MILLER–MORITA–MUMFORD CLASSES VANISH ON THE MODULI SPACE OF HYPERSURFACES

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The following is now redundant in view of the work of Aumonier [Aum23], especially Section 8.2 of that paper. Namely, he shows that the universal family of smooth hypersurfaces admits a particular tangential structure, from which it is easy to see that all Miller–Morita–Mumford classes vanish.

Let us write

$$U_d^n \subset H^0(\mathbb{CP}^{n+1}; \mathcal{O}(d))$$

for the open subset of those homogeneous polynomials of degree d which define a smooth hypersurface. The group $GL_{n+1}(\mathbb{C})$ acts on the space of such polynomials and hence on U_d^n, and this action has finite stabilisers. We let

$$M_d^n := U_d^n /\!\!/ GL_{n+1}(\mathbb{C})$$

denote the orbifold, or stack, quotient of this action. This is the moduli space of degree d hypersurfaces of dimension n. Tommasi [Tom14] has proved the following surprising vanishing result.

Theorem 0.1 (Tommasi)**.** *$H^i(M_d^n; \mathbb{Q}) = 0$ for $0 < i < \frac{d+1}{2}$ and $d \geq 3$.*

The space

$$\bar{U}_d^n := \{(f, x) \in U_d^n \times \mathbb{CP}^{n+1} \mid f(x) = 0\}.$$

has a forgetful map $p : \bar{U}_d^n \to U_d^n$, which is a smooth fibre bundle with fibre over $f \in U_d^n$ given by the projective hypersurface defined by f. The map p is $GL_{n+1}(\mathbb{C})$-equivariant and descends to a morphism

$$\pi : E_d^n := \bar{U}_d^n /\!\!/ GL_{n+1}(\mathbb{C}) \longrightarrow M_d^n,$$

which is the universal family of degree d hypersurfaces of dimension n. This is equipped with a vertical tangent bundle $T_v E_d^n \to E_d^n$, a complex vector bundle of dimension n, and for any monomial c_I in Chern classes we can therefore define, by analogy with the classical Miller–Morita–Mumford classes,

$$\kappa_I := \int_\pi c_I(T_v E_d^n) \in H^{|c_I|-2n}(M_d^n; \mathbb{Q}).$$

Our goal is to show that all such classes of non-zero degree vanish.

Theorem 0.2. *If $|c_I| - 2n > 0$ then $\kappa_I = 0 \in H^{|c_I|-2n}(M_d^n; \mathbb{Q})$.*

(i) If $|c_I| = 2n$ then κ_I is just a scalar and is simply the corresponding Chern number of the degree d hypersurface, which need not vanish.

(ii) If $|c_I| - 2n < \frac{d+1}{2}$ then Tommasi's theorem means that the relevant cohomology group vanishes: the point of our result is that the κ_I vanish universally.

(iii) It is more usual to consider the Miller–Morita–Mumford classes associated to polynomials in the Euler class e and Pontrjagin classes $p_1, p_2, \ldots, p_{n-1}$ of the underlying oriented real vector bundle of $T_v E_d^n$. However, these may be expressed as certain polynomials in the Chern classes, so the Miller–Morita–Mumford classes $\kappa_{e^i p_I}$ also vanish.

(iv) If V^6_d is the underlying smooth manifold of a degree d projective hypersurface of complex dimension 3, then in [GRW18, Section 5.3] Galatius and I have computed the cohomology of $B\mathrm{Diff}^+(V^6_d)$ in degrees $* \leq \frac{d^4-5d^3+10d^2-10d+4}{4}$, where it is highly non-trivial and for example contains the polynomial algebra on the classes κ_c with $c \in \mathbb{Q}[e, p_1, p_2]$ of degree > 6. Thus the map
$$M^3_d \longrightarrow B\mathrm{Diff}^+(V^6_d)$$
recording the underlying smooth structure is highly trivial on cohomology. Analogous conclusions can be made for all $2n \geq 6$.

