

# MILLER–MORITA–MUMFORD CLASSES VANISH ON THE MODULI SPACE OF HYPERSURFACES

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Let us write

$$U_d^n \subset H^0(\mathbb{C}\mathbb{P}^{n+1}; \mathcal{O}(d))$$

for the open subset of those homogeneous polynomials of degree  $d$  which define a smooth hypersurface. The group  $GL_{n+1}(\mathbb{C})$  acts on the space of such polynomials and hence on  $U_d^n$ , and this action has finite stabilisers. We let

$$M_d^n := U_d^n // GL_{n+1}(\mathbb{C})$$

denote the orbifold, or stack, quotient of this action. This is the moduli space of degree  $d$  hypersurfaces of dimension  $n$ . Tommasi [Tom14] has proved the following surprising vanishing result.

**Theorem 0.1** (Tommasi).  $H^i(M_d^n; \mathbb{Q}) = 0$  for  $0 < i < \frac{d+1}{2}$  and  $d \geq 3$ .

The space

$$\bar{U}_d^n := \{(f, x) \in U_d^n \times \mathbb{C}\mathbb{P}^{n+1} \mid f(x) = 0\}.$$

has a forgetful map  $p : \bar{U}_d^n \rightarrow U_d^n$ , which is a smooth fibre bundle with fibre over  $f \in U_d^n$  given by the projective hypersurface defined by  $f$ . The map  $p$  is  $GL_{n+1}(\mathbb{C})$ -equivariant and descends to a morphism

$$\pi : E_d^n := \bar{U}_d^n // GL_{n+1}(\mathbb{C}) \longrightarrow M_d^n,$$

which is the universal family of degree  $d$  hypersurfaces of dimension  $n$ . This is equipped with a vertical tangent bundle  $T_v E_d^n \rightarrow E_d^n$ , a complex vector bundle of dimension  $n$ , and for any monomial  $c_I$  in Chern classes we can therefore define, by analogy with the classical Miller–Morita–Mumford classes,

$$\kappa_I := \int_{\pi} c_I(T_v E_d^n) \in H^{|c_I|-2n}(M_d^n; \mathbb{Q}).$$

Our goal is to show that all such classes of non-zero degree vanish.

**Theorem 0.2.** *If  $|c_I| - 2n > 0$  then  $\kappa_I = 0 \in H^{|c_I|-2n}(M_d^n; \mathbb{Q})$ .*

- (i) If  $|c_I| = 2n$  then  $\kappa_I$  is just a scalar and is simply the corresponding Chern number of the degree  $d$  hypersurface, which need not vanish.
- (ii) If  $|c_I| - 2n < \frac{d+1}{2}$  then Tommasi's theorem means that the relevant cohomology group vanishes: the point of our result is that the  $\kappa_I$  vanish universally.
- (iii) It is more usual to consider the Miller–Morita–Mumford classes associated to polynomials in the Euler class  $e$  and Pontrjagin classes  $p_1, p_2, \dots, p_{n-1}$  of the underlying oriented real vector bundle of  $T_v E_d^n$ . However, these may be expressed as certain polynomials in the Chern classes, so the Miller–Morita–Mumford classes  $\kappa_{e^i p_I}$  also vanish.
- (iv) If  $V_d^6$  is the underlying smooth manifold of a degree  $d$  projective hypersurface of complex dimension 3, then in [GRW18, Section 5.3] Galatius and I have computed the cohomology of  $B\text{Diff}^+(V_d^6)$  in degrees  $*$   $\leq \frac{d^4 - 5d^3 + 10d^2 - 10d + 4}{4}$ , where it is highly non-trivial and for example contains the polynomial algebra on the classes  $\kappa_c$  with  $c \in \mathbb{Q}[e, p_1, p_2]$  of degree  $> 6$ . Thus the map

$$M_d^3 \longrightarrow B\text{Diff}^+(V_d^6)$$

recording the underlying smooth structure is highly trivial on cohomology. Analogous conclusions can be made for all  $2n \geq 6$ .

