Monoids of moduli spaces of manifolds

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(joint work with Søren Galatius)

The work of Tillmann, Madsen, Weiss and Galatius [5, 3, 4, 1] has firmly linked the study of diffeomorphism groups (or mapping class groups) of oriented surfaces to the spectrum known as $\mathbf{MTSO}(2)$, which is the Thom spectrum of the complement to the canonical vector bundle over $BSO(2)$, and to *cobordism categories.*

Let $\theta : \mathbf{X} \to BO(d)$ be a fibration, and define a θ-structure on a d-dimensional vector bundle $V \to B$ is a fibrewise linear isomorphism $V \to \theta^*\gamma_d$. Let $\mathrm{Bun}(V, \theta^*\gamma_d)$ denote the space of such maps. Recall from [1] that the cobordism category $\mathcal{C}_\theta$ is a topological category roughly defined as follows. It has objects subsets $M \subset \mathbb{R}^\infty$ which are compact $(d-1)$-dimensional submanifolds, along with a θ-structure on $\epsilon^1 \oplus TM$. It has as morphisms subsets $W \subset [0, a] \times \mathbb{R}^\infty$ which are compact, collared d-manifolds, having boundary in $\{0, a\} \times \mathbb{R}^\infty$, along with a θ-structure on TW. Composition is via union of subsets of $\mathbb{R} \times \mathbb{R}^\infty$. Galatius, Madsen, Tillmann and Weiss [1] proved that there is a homotopy equivalence

$$B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

where $\mathbf{MT}\theta$ is the Thom spectrum of the complement to the bundle classified by the map θ.

We define another category $\mathcal{C}_\theta^\bullet$, the *pointed cobordism category*, which is roughly the subcategory of $\mathcal{C}_\theta$ whose objects are connected and contain the origin, and whose morphisms are connected and contain the interval $[0, a] \times \{0\}$. Our first theorem is as follows.

Theorem A. *Let $d \geq 2$ and $\theta : \mathbf{X} \to BO(d)$ be a tangential structure such that $\mathbf{X}$ is path connected and S^d admits a θ-structure. Then the map*

$$B\mathcal{C}_\theta^\bullet \to B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

is a weak homotopy equivalence.

In dimension two, this can be improved.

Theorem B. *Let $\theta : \mathbf{X} \to BO(2)$ be a tangential structure such that $\mathbf{X}$ is path connected and that S^2 admits a θ-structure. Let $\mathcal{D} \subseteq \mathcal{C}_\theta^\bullet$ be a full subcategory. Then the inclusion*

$$B\mathcal{D} \to B\mathcal{C}_\theta^\bullet$$

is a weak homotopy equivalence of each component of $B\mathcal{D}$ onto a component of $B\mathcal{C}_\theta^\bullet$. Additionally, the monoid of endomorphisms of any object $c \in \mathcal{C}_\theta^\bullet$ is homotopy commutative.

The implication of these theorems are as follows. Suppose θ is a tangential structure satisfying the hypotheses of Theorem B. Pick an object $c \in \mathcal{C}_\theta^\bullet$. Then

$$\mathrm{End}_{\mathcal{C}_\theta^\bullet}(c) \simeq \coprod_{[F]} \mathrm{Bun}_c(TF, \theta^*\gamma_2) /\!\!/ \mathrm{Diff}(F)$$

is a full subcategory of $\mathcal{C}_\theta^\bullet$, where the disjoint union is over diffeomorphism types of θ-surfaces with one boundary circle and Bun_c denotes the space of those bundle maps which agree with that of c on the boundary of F. By Theorems A and B the group completion of this topological monoid is homotopy equivalent to a collection of path components of $\Omega^\infty \mathbf{MT}\theta$.

In cohomology (certainly over a field $\mathbb{F}$, and possibly more generally), the group completion theorem of McDuff–Segal [6] asserts that for a homotopy commutative monoid $\mathcal{M}$, the natural map $\mathcal{M} \to \Omega B\mathcal{M}$ induces a map on cohomology which is an isomorphism after taking $\pi_0(\mathcal{M})$-invariants:

$$H^*(\mathcal{M};\mathbb{F})^{\pi_0(\mathcal{M})} \xleftarrow{\simeq} H^*(\Omega_0 B\mathcal{M};\mathbb{F}).$$

In our setting we obtain that

$$H^*(\Omega_0^\infty \mathbf{MT}\theta;\mathbb{F}) \longrightarrow H^*\left(\coprod_{[F]} \mathrm{Bun}_c(TF, \theta^*\gamma_2)/\!\!/\mathrm{Diff}(F);\mathbb{F}\right)$$

is an isomorphism onto the $\pi_0(\mathrm{End}_{\mathcal{C}_\theta^\bullet}(c))$-invariants; that is, that this map is injective and that its image is precisely those characteristic classes of bundles of θ-surfaces with boundary which are invariant under gluing on trivial bundles. This statement can perhaps be generalised to higher-dimensional manifolds.

Question. Is there a homotopy commutative monoid $\mathcal{M}$ constructed from classifying spaces of diffeomorphism groups of oriented d-manifolds (with boundary) and composition given by gluing manifolds along their boundary, such that the natural map

$$H^*(\Omega_0^\infty \mathbf{MTSO}(d)) \longrightarrow H^*(\mathcal{M})^{\pi_0(\mathcal{M})}$$

to the $\pi_0(\mathcal{M})$-invariants is surjective, with kernel as conjectured by J. Ebert in this volume?

References

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