ERRATUM TO: THE HOMOLOGY OF THE STABLE NONORIENTABLE MAPPING CLASS GROUP

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Abstract. We correct the results of Section 4 of the paper mentioned in the title. The error is a technical one, and the main results of the paper are unaffected.

T. Kashiwabara has pointed out that the formula of [RW08, Theorem 4.3] cannot be correct, as $\bar{\omega}_* \partial_* d_*(e_i \otimes e_j)$ should be invariant under interchanging the role of i and j. Our error occurs in [RW08, Proposition 4.2], where we have misapplied the correction [MT01, §3.2] to the Brumfiel–Madsen formula [BM76, Theorem 2.10] for the transfer. Here we fix that error. Fortunately the goal of Section 4, Proposition 4.5, is unaffected, and so the main results of the paper hold without modification.

The following four statements replace those of the corresponding name.

Proposition 4.2′. *Let $\pi : EO_1 \to BO_1$ be the universal cover, and $t : BO_1 \to QS^0$ be its associated transfer map. Let $i : BO_1 \to Q(BO_{1+})$ be the natural inclusion and $\iota : QS^0 \to Q(BO_{1+})$ be the inclusion of a basepoint. Write $\wedge$ for the natural map $Q(X) \times Q(X') \to Q(X \wedge X')$.*

Then in the group $[BO_1 \times BO_1, Q(BO_{1+})]$ the composition $BO_1 \times BO_1 \xrightarrow{d} BO_2 \xrightarrow{T} Q(BO_{1+})$ is homotopic to $t \wedge i + i \wedge t - \iota \circ (t \wedge t)$.

Proof. This is an application of Theorem 2.10 of [BM76], using the correction to the index described in [MT01, §3.2].

There is a $O_1 \times O_1$-invariant vector field ξ on S^1 with 4 zeroes, split into two orbits of two each. The stabilisers of these orbits are the subgroups $O_1 \times \{e\}$ and $\{e\} \times O_1$ respectively, and we may suppose (by changing the sign of the vector field if necessary) that on the orbit stabilised by $\{e\} \times O_1$ the vector field is pointing outwards, and on the orbit stabilised by $O_1 \times \{e\}$ the vector field is pointing inwards. Following [MT01, §3.2], there is a commutative diagram

$$\begin{array}{ccccc}
BO_1 \times BO_1 & \xrightarrow{d} & Q(BO_{2+}) & \xrightarrow{T} & Q(BO_{1+}) \\
\downarrow{\scriptstyle (t\wedge i)\times(i \wedge t)} & & & & \uparrow{\scriptstyle \mu} \\
Q(BO_{1+}) \times Q(BO_{1+}) & & \xrightarrow{IND(1)\times IND(2)} & & Q(BO_{1+}) \times Q(BO_{1+})
\end{array}$$

where $IND(1)$ and $IND(2)$ are units of the Burnside rings $A(\{e\} \times O_1)$ and $A(O_1 \times \{e\})$ respectively, which we must determine.

For a point $x \in S^1$ in the orbit stabilised by $\{e\} \times O_1$, the derivative of the vector field ξ gives a $\{e\} \times O_1$-equivariant automorphism of the one-point compactification $(T_x S^1)^+$, which is a circle. This is a degree $+1$ map, and restricted to the $\{e\} \times O_1$ fixed points is the identity, so also a degree $+1$ map. Thus the corresponding element of the Burnside ring is $1 \in A(\{e\} \times O_1)$. The automorphism of $Q(BO_{1+})$ induced by this element is the identity map, so $IND(1) = id$.

For a point $x \in S^1$ in the orbit stabilised by $O_1 \times \{e\}$, the $O_1 \times \{e\}$-equivariant automorphism of $(T_x S^1)^+$ given by the vector field ξ has degree -1, but restricted

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to the $O_1 \times \{e\}$ fixed points is the identity, so has degree $+1$. The corresponding element of the Burnside ring is $1 - O_1 \in A(O_1 \times \{e\})$. The automorphism of $Q(BO_{1+})$ induced by this element is $IND(2) = id - \iota \circ t$, the identity map minus the composition $Q(BO_{1+}) \stackrel{\mathrm{trf}_\pi}{\to} Q(EO_{1+}) \stackrel{\pi}{\to} Q(BO_{1+})$ where $\pi : EO_1 \to BO_1$ is the covering space corresponding to the O_1-set O_1.

Thus, $T \circ d = t \wedge i + (id - \iota \circ t) \circ (i \wedge t) = t \wedge i + i \wedge t - \iota \circ (t \wedge t)$. □

This description may also be proved using the "Mayer–Vietoris" or "inclusion-exclusion" property of the transfer, as explained in e.g. [MP89, §1].