*Proof.* The stack  $M_d^n$  carries a  $(n+1)$ -dimensional complex vector bundle  $V \rightarrow M_d^n$ , which is equipped with a section  $s$  of  $\mathcal{O}(d) \rightarrow \mathbb{P}(V)$  defining a smooth hypersurface in each fibre. The zero locus  $s^{-1}(0) \subset \mathbb{P}(V)$  is identified with  $E_d^n$ , and we write  $i : E_d^n \rightarrow \mathbb{P}(V)$  for the inclusion. We write  $p : \mathbb{P}(V) \rightarrow M_d^n$  and  $\pi : E_d^n \rightarrow M_d^n$  for the bundle projections.

The usual description of the (stable) tangent bundle of a hypersurface extends to families as

$$T_v E_d^n \oplus i^* \mathcal{O}(d) \cong i^* T_v \mathbb{P}(V).$$

The usual description of the (stable) tangent bundle of a projective space extends to families as

$$T_v \mathbb{P}(V) \oplus \mathbb{C} = p^*(V) \otimes \mathcal{O}(1).$$

Writing  $x := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(V); \mathbb{Q})$ , we have

$$c(T_v H) = i^* \left( \frac{c(p^*(V) \otimes \mathcal{O}(1))}{1 + dx} \right).$$

Now  $c(p^*(V) \otimes \mathcal{O}(1))$  is some polynomial in  $x$  and  $p^*c_1(V), p^*c_2(V), \dots, p^*c_{n+1}(V)$ , and so each  $c_j(T_v E_d^n)$  is a polynomial in  $i^*x$  and  $\pi^*c_1(V), \pi^*c_2(V), \dots, \pi^*c_{n+1}(V)$ . Thus, by the projection formula, each  $\kappa_I$  is a polynomial in  $c_1(V), c_2(V), \dots, c_{n+1}(V)$  and the classes  $\delta_j := \pi_!(i^*(x^j))$ . We may compute the pushforward for the map

$$\pi : E_d^n \xrightarrow{i} \mathbb{P}(V) \xrightarrow{p} M_d^n$$

in two steps, giving

$$\delta_j = \pi_!(i^*(x^j)) = p_! i_!(i^*(x^j)) = p_!(i_!(1) \smile x^j).$$

Now  $i_!(1)$  is the Poincaré dual of  $i(E_d^n) \subset \mathbb{P}(V)$ , so is  $e(\mathcal{O}(d)) = c_1(\mathcal{O}(d)) = d \cdot x$ , and so

$$\delta_j = d \cdot p_!(x^{j+1}).$$

But by the identity  $0 = \sum_{i=0}^{n+1} p^*c_i(V)x^{n+1-i} \in H^*(\mathbb{P}(V); \mathbb{Q})$  one can write  $x^{j+1}$  as a linear combination of  $1, x, x^2, \dots, x^n$  with coefficients given by polynomials in  $p^*c_1(V), p^*c_2(V), \dots, p^*c_{n+1}(V)$ . Then, using the projection formula and the fact that  $p_!(x^n) = 1$ , we see that each  $\delta_j$  is also a polynomial in  $c_1(V), c_2(V), \dots, c_{n+1}(V)$ .

The conclusion of the above discussion is that each  $\kappa_I(\pi)$  may be expressed as a polynomial in the classes  $c_1(V), c_2(V), \dots, c_{n+1}(V)$ .

Now we use the theorem of Peters and Steenbrink [PS03, Theorem 1], that the Leray spectral sequence for the quotient map

$$U_d^n \longrightarrow M_d^n = U_d^n // GL_{n+1}(\mathbb{C})$$

in  $\mathbb{Q}$ -cohomology degenerates at  $E_2$ . In particular this map is injective in  $\mathbb{Q}$ -cohomology, and so the associated map

$$M_d^n = U_d^n // GL_{n+1}(\mathbb{C}) \longrightarrow * // GL_{n+1}(\mathbb{C}) = BGL_{n+1}(\mathbb{C})$$

is trivial in  $\mathbb{Q}$ -cohomology in positive degrees. This is the map which classifies the vector bundle  $V$ , meaning that  $c_i(V) = 0 \in H^{2i}(M_d^n; \mathbb{Q})$  for all  $i > 0$ . As the Miller–Morita–Mumford classes are polynomials in these, they vanish as claimed.  $\square$

## REFERENCES

- [GRW18] S. Galatius and O. Randal-Williams, *Moduli spaces of manifolds: a user's guide*, 2018, arXiv:1811.08151, to appear in the Handbook of Homotopy Theory.
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