb{Q}) = 0$ so there are no nontrivial homology classes of this dimension supported in a disc).

The target of the scanning map in this case is $\mathrm{map}_n^\partial([0,1] \times S^{d-1}, S^d)$, the space of continuous maps $f : [0,1] \times S^{d-1} \to S^d$ which send $\{0,1\} \times S^{d-1}$ to the basepoint $* \in S^d$, and which have degree n in the sense that the induced map

$$f_* : H_d([0,1] \times S^{d-1}, \{0,1\} \times S^{d-1}; \mathbb{Z}) \longrightarrow H_d(S^d, *; \mathbb{Z})$$

sends the relative fundamental class to n times the fundamental class. The collection of all these mapping spaces fit into a fibration sequence

$$\Omega^d S^d \longrightarrow \mathrm{map}^\partial([0,1] \times S^{d-1}, S^d) \xrightarrow{p} \Omega S^d, \tag{3.3}$$

where p restricts a map to the interval $[0,1] \times \{*\}$. By taking the adjoint in the middle mapping space, we can express it as $\Omega \mathrm{map}(S^{d-1}, S^d)$. This exhibits the fibration sequence (3.3) as obtained from looping the evaluation fibration

$$\Omega^{d-1} S^d \longrightarrow \mathrm{map}(S^{d-1}, S^d) \longrightarrow S^d$$

so it is a principal fibration. Moreover, the evaluation fibration has a section, given by the inclusion of the constant maps, so after looping it splits as a product: thus (3.3) is a trivial fibration.

Let us compute the rational homology of the H-space $\mathrm{map}^\partial([0,1] \times S^{d-1}, S^d)$ with its Pontrjagin ring structure. Suppose first that d is odd. Then $\Omega^d_n S^d$ has trivial rational homology, and $\Omega S^d = \Omega\Sigma(S^{d-1})$ so by the Bott–Samelson theorem has rational Pontrjagin ring the free (non-commutative) algebra on $\widetilde{H}_*(S^{d-1}; \mathbb{Q})$, i.e. $\mathbb{Q}[u]$

where the class u is obtained from the map $S^{d-1} \to \Omega S^d$ adjoint to the identity map. As $p_*(\mathcal{S}_*(\Delta)) = u$ and p and $\mathcal{S}$ are H-space maps, it follows that

$$\mathbb{Q}[[1]^{\pm 1}, \mathcal{S}_*(\Delta)] \longrightarrow H_*(\mathrm{map}^\partial([0,1] \times S^{d-1}, S^d); \mathbb{Q})$$

is an isomorphism of rings.

Suppose now that d is even. We again have that $H_*(\Omega S^d; \mathbb{Q}) \cong \mathbb{Q}[u]$ as a ring. We have the path components $[n] \in H_0(\mathrm{map}^\partial([0,1] \times S^{d-1}, S^d); \mathbb{Q})$ for $n \in \mathbb{Z}$ as well as the classes $\mathcal{S}_*(\Delta)$ and $\mathcal{S}_*(\tau)$, which we will simply call Δ and τ again to save space, and we obtain an induced map

$$\mathbb{Q}\langle [1]^{\pm 1}, \Delta, \tau \rangle / (\tau^2, [[1], \Delta] = -\tau, \tau \,\mathrm{central}) \longrightarrow H_*(\mathrm{map}^\partial([0,1] \times S^{d-1}, S^d); \mathbb{Q}).$$

It follows from the fact that $p_*(\mathcal{S}_*(\Delta)) = u$ and that (3.3) is a fibration sequence of H-spaces which is trivial as a fibration of spaces (so the homology Serre spectral sequence is one of rings, in fact even of Hopf algebras c.f. [**2**, §5]) that this map is surjective, but by the splitting of (3.3) and counting dimensions it follows that in fact it is an isomorphism.

