Cohomology of $\mathrm{Aut}(F_n)$ with twisted coefficients

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The stable (co)homology of $\mathrm{Aut}(F_n)$ (with constant coefficients) has been computed by Galatius [1]. His argument exploits a model $\mathcal{G}_n^1 \simeq B\mathrm{Aut}(F_n)$ given by the space of rank n graphs in $\mathbb{R}^\infty$ equipped with a marked point. These form morphism spaces of a graph cobordism category $\mathcal{G}$, and on one hand the group completion theorem and "parametrised surgery on graphs" relate $\mathcal{G}_\infty^1$ with the K-theory $\Omega B|\mathcal{G}|$ of the symmetric monoidal category $\mathcal{G}$. On the other hand, using geometric arguments concerning spaces of non-compact graphs, Galatius identifies this K-theory space with $Q(S^0)$, the free infinite loop space on a point; this is a well-studied object in homotopy theory, and its homology is completely known.

Fixing a based space X, one may consider analogous spaces $\mathcal{G}_n^1(X)$ of graphs with a marked point which are additionally equipped with a based map to X. Using homological stability for $\mathrm{Aut}(F_n)$ with finite-degree twisted coefficients (which we have recently established in joint work with Wahl [6]) and borrowing an argument of Cohen–Madsen [2] from the case of mapping class groups, one can establish homological stability and the analogue of Galatius' theorem in this case.

Theorem A. *There is a map*

$$\mathcal{G}_n^1(X) \longrightarrow Q_0(X_+)$$

which, if X is simply-connected, is an isomorphism on homology in degrees $* \leq \frac{n-3}{2}$. (The map is induced by the Becker–Gottlieb transfer for the universal family of graphs over $\mathcal{G}_n^1(X)$.)

The relevance of Theorem A to the question of twisted coefficients for $\mathrm{Aut}(F_n)$ is the fibration

$$\mathrm{map}_*(\vee^n S^1, X) \longrightarrow \mathcal{G}_n^1(X) \longrightarrow \mathcal{G}_n^1 \simeq B\mathrm{Aut}(F_n)$$

and its associated Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Aut}(F_n); H^q(\mathrm{map}_*(\vee^n S^1, X))) \Rightarrow H^{p+q}(\mathcal{G}_n^1(X)),$$

whose target may be identified with $H^{p+q}(Q_0(X_+))$ in a range of degrees by Theorem A. Of course, the behaviour of this spectral sequence for any particular X might be very complicated. However, by exploiting the functoriality of the spectral sequence in the variable X, one can severely restrict its behaviour in many situations.

In particular, choosing a $\mathbb{Q}$-vector space V and setting $X = K(V^*, 2)$, one obtains a spectral sequence of $GL(V)$-representations. Analysing the weight decomposition on this spectral sequence induced by the action of $\mathbb{Q}^\times \leq GL(V)$ one may show that it collapses rationally, and analysing the weight decomposition on $H^*(Q_0(K(V^*,2)_+);\mathbb{Q}) = \mathrm{Sym}^*(\mathrm{Sym}^{*>0}(V[2]))$ allows one to compute the E_2-page completely. It them becomes a problem in representation theory to combine the information so obtained in a useful way, which eventually leads to the following, where we write $H := H_1(F_n;\mathbb{Z})$ and $H_\mathbb{Q} = H \otimes \mathbb{Q}$, considered as $\mathrm{Aut}(F_n)$-modules.

Theorem B. *As a* $[\Sigma_q]$*-module we have*

$$H^q(\mathrm{Aut}(F_\infty); H_{\mathbb{Q}}^{\otimes q}) = \mathbb{Q}\{\textit{partitions of } \{1,2,\ldots,q\}\} \otimes \mathbb{Q}^-$$

and the cohomology in all other degrees vanishes.

The Schur–Weyl decomposition $H_{\mathbb{Q}}^{\otimes q} = \bigoplus_\lambda S^\lambda \otimes S_\lambda(H_{\mathbb{Q}})$ in terms of Schur functors $S_\lambda(-)$ shows that for a partition λ of q the dimension of the twisted cohomology $H^q(\mathrm{Aut}(F_\infty); S_\lambda(H_{\mathbb{Q}}))$ is the multiplicity of the Specht module S^λ in the Σ_q-representation given by $\mathbb{Q}\{\text{partitions of } \{1,2,\ldots,q\}\} \otimes \mathbb{Q}^-$. This may be calculated algorithmically by character theory.

The result of Theorem B can also be obtained by combining work of Djament [3] and Vespa [7], who use techniques of functor homology. However the technique we have described is quite general and can be used to obtained related results in several directions. In one direction, one may study $\mathrm{Out}(F_n)$ by the same methods, giving

$$H^q(\mathrm{Out}(F_\infty); H_{\mathbb{Q}}^{\otimes q}) = \mathbb{Q}\{\text{partitions of } \{1,2,\ldots,q\} \text{ with no parts of size } 1\} \otimes \mathbb{Q}^-.$$

In another direction, one may obtain results about torsion in the twisted cohomology too. For example, if λ is a partition of q, and $p > q$ is a prime number, then one may still make sense of the Schur functor $S_\lambda(-)$ on $\mathbb{Z}_{(p)}$-modules, and similar techniques show that $H^*(\mathrm{Aut}(F_\infty); S_\lambda(H_{(p)}))$ is a free $H^*(\mathrm{Aut}(F_\infty); \mathbb{Z}_{(p)})$-module (with module generators in degrees which may be deduced from Theorem B). In particular, as all prime numbers are greater than 1 we find that $H^*(\mathrm{Aut}(F_\infty); H)$ is a free $H^*(\mathrm{Aut}(F_\infty); \mathbb{Z})$-module on a single generator in degree 1.

In a third direction, the general strategy we have employed may be attempted whenever one has a “Madsen–Weiss theorem with maps to a background space”. This is available in many situations, including mapping class groups of surfaces and diffeomorphism groups of high-dimensional manifolds. In particular, one may use this strategy to recover Looijenga’s calculation [4] of the stable cohomology with twisted coefficients for mapping class groups of closed surfaces, and to obtain new results for mapping class groups of surfaces with boundary.

References

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