**Monoids of moduli spaces of manifolds, II**

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(joint work with Søren Galatius)

In my talk I announced new results on the homology of stable diffeomorphism groups of highly-connected manifolds. In order to motivate these results, we consider first the diffeomorphism groups of compact 0-manifolds: these are the symmetric groups and have classifying spaces $B\Sigma_n$. Adding an extra point to a 0-manifold induces maps $B\Sigma_n \to B\Sigma_{n+1}$, and theorems of Barratt–Priddy [1] and Quillen [7] provide a homology equivalence

$$\mathrm{hocolim}_{n\to\infty} B\Sigma_n \longrightarrow \Omega^\infty_0 \mathbf{S}$$

from the direct limit to a component of the infinite loop space corresponding to the sphere spectrum.

Moving up two dimensions, let us write $B\mathrm{Diff}^+_\partial(\Sigma_{g,1})$ for the classifying space of the group of diffeomorphisms of a connected, oriented surface of genus $g$ with a single boundary component, where the diffeomorphisms are required to restrict to the identity on the boundary. Gluing on a torus with two boundary components along a single boundary gives a map $B\mathrm{Diff}^+_\partial(\Sigma_{g,1}) \to B\mathrm{Diff}^+_\partial(\Sigma_{g+1,1})$. The theorem of Madsen and Weiss [5] provides a homology equivalence

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}^+_\partial(\Sigma_{g,1}) \longrightarrow \Omega^\infty_0 \mathbf{MTSO}(2)$$

from the direct limit to a component of the infinite loop space corresponding to the negative of the tautological bundle over $BSO(2)$, which was used to compute the stable rational homology of Riemann's moduli space of curves, and hence prove the Mumford conjecture.

Our results give an extension of these theorems to all higher even dimensions, except dimension 4. In order to state them, let us write

$$\theta : BO(2n)[n+1,\infty) \longrightarrow BO(2n) \qquad \bar\theta : BO(2n)[n,\infty) \longrightarrow BO(2n)$$

for the $n$- and $(n-1)$-connected covers of $BSO(2n)$, and let $\mathbf{MT}\theta$ and $\mathbf{MT}\bar\theta$ be the Thom spectra of the virtual bundles $-\theta^*\gamma_{2n}$ and $-\bar\theta^*\gamma_{2n}$ respectively, where $\gamma_{2n} \to BO(2n)$ is the universal bundle.

**Theorem A.** *Let $W_g := \#^g S^n \times S^n$, and $n \neq 2$. The map*

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g, D^{2n}) \longrightarrow \Omega^\infty_0 \mathbf{MT}\theta$$

*is a homology equivalence.*

**Theorem B.** *Suppose $W$ is a $(n-1)$-connected $2n$-manifold such that $\pi_n(W) \to \pi_n(BO)$ is surjective. Let $\overline{W}_g := W \# W_g$, and $n \neq 2$. The map*

$$\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(\overline{W}_g, D^{2n}) \longrightarrow \Omega^\infty_0 \mathbf{MT}\bar\theta$$

*is a homology equivalence.*

In both of these statements, we may replace the homotopy colimit by the classifying space of a diffeomorphism group with compact support, and so compute the homology of $B\mathrm{Diff}_c(W_\infty)$ and $B\mathrm{Diff}_c(\overline{W}_\infty)$ respectively.

These theorems follow from a more technical statement about *cobordism categories*. Let $\theta : \mathbf{X} \to BO(d)$ be a map and $\mathcal{C}_\theta$ be the associated cobordism category. Roughly speaking, it's morphisms are $d$-dimensional cobordisms in $[0,t]\times\mathbb{R}^\infty$ equipped with a $\theta$-structure on their tangent bundles, and objects are closed $(d-1)$-manifolds in $\mathbb{R}^\infty$ with a $\theta$-structure on their once-stabilised tangent bundles. For technical reasons it is convenient to also assume that the line $[0,t]\times\{0\}$ is contained in every cobordism, and the origin is contained in every object. A full and detailed definition appears in [3] (based on the definition in [2]) where the category is called $\mathcal{C}_\theta^\bullet$.

We first filter the category $\mathcal{C}_\theta$ by subcategories $\mathcal{C}_\theta^\kappa$ containing all objects, but only those morphisms $W$ such that $(W, \partial_{out} W)$ is $\kappa$-connected. We further filter each $\mathcal{C}_\theta^\kappa$ by the full subcategories $\mathcal{C}_\theta^{\kappa,\ell}$ on the objects which are $\ell$-connected: that is, $M \subset \mathbb{R}^\infty$ containing the origin is an object of $\mathcal{C}_\theta^{\kappa,\ell}$ precisely if $\pi_{\leq \ell}(M, 0) = 0$.

**Theorem C.** *For $d \neq 4$, the inclusion*

$$B\mathcal{C}_\theta^{\kappa,\ell} \longrightarrow B\mathcal{C}_\theta \simeq \Omega^{\infty-1}\mathbf{MT}\theta$$

*is a weak homotopy equivalence as long as*

(1) $2\kappa \leq d-1$,
(2) $\ell \leq \kappa$,
(3) $\ell + \kappa \leq d-2$
(4) $\mathbf{X}$ *is $\ell$-connected.*

*Furthermore, we require the technical assumption that $\theta : \mathbf{X} \to BO(d)$ is the homotopy pullback of a map to $BO(d+1)$.*

In the case $d = 2n \neq 4$, the objects of $\mathcal{C}_\theta^{n-1,n-1}$ are $(n-1)$-connected $(2n-1)$-manifolds and hence homotopy spheres, but we can say more. Let us restrict ourselves to the case where $\theta : \mathbf{X} \to BO(2n)$ is either the $n$- or $(n-1)$-connected cover of $BO(2n)$, and choose an object of $\mathcal{C}_\theta^{n-1,n-1}$ diffeomorphic to the standard sphere. Let $\mathcal{E}$ denote the monoid of endomorphisms of $S^{2n-1}$ in the category $\mathcal{C}_\theta^{n-1,n-1}$. We prove that in this case the inclusion

$$B\mathcal{E} \longrightarrow B\mathcal{C}_\theta^{n-1,n-1}$$

is a weak homotopy equivalence onto the path component it hits, which along with Theorem C identifies the group-completion

$$\Omega B\mathcal{E} \simeq \Omega^\infty \mathbf{MT}\theta.$$

It is not difficult to verify that the monoid $\mathcal{E}$ is homotopy-commutative, and so the group-completion theorem [6] may be applied to it. Theorems A and B follow from two observations about the monoid $\mathcal{E}$: firstly that the submonoid $\mathcal{E}'$ consisting of those path components represented by manifolds $W$ such that $\pi_d(W) \to \pi_d(BO)$ is surjective is a cofinal submonoid (and hence has the same group completion), and secondly that the discrete monoid $\pi_0(\mathcal{E}')$ may be group-completed by inverting multiplication by $S^d \times S^d$, which follows by a theorem of Kreck [4].

## References


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