Proof. The stack M^n_d carries a $(n+1)$-dimensional complex vector bundle $V \to M^n_d$, which is equipped with a section s of $\mathcal{O}(d) \to \mathbb{P}(V)$ defining a smooth hypersurface in each fibre. The zero locus $s^{-1}(0) \subset \mathbb{P}(V)$ is identified with E^n_d, and we write $i : E^n_d \to \mathbb{P}(V)$ for the inclusion. We write $p : \mathbb{P}(V) \to M^n_d$ and $\pi : E^n_d \to M^n_d$ for the bundle projections.

The usual description of the (stable) tangent bundle of a hypersurface extends to families as
$$T_v E^n_d \oplus i^*\mathcal{O}(d) \cong i^* T_v \mathbb{P}(V).$$
The usual description of the (stable) tangent bundle of a projective space extends to families as
$$T_v \mathbb{P}(V) \oplus \mathbb{C} = p^*(V) \otimes \mathcal{O}(1).$$
Writing $x := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(V); \mathbb{Q})$, we have
$$c(T_v H) = i^* \left(\frac{c(p^*(V) \otimes \mathcal{O}(1))}{1 + dx} \right).$$
Now $c(p^*(V) \otimes \mathcal{O}(1))$ is some polynomial in x and $p^*c_1(V), p^*c_2(V), \ldots, p^*c_{n+1}(V)$, and so each $c_j(T_v E^n_d)$ is a polynomial in i^*x and $\pi^*c_1(V), \pi^*c_2(V), \ldots, \pi^*c_{n+1}(V)$. Thus, by the projection formula, each κ_I is a polynomial in $c_1(V), c_2(V), \ldots, c_{n+1}(V)$ and the classes $\delta_j := \pi_!(i^*(x^j))$. We may compute the pushforward for the map
$$\pi : E^n_d \xrightarrow{i} \mathbb{P}(V) \xrightarrow{p} M^n_d$$
in two steps, giving
$$\delta_j = \pi_!(i^*(x^j)) = p_! i_!(i^*(x^j)) = p_!(i_!(1) \smile x^j).$$
Now $i_!(1)$ is the Poincaré dual of $i(E^n_d) \subset \mathbb{P}(V)$, so is $e(\mathcal{O}(d)) = c_1(\mathcal{O}(d)) = d \cdot x$, and so
$$\delta_j = d \cdot p_!(x^{j+1}).$$
But by the identity $0 = \sum_{i=0}^{n+1} p^*c_i(V) x^{n+1-i} \in H^*(\mathbb{P}(V); \mathbb{Q})$ one can write x^{j+1} as a linear combination of $1, x, x^2, \ldots, x^n$ with coefficients given by polynomials in $p^*c_1(V), p^*c_2(V), \ldots, p^*c_{n+1}(V)$. Then, using the projection formula and the fact that $p_!(x^n) = 1$, we see that each δ_j is also a polynomial in $c_1(V), c_2(V), \ldots, c_{n+1}(V)$.

The conclusion of the above discussion is that each $\kappa_I(\pi)$ may be expressed as a polynomial in the classes $c_1(V), c_2(V), \ldots, c_{n+1}(V)$.

Now we use the theorem of Peters and Steenbrink [PS03, Theorem 1], that the Leray spectral sequence for the quotient map
$$U^n_d \longrightarrow M^n_d = U^n_d /\!\!/ GL_{n+1}(\mathbb{C})$$
in $\mathbb{Q}$-cohomology degenerates at E_2. In particular this map is injective in $\mathbb{Q}$-cohomology, and so the associated map
$$M^n_d = U^n_d /\!\!/ GL_{n+1}(\mathbb{C}) \longrightarrow * /\!\!/ GL_{n+1}(\mathbb{C}) = BGL_{n+1}(\mathbb{C})$$
is trivial in $\mathbb{Q}$-cohomology in positive degrees. This is the map which classifies the vector bundle V, meaning that $c_i(V) = 0 \in H^{2i}(M^n_d; \mathbb{Q})$ for all $i > 0$. As the Miller–Morita–Mumford classes are polynomials in these, they vanish as claimed. $\square$

References

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