THE COMPLEX OF INJECTIVE WORDS IS HIGHLY CONNECTED

OSCAR RANDAL-WILLIAMS

Let S be a finite set. The *complex of injective words* $X(S)_\bullet$ is the semi-simplicial set

$$X(S)_p := \mathrm{Inj}([p], S),$$

in which the (semi-)simplicial structure maps are given by precomposition. We think of elements of $X(S)_p$ as being words of length $(p+1)$ in the alphabet S, in which each letter occurs at most once.

The geometric realisation $|X(S)_\bullet|$ clearly has dimension $(|S|-1)$. It has been shown by F. D. Farmer that $|X(S)_\bullet|$ is in fact $(|S|-2)$-connected, so that it is in fact a (finite) wedge of $(|S|-1)$-spheres. This note gives what must surely be a geodesic proof of the easier fact that

Theorem 1. *The space $|X(S)_\bullet|$ has trivial homology in degrees $0 < * \leq \frac{|S|-1}{2}$.*

This worse range is sufficient to prove homological stability for symmetric groups with the optimum range.

Proof. The claim holds trivially for $|S| \leq 2$ and can be shown by hand for $|S| = 3$, so let $|S| \geq 4$. Let $X := |X(S)_\bullet|$, and for each $\alpha \in S$ let $X_\alpha := |X(S \setminus \alpha)_\bullet| \subset X$ be the sub-CW-complex of those injective words which do not use the letter α. The map

$$Y := \bigcup_{\alpha \in S} X_\alpha \hookrightarrow X$$

is the inclusion of the $(|S|-2)$-skeleton (every word of length strictly less than $|S|$ misses some letter), so induces a surjection on homology in degrees $* \leq |S|-2$.

Apply the Mayer–Vietoris spectral sequence to the cover of Y by the X_α (these are not open, but are sub-CW-complexes so we are safe), giving

$$E^1_{p,q} = \bigoplus_{\{\alpha_0, \alpha_1, \ldots, \alpha_p\} \subset S} H_q(X_{\alpha_0} \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_p}) \Longrightarrow H_{p+q}(Y).$$

Note that $X_{\alpha_0} \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_p} = |X(S \setminus \{\alpha_0, \alpha_1, \ldots, \alpha_p\})_\bullet|$, which we may suppose by induction to have trivial homology in degrees $0 < * \leq \frac{|S|-p-2}{2}$. Thus $E^1_{p,q} = 0$ for $0 < q \leq \frac{|S|-p-2}{2}$. Furthermore, we recognise the row $(E^1_{*,0}, d^1)$ as the simplicial chain complex of the boundary of the simplex on the set of vertices S, so it just has homology $\mathbb{Z}$ in degrees 0 and $(|S|-2)$. Now $\frac{|S|-1}{2} < |S|-2$ by our assumption that $|S| \geq 4$, so $E^2_{p,q} = 0$ for $p \geq 1$ and $p+q \leq \frac{|S|-1}{2}$.

It follows that the edge homomorphism $\bigoplus_{\alpha \in S} H_*(X_\alpha) \to H_*(Y)$ is onto in degrees $0 \leq * \leq \frac{|S|-1}{2}$. However the inclusion $X_\alpha \hookrightarrow X$ is nullhomotopic: adding the letter α last gives a semi-simplicial contraction. Thus $H_*(Y) \to H_*(X)$ is zero for $0 < * \leq \frac{|S|-1}{2}$, but is also surjective for $* \leq |S|-2$. As the former range is included in the latter as long as $|S| \geq 3$, it follows that X has the claimed vanishing of its homology. □

Email address: o.randal-williams@dpmms.cam.ac.uk

Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK