Induction and Mackey Theory

I'm writing this short handout to try and explain what the idea of Mackey theory is. The aim of this is not to replace proofs/definitions in the lecture notes, but rather to give an explanation of what's going on. Technical details might be omitted. Throughout, G denotes a finite group, and all vector spaces and representations are over $\mathbb{C}$. When H is a (not necessarily normal) subgroup of G, we will write G/H to denote the set of left cosets of H.

1 Motivation

Mackey theory is the most conceptually difficult subject tackled in the Cambridge Part II course on Representation theory. The reason for this is that to really understand it, one needs to understand how induction works, and to do that properly one needs to work with tensor products over rings, which is not covered until Part III. However it is still possible to talk about Mackey theory and induction, it's just that the bigger picture of what's going on is a bit obscure. Now, you should think of Mackey theory as the way to answer the following:

Problem. Suppose that H is a subgroup of G, and that W is a representation of H. When is $\mathrm{Ind}_H^G W$ irreducible as a G-module?

Let's try to answer this using characters. If χ is the character afforded by W, then we have

$$\langle \mathrm{Ind}_H^G \chi, \mathrm{Ind}_H^G \chi \rangle_G = \langle \chi, \mathrm{Res}_H^G \mathrm{Ind}_H^G \chi \rangle_H \tag{1}$$

by Frobenius reciprocity. So the question becomes: how does $\mathrm{Res}_H^G \mathrm{Ind}_H^G W$ split into irreducibles? If we can determine this, we can compute the above inner product and the irreducibility of $\mathrm{Ind}_H^G W$ can be determined completely (assuming we know how χ decomposes). Moreover, we might as well be a bit more general and study $\mathrm{Res}_K^G \mathrm{Ind}_H^G W$ where K is another subgroup of G. As it turns out, this doesn't make the theory much more complicated.

2 What's induction?

In order to understand Mackey theory, we will need to understand quite well how induction works. As far as the Cambridge course goes, the only thing that you need to know in order to get going most of the time is the formula to compute the induced characters. However for Mackey theory to make sense, we need to remind ourselves of how the induced module itself is constructed, rather than working with characters.

What is the idea of induction? Its aim is to construct representations of G from representations of a given subgroup H. Let W be a representation of H, say we have $\rho : H \to \mathrm{GL}(W)$. The aim is to construct a representation of G which extends W in some way. So, more precisely, we want to look at representations $\sigma : G \to \mathrm{GL}(V)$, where V contains W as a vector subspace in such a way that the two H-actions on W coming from ρ and σ are the same, i.e $\sigma(h)|_W = \rho(h)$ for all $h \in H$. In other words we seek a representation V of G so that $\mathrm{Res}^G_H V$ contains W as a subrepresentation. Moreover, we would like to be able to make a canonical choice of such a representation.

To understand how this works, suppose we have a representation V of G which contains W as a vector subspace, in such a way that the two H-actions on W coincide. Denote the elements of W inside V by the formal symbols $1 \otimes w$, $w \in W$. Since V is a G-representation, the group G acts on those and we denote the element $g \cdot (1 \otimes w) \in V$ by the formal symbol $g \otimes w$. Since the two H-actions coincide, this must satisfy the rule that $h \otimes w = 1 \otimes (hw)$ for all $h \in H$. Also, since V is a representation of G we must have $g_1 \cdot (g_2 \otimes w) = (g_1 g_2) \otimes w$.

Now let $1 = t_1, t_2, \ldots, t_r$ be a left transversal of H, i.e a complete collection of coset representatives. Then $g = t_j h$ for some $1 \leq j \leq r$ and some $h \in H$. Since V is a representation of G, we must have $gw = (t_j h)w = t_j(hw)$. In other words, more formally, we have $g \otimes w = (t_j h) \otimes w = t_j \otimes hw$. Moreover, if we choose a basis $w_1, \ldots, w_n$ for W, every $w \in W$ can be written as a linear combination $\sum_{i=1}^n \lambda_i w_i$, and as the G-action on V is linear, we must have

$$g \cdot w = g \cdot \left(\sum_{i=1}^n \lambda_i w_i \right) = \sum_{i=1}^n \lambda_i (g \cdot w_i).$$

In other words, more formally,

$$g \otimes w = \sum_{i=1}^n \lambda_i (g \otimes w_i) = \sum_{i=1}^n \lambda_i (t_j \otimes h w_i).$$

Writing each hw_i in terms of the basis, we can break down further to get an expression of the form

$$g \otimes w = \sum_{i=1}^n \mu_i (t_j \otimes w_i)$$

for some $\mu_i \in \mathbb{C}$, by using the rule $t_j \otimes (\sum_{i=1}^n \alpha_i w_i) = \sum_{i=1}^n \alpha_i (t_j \otimes w_i)$.

Hence we see that every element of the form $g \otimes w$ is a linear combination of the elements $t_j \otimes w_i$. Moreover, clearly $g \cdot (t_j \otimes w_i) = (g t_j) \otimes w_i$ is of that form for all $g \in G$ and all i, j. Therefore, if we denote by U the span of the elements $t_j \otimes w_i$ ($1 \leq j \leq r$, $1 \leq i \leq n$), then we see that U is a subrepresentation of V.

At this point, we do not yet know that such a representation V exists, as we haven't constructed one. But if there is one, the above gives us an idea of what it should look like, and this U we've constructed looks promising on our quest of a canonical choice of such representations. This should motivate the following

Definition. The induced representation $\mathrm{Ind}_H^G W$ is defined to be the vector space with basis given by the formal symbols $t_j \otimes w_i$ ($1 \leq j \leq r$, $1 \leq i \leq n$), and G-action given by

$$g \cdot (t_j \otimes w_i) = \sum_{k=1}^{n} \lambda_{ik} t_s \otimes w_k$$

where $gt_j = t_s h$ with $h \in H$, and $hw_i = \sum_{k=1}^{n} \lambda_{ik} w_k$. It can be easily checked that this gives a well-defined representation. The proof is the same as the one given in the notes: note that $h = t_s^{-1} g t_j$ and write the above action as $g \cdot (t_j w_i) = t_s(hw_i) = t_s((t_s^{-1} g t_j) w_i)$ where t_s is the unique coset representative such that $gt_j \in t_s H$. You can then work with that to check that $g_1 \cdot (g_2 \cdot (t_j w_i)) = (g_1 g_2) \cdot (t_j w_i)$.

Hence, the idea in constructing $\mathrm{Ind}_H^G W$ is to take a formal element $g \otimes w$, not necessarily in W, for every $g \in G$ and $w \in W$, and to let the induced module be the span of those under some set of relations which make the G-action well-defined and extend the H-action on W. Clearly we can identify W with the vector subspace of $\mathrm{Ind}_H^G W$ consisting of elements of the form $1 \otimes w$ (it is the span of $\{t_1 \otimes w_i : 1 \leq i \leq n\}$). Moreover we see that W is a subrepresentation of $\mathrm{Res}_H^G \mathrm{Ind}_H^G W$, because $h \cdot (1 \otimes w) = h \otimes w = 1 \otimes hw$. So we have indeed found a representation of G with the required properties.

Remark. Note that if V is a representation of G such that W is a subrepresentation of $\mathrm{Res}_H^G V$, then the above argument shows that V contains a quotient of $\mathrm{Ind}_H^G W$ as a subrepresentation (the thing we called U). It turns out that this property characterises $\mathrm{Ind}_H^G W$ uniquely up to isomorphism. Hence, in some sense, the induced representation is the "universal" representation with that property.

Example. Let $G = D_6 = \{a, b : a^3 = b^2 = 1, bab = a^2\}$, $H = \langle a \rangle \cong C_3$ and $W =$ trivial representation $= \langle v \rangle$ say. Then $V = \mathrm{Ind}_H^G W$ is the vector space with basis $\{v, bv\}$, where a acts trivally on it and b swaps the two basis vectors. We can decompose explicitly $V = V_1 \oplus V_2$, where $V_1 = \langle v + bv \rangle$ and $V_2 = \langle v - bv \rangle$. Clearly V_1 is just the trivial representation while V_2 is the non-trivial representation lifted from $G/H \cong C_2$. Both of these are representations whose restriction to H is (isomorphic to) W, and they are indeed both quotients of V.

Now, going back to equation (1), we see that χ is a summand of $\mathrm{Res}_H^G \mathrm{Ind}_H^G \chi$, and so the only way that we could have

$$1 = \langle \mathrm{Ind}_H^G \chi, \mathrm{Ind}_H^G \chi \rangle_G = \langle \chi, \mathrm{Res}_H^G \mathrm{Ind}_H^G \chi \rangle_H$$

is if W is irreducible and has multiplicity one inside $\mathrm{Res}_H^G \mathrm{Ind}_H^G W$. We now move on to determine when exactly that happens.

3 Mackey's restriction formula

Ok, we are now better equipped to try and understand what Mackey theory says. From the above description of induction, we see that every element of $\mathrm{Ind}_H^G W$ is a linear combination of terms of the form $g \otimes w$, i.e $\mathrm{Ind}_H^G W$ is the $\mathbb{C}$-span of $G \otimes W = \{g \otimes w : g \in G, w \in W\}$. As we saw, we can break this further into left coset representatives for H, i.e terms of the form $t_j \otimes w$, because of the rule that $gh \otimes w = g \otimes hw$ for any $h \in H$. In other words we have that

$$\mathrm{Ind}_H^G W = \bigoplus_{i=1}^{r} t_i \otimes W$$

where $t_i \otimes W = \{t_i \otimes w : w \in W\}$.