From these two calculations it follows that the induced map ϕ in the statement of the proposition is injective. To see that it is an isomorphism, note that the cokernel vanishes after stabilisation by $[n]$ (as ϕ induces an isomorphism after inverting $[1]$), so it is enough to show that if x, of bidegree $(k(d-1), m)$, is such that

$$x \cdot [n] = A \cdot \Delta^k \cdot [n+m-k] + B \cdot \Delta^{k-1} \cdot \tau \cdot [n+m-k-1] \tag{3.4}$$

for $n \gg 0$, then if $m-k<0$ then A and B are zero, and if $m-k=0$ then B is zero. We prove this by induction on the multiplicity grading of x: if $m=0$ then the class x is in the homology of $C_0((0,1) \times S^{d-1}) = *$, and the claim follows.

For the induction step, we use a map

$$t_* : H_*(C_n((0,1) \times S^{d-1}); \mathbb{Q}) \longrightarrow H_*(C_{n-1}((0,1) \times S^{d-1}); \mathbb{Q})$$

constructed as follows. Let $\pi : C_{n,1}((0,1) \times S^{d-1}) \to C_n((0,1) \times S^{d-1})$ denote the n-fold covering space whose total space consists of a configuration of n points in $(0,1) \times S^{d-1}$ with one distinguished point, and π forgets which point is distinguished. There is a map $f : C_{n,1}((0,1) \times S^{d-1}) \to C_{n-1}((0,1) \times S^{d-1})$ which removes the distinguished point, and we let t_* be the composition of the transfer map for the finite covering π followed by f_*. The construction of t_* shows that it is a derivation for the H-space multiplication, and it is easy to compute that

$$t_*(\Delta) = 0 \qquad t_*(\tau) = 0 \qquad t_*([1]) = [0].$$

If $m-k<0$ then applying t_* to (3.4) n times annihilates $\Delta^k \cdot [n+m-k]$ and $\Delta^{k-1} \cdot \tau \cdot [n+m-k-1]$, and we obtain

$$n! \cdot x + y \cdot [1] = 0$$

for y some expression in iterated applications of t_* to x. The class y is of bidegree $(k(d-1), m-1)$ and satisfies the analogue of (3.4), so by induction both A and B are zero, which finishes the proof in this case.

If $m-k=0$ then applying t_* to (3.4) n times gives

$$n! \cdot x + y \cdot [1] = A \cdot n! \cdot \Delta^k$$

and substituting back into (3.4) gives $y \cdot [n+1] = -n! \cdot B \cdot \Delta^{k-1} \cdot \tau \cdot [n-1]$. But y has bidegree $(k(d-1), m-1)$ so by induction $B = 0$, which finishes the proof in this case. □

3.3. D as an A-module (d even). — The left A-module structure on D is given by

$$[k] \bullet (\tau^\epsilon \cdot [n]) = \tau^\epsilon \cdot [n+k] \qquad \tau \bullet (\tau^\epsilon \cdot [n]) = \tau^{\epsilon+1} \cdot [n]$$

and

$$\Delta \bullet (\tau^\epsilon \cdot [n]) = n \cdot \tau^{\epsilon+1} \cdot [n-1].$$

The first two are clear and the last follows from $\Delta \bullet [0] = 0$ (as $C_1(\mathbb{R}^d)$ is contractible, so has trivial homology in degree $(d-1)$), $\Delta \bullet \tau = 0$ (as $C_3(\mathbb{R}^d)$ has trivial homology in dimension $2(d-1)$), and the commutation relation $[\Delta, [1]] = \tau$. One verifies that

$$0 \longrightarrow \Sigma^{d-1,1} A \xrightarrow{\cdot \Delta} A \xrightarrow{[0] \mapsto [0]} D \longrightarrow 0$$

is an exact sequence of left A-modules.

3.4. D as an A-module (d odd). — The left A-module structure on D is given by $[k] \bullet [n] = [n+k]$ and $\Delta \bullet [n] = 0$. One verifies that

$$0 \longrightarrow \Sigma^{d-1,1} A \xrightarrow{\cdot \Delta} A \xrightarrow{[0] \mapsto [0]} D \longrightarrow 0$$

is an exact sequence of left A-modules.