Theorem 4.3′. *The composition*

$$H_*(BO_1 \times BO_1) \stackrel{d_*}{\longrightarrow} H_*(BO_2) \longrightarrow H_*(Q_1(BO_{2+})) \stackrel{\bar{\omega}_* \partial_*}{\longrightarrow} H_*(Q_0(BO_{1+}))$$

sends $e_i \otimes e_j$ *to*

$$\sum_{\substack{a+b+c=i \\ x+y+z=j}} \left(\sum_{s \geq 0} \binom{x-s}{s} Q^{a+s}(e_{x-s}) \right) * \left(\sum_{t \geq 0} \binom{b-t}{t} Q^{y+t}(e_{b-t}) \right) * \chi \left(\sum_{u \geq 0} \binom{z-u}{u} Q^{c+u} Q^{z-u}([1]) \right)$$

which modulo decomposables is $[-2]$ *times*

$$\sum_{s \geq 0} \binom{j-s}{s} Q^{i+s}(e_{j-s}) + \sum_{t \geq 0} \binom{i-t}{t} Q^{j+t}(e_{i-t}) + \sum_{u \geq 0} \binom{j-u}{u} Q^{i+u} Q^{j-u}([1])[-2].$$

Proof. This is as in [RW08, Theorem 4.3], but using Proposition 4.2′. The extra factor is that induced by the composition

$$Q(BO_1 \times BO_{1+}) \stackrel{t \wedge t}{\longrightarrow} Q(EO_1 \times EO_{1+}) \simeq QS^0 \stackrel{\iota}{\longrightarrow} Q(BO_{1+}),$$

which sends $e_c \otimes e_z$ to $\iota_*(Q^c([1]) \wedge Q^z([1]))$. To express this is elementary terms, we use the formula

$$Q^c([1]) \wedge Q^z([1]) = \sum_{u \geq 0} Q^{c+u}([1] \wedge Sq^u_* Q^z([1])) = \sum_{u \geq 0} Q^{c+u}(Sq^u_* Q^z([1]))$$

from [CLM76, p. 15] and the Nishida relation $Sq^u_* Q^z([1]) = \binom{z-u}{u} Q^{z-u}([1])$ from [CLM76, p. 6]. □

Corollary 4.4′. *In* $\mathrm{Ker}(Q\bar{\partial}_*)$ *the element* $Q\bar{\omega}_* \partial_* d_*(e_i \otimes e_j)$ *is*

$$\sum_{s \geq 0} \binom{j-s}{s} v^{i+s, j-s} + \sum_{t \geq 0} \binom{i-t}{t} v^{j+t, i-t} + \sum_{u \geq 0} \binom{j-u}{u} Q^{i+u}(v^{j-u, 0}).$$

In particular $Q\bar{\omega}_* \partial_* d_*(e_i \otimes e_0) = v^{i,0}$.

Proof. The first two terms are treated as in [RW08, Corollary 4.4]. For the last term, consider the Adem relation $Q^x Q^0 = \sum_{b \geq 0} \binom{x-b-1}{b-1} Q^b Q^{x-b}$ (cf. [CLM76, p. 6]) which shows that

$$v^{x,0} = Q^x([1])[-2] + \sum_{b \geq 0} \binom{x-b-1}{b-1} Q^b(e_{x-b})[-2].$$

Now, if $b > x-b$ then $b-1 > x-b-1$ so the binomial coefficient vanishes, but if $b \leq x-b$ then $Q^b(e_{x-b})$ is either zero (if $b < x-b$) or e^2_{x-b} (if $b = x-b$), and in either case is decomposable. Thus $v^{x,0} = Q^x([1])[-2]$, and so $Q^y(v^{x,0}) = Q^y Q^x([1])[-4]$.

The second claim follows from the observation that $v^{0,0} = 0$ and that the sum $\sum_{t \geq 0} \binom{i-t}{t} v^{t, i-t}$ is trivial: if $t > i-t$ then the binomial coefficient vanishes, and if $t \leq i-t$ then the sequence $(t, i-t)$ is admissible, so by our convention $v^{t,i-t}$ is zero. □

Proposition 4.5′. *The map* $Q(\partial_*) : QH_*(Q_0(BO_{2+})) \longrightarrow QH_*(\Omega_0^\infty \mathbf{MTO}(1))$ *is surjective.*

Proof. We require a small modification to the proof of [RW08, Proposition 4.5], due to the corrected formula in Corollary 4.4′. Namely, the element $Q(\partial_* d_*)(e_a \otimes e_{i-a})$ is now

$$\sum_{s\geq 0} \binom{i-a-s}{s} V^{a+s,i-a-s} + \sum_{t\geq 0} \binom{a-t}{t} V^{i-a+t,a-t} + \sum_{u\geq 0} \binom{i-a-u}{u} Q^{a+u}(V^{i-a-u,0}),$$

the first sum is still zero, the second sum is still $V^{i-a,a} + G^{a-1}$, and the new, last, sum lies in $\mathcal{R} \cdot G^0$. As G^0 lies in $\mathrm{Im}(Q(\partial_*))$, and the image of $Q(\partial_*)$ is closed under the action of the Dyer–Lashof algebra, the third sum lies in $\mathrm{Im}(Q(\partial_*))$. Hence if G^{a-1} lies in $\mathrm{Im}(Q(\partial_*))$ so does $V^{i-a,a}$; by induction G^∞ lies in $\mathrm{Im}(Q(\partial_*))$. We then continue as in the proof of [RW08, Proposition 4.5]. □

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