Now we want to restrict $\mathrm{Ind}_H^G W$ to some subgroup $K \leq G$ (e.g can take $K = H$ for the purpose of irreducibility). That means breaking this up into subspaces which are fixed uner the action of K. What do we mean by that? Well, first, if we let K act on G by left multiplication, then the orbits of that action are the right cosets Kg. Similarly, if H acts on G by right multiplication (this is a right action), then the orbits are the left cosets gH. So if we consider both actions simultaneously, the "double orbits" are the double cosets KgH. Another way to think about this is to consider the orbits under the action of K by left multiplication on G/H. These can be naturally identified with the double cosets KgH. Write $K\backslash G/H$ for the set of all double cosets, and we write $g \in K\backslash G/H$ to mean 'g is a representative of KgH'.

The idea is that we saw above that $\mathrm{Ind}_H^G W$ breaks up into 'left cosets' $t_i \otimes W$, and we now further break down under the left action of K. So we obtain that

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G W = \bigoplus_{K\backslash G/H} \mathbb{C}KgH \otimes W.$$

We need to write $\mathbb{C}KgH$ here to make the summands into vector spaces and so representations of K (e.g we didn't have that $\mathrm{Ind}_H^G W$ was equal to $G \otimes W$ but instead the $\mathbb{C}$-span of it).

Now, each $\mathbb{C}KgH \otimes W$ is just $\mathbb{C}K \cdot (gH \otimes W)$. Note that $gH \otimes W = g \otimes W$ is a vector space since $gh_1 \otimes w_1 + gh_2 \otimes w_2 = g \otimes (h_1 w_1 + h_2 w_2)$. We can identify it with W by mapping $gh \otimes w \mapsto hw$. Moreover, both $\mathbb{C}K$ and $gH \otimes W$ are representations of $H_g := K \cap gHg^{-1}$. We denote by W_g the representation of H_g which is W as a vector space, but with action $x \cdot w = (g^{-1}xg)w$ where now $g^{-1}xg \in H$ by definition of H_g. Clearly we have $gH \otimes W \cong W_g$ as representations of H_g: if we think about how $x = gh_1g^{-1} \in H_g$ acts on $gh_2 \otimes w$, we see that it sends it to $gh_1h_2 \otimes w$, which gives the action on W_g if you translate everything to W by the above isomorphism $gH \otimes W \cong W$ of vector spaces.

Now by the above we can identify each $\mathbb{C}KgH \otimes W$ with $\mathbb{C}K \otimes W_g$, by which we mean the $\mathbb{C}$-span of formal symbols $k \otimes w$ with $k \in K$ and $w \in W_g$, with the rule that $kx \otimes w = k \otimes xw$ for all $x \in H_g$. This looks suspiciously like $\mathrm{Ind}_{H_g}^K W_g$. And it is! Hence, in summary, we have:

Theorem. (Mackey's restriction formula)

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H W \cong \bigoplus_{K\backslash G/H} \mathrm{Ind}^K_{H_g} W_g$$

.

You'll have to trust me that everything described above works. This result is not necessarily very pretty at first sight, however I think that once you start thinking about it for a bit, you realise that a lot of its aspects are sort of what we should expect to get. For example, induction splits naturally as a vector space into summands corresponding to the left cosets of H in G, and so if we restrict the action from G to K, it makes sense that it should split into summands that correspond to double cosets.

4 Mackey's irreducibility criterion

We can now finally determine when the induced representation is irreducible. As we said already, we need W to be irreducible and have multiplicity one inside $\mathrm{Res}^G_H W$. So take $K = H$ in Mackey's restriction formula:

$$\mathrm{Res}^G_H \mathrm{Ind}^G_H W \cong \bigoplus_{H\backslash G/H} \mathrm{Ind}^H_{H_g} W_g$$

where $H_g = H \cap gHg^{-1}$. Write χ_g for the character of H_g afforded by W_g. Now clearly $H_1 = H$ and $\chi_1 = \chi$. So we have

$$\begin{aligned}\langle \mathrm{Ind}^G_H \chi, \mathrm{Ind}^G_H \chi \rangle_G &= \langle \chi, \mathrm{Res}^G_H \mathrm{Ind}^G_H \chi \rangle_H \\ &= \langle \chi, \chi \rangle_H + \sum_{1 \neq g \in H\backslash G/H} \langle \chi, \mathrm{Ind}^H_{H_g} \chi_g \rangle_H \\ &\geq 1 + \sum_{1 \neq g \in H\backslash G/H} \langle \mathrm{Res}^H_{H_g} \chi, \chi_g \rangle_{H_g}\end{aligned}$$

by Frobenius reciprocity. Thus we see that $\langle \mathrm{Ind}^G_H \chi, \mathrm{Ind}^G_H \chi \rangle_G = 1$ if and only if $\langle \chi, \chi \rangle_H = 1$ and $\langle \mathrm{Res}^H_{H_g} \chi, \chi_g \rangle_{H_g} = 0$ for all $1 \neq g \in H\backslash G/H$. This gives the following

Theorem. (Mackey's irreducibility criterion) *Let G be a finite group and H a subgroup of G. Given a representation W of H, the induced representation $\mathrm{Ind}^G_H W$ is irreducible if and only if*

(i) W is irreducible, and

(ii) for each $g \in G \setminus H$, the representations $\mathrm{Res}^H_{H_g} W$ and W_g have no irreducible factors in common.