4. Configuration spaces of spheres

We apply the spectral sequence (2.1) to $M = S^d$, for d even. In this case $\mathring{M} = D^d$ so $H_*(C(\mathring{M}); \mathbb{Q}) = D$, but it has a right A-module structure. This is induced from the left A-module structure by the antiinvolution of A, which in turn is induced by reflecting the first coordinate of the cylinder around $\frac{1}{2}$. It is easy to see that this antiinvolution $\overline{} : A \to A$ is given on generators by

$$\overline{[n]} = [n] \qquad \overline{\tau} = -\tau \qquad \overline{\Delta} = \Delta.$$

Concretely, the right module structure is given by

$$(-) \bullet (\Delta^i \cdot \tau^\epsilon \cdot [n]) := ([n] \cdot (-\tau)^\epsilon \cdot \Delta^i) \bullet (-).$$

We have shown that D has a length 1 resolution by free A-modules, so the complex $\Sigma^{d-1,1} D \xrightarrow{- \bullet \Delta} D$ computes the E^2 page of the spectral sequence. This map is given explicitly by

$$(\tau^\epsilon \cdot [n]) \mapsto (\tau^\epsilon \cdot [n]) \bullet \Delta = \Delta \bullet (\tau^\epsilon \cdot [n]) = n \cdot \tau^{\epsilon+1} \cdot [n-1],$$

so is as shown in Figure 1.

By inspection there can be no further differentials in the spectral sequence, and we immediately see the correct rational homology for $C_0(S^d) = *$, $C_1(S^d) = S^d$, and $C_2(S^d) \simeq \mathbb{R}P^d$, and also deduce that for all $n \geq 3$, $C_n(S^d)$ has the rational homology of S^{2d-1}.

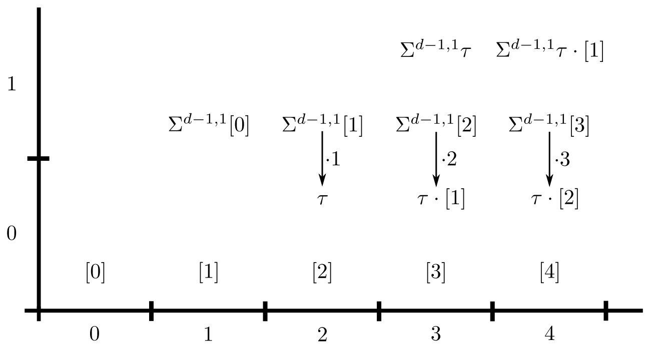


FIGURE 1. A complex computing the E^2 page of the spectral sequence for d even, with multiplicity grading along the horizontal axis, and Tor grading along the vertical axis. The homological grading is not shown.

The analysis for d odd is the same, but a little easier as the element τ does not appear.

5. Final remarks

The reader will realise that the multiplicative decomposition technique described in Section 2 admits many variations. Let us describe one, which leads to what appears to be a difficult calculation in homological algebra. For the manifold $M = S^1 \times S^{d-1}$ we may let

$$B_n := \left\{ (p_0, \ldots, p_n; c) \in (S^1)^{n+1} \times C(M) \;\middle|\; \begin{array}{l} p_i \text{ distinct, cyclically ordered} \\ c \cap \{p_i\} \times S^{d-1} = \emptyset \end{array} \right\}$$

and as in Section 2 we may show that the augmentation $|B_\bullet| \to C(M)$ is a homotopy equivalence. Following the proof of Proposition 2.2, we find that the E^1 page of the resulting spectral sequence is now the *cyclic* bar complex for the algebra $A = H_*(C((0,1) \times S^{d-1}); \mathbb{F})$, so we have a spectral sequence

$$E^2_{s,*} = HH_s(A, A) \Longrightarrow H_*(C(S^1 \times S^{d-1}); \mathbb{F}).$$

starting with the Hochschild homology of the algebra A with coefficients in itself. For $\mathbb{F} = \mathbb{Q}$, one ought to be able to use the calculation in Section 3.2 of the algebra A to study this spectral sequence, but the homological algebra seems to be a lot harder.

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