A COMBINATORIAL IDENTITY

by

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Abstract. — We record a combinatorial identity which arises in the the study of relations in the cohomology of the moduli space of Riemann surfaces.

Define numbers $C_{r,n}$ by the formula

$$\sum C_{r,n} \frac{t^r x^n}{r!n!} = \log \left(\sum \binom{n+1}{2}^r \frac{t^r x^n}{r!n!} \right),$$

and polynomials

$$p_{R,N}(y) = \left[\exp\left(\sum_{n=1}^{\infty} C_{n,n} \frac{t^n x^n y}{n!n!} \right) \cdot \left(\sum_{r>n} C_{r,n} \frac{t^r x^n}{r!n!} \right) \right]_{t^R x^N}.$$

Theorem 0.1. — *If $2 \leq R-N \leq \frac{N+1}{3}$ then $-2N$ is a root of the polynomial $p_{R,N}(y)$.*

We have not been able to find a combinatorial proof, nor interpretation, of this theorem.

Proof. — By the Wick form relation of Pandharipande and the author (see [**4**, p. 5] or [**5**, §2.7]), for $r \geq 2$ the cohomology class

$$p_{r+g-1,g-1}(2-2g) \cdot \kappa_r \in H^{2r}(\mathbf{M}_g; \mathbb{Q})$$

is decomposable in terms of lower κ_i. However, by Boldsen's [**1**] or the author's [**6**] improvement of Harer's stability theorem, there are no relations among κ_i in degrees $* \leq \lfloor \frac{2g}{3} \rfloor$. In particular, $p_{r+g-1,g-1}(2-2g)$ must be zero if $3r \leq g$. The result follows by the substitution $N = g-1$ and $R = r+g-1$. □

Computer calculations suggest the following conjecture.

Conjecture 0.2. — If $R - N > \frac{N+1}{3}$ then $p_{R,N}(-2N) > 0$.

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A proof of this conjecture would—with the relations of Pandharipande and the author [**4, 5**]—lead to a new proof of the theorems of Morita [**3**] and Ionel [**2**] that the classes $\kappa_1, \ldots, \kappa_{\lfloor \frac{g}{3} \rfloor}$ generate the tautological ring of $\mathbf{M}_g$.

References

[1] S. Boldsen *Improved homological stability for the mapping class group with integral or twisted coefficients*, Math. Z., Volume 270, Number 1 (2012), 297–329.

[2] E.-N. Ionel *Relations in the tautological ring of* $\mathbf{M}_g$, Duke Math. J., Volume 129, Number 1 (2005), 157–186.

[3] S. Morita *Generators for the tautological algebra of the moduli space of curves*, Topology 42 (2003), 787–819.

[4] R. Pandharipande, *Morita's* κ *relations.* `http://www.math.ethz.ch/~rahul/MRW.pdf`.

[5] O. Randal-Williams *Relations among tautological classes revisited*, Adv. Math. 231 (3-4) (2012) 1773-1785.

[6] O. Randal-Williams *Resolutions of moduli spaces and homological stability*, `http://arxiv.org/abs/0909.4278`

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