Mirror symmetry

Mark Gross

University of Cambridge

July 24, 2015

Mirror symmetry: A very brief and biased history.

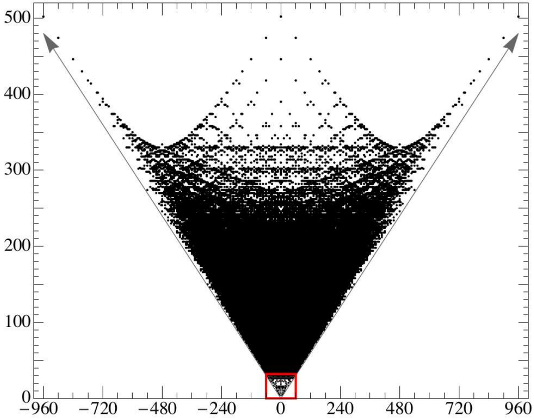
A search for examples of compact Calabi-Yau three-folds by Candelas, Lynker and Schimmrigk (1990) as crepant resolutions of hypersurfaces in weighted projective 4-space provided the following scatter plot of invariants, with the x-axis being Euler characteristic,

$$\chi = 2(h^{1,1} - h^{1,2})$$

and y-axis

$$h^{1,1} + h^{1,2}.$$

Mirror symmetry: A very brief and biased history.



(Thanks to Philip Candelas)

Mirror symmetry: A very brief and biased history.

Independently, Greene and Plesser (1990) provided an explicit mirror $\check{X}$ to the non-singular quintic three-fold $X \subseteq \mathbb{P}^4$.

Mirror symmetry: A very brief and biased history.

Independently, Greene and Plesser (1990) provided an explicit mirror $\check{X}$ to the non-singular quintic three-fold $X \subseteq \mathbb{P}^4$.
Here

$$h^{1,1}(X) = 1, \qquad h^{1,2}(X) = 101.$$

Mirror symmetry: A very brief and biased history.

Independently, Greene and Plesser (1990) provided an explicit mirror $\check{X}$ to the non-singular quintic three-fold $X \subseteq \mathbb{P}^4$.
Here

$$h^{1,1}(X) = 1, \qquad h^{1,2}(X) = 101.$$

while

$$h^{1,1}(\check{X}) = 101, \qquad h^{1,2}(\check{X}) = 1.$$

Mirror symmetry: A very brief and biased history.

1990: Candelas-de la Ossa-Green-Parkes: Amazing calculation, following predictions of string theory.

Mirror symmetry: A very brief and biased history.

1990: Candelas-de la Ossa-Green-Parkes: Amazing calculation, following predictions of string theory.

By carrying out period calculations on the mirror quintic, they were able to predict the "number" N_d of rational curves of degree d in X.

Mirror symmetry: A very brief and biased history.

1990: Candelas-de la Ossa-Green-Parkes: Amazing calculation, following predictions of string theory.

By carrying out period calculations on the mirror quintic, they were able to predict the "number" N_d of rational curves of degree d in X.

Technically, these numbers are defined as Gromov-Witten invariants.

Mirror symmetry: A very brief and biased history.

1990: Candelas-de la Ossa-Green-Parkes: Amazing calculation, following predictions of string theory.

By carrying out period calculations on the mirror quintic, they were able to predict the "number" N_d of rational curves of degree d in X.

Technically, these numbers are defined as Gromov-Witten invariants. ($N_1 = 2875$, $N_2 = 609250$, $N_3 = 317206375$.)

Mirror symmetry: A very brief and biased history.

1990: Candelas-de la Ossa-Green-Parkes: Amazing calculation, following predictions of string theory.

By carrying out period calculations on the mirror quintic, they were able to predict the "number" N_d of rational curves of degree d in X.

Technically, these numbers are defined as Gromov-Witten invariants. ($N_1 = 2875$, $N_2 = 609250$, $N_3 = 317206375$.)

While these formulas have been proved in the 1990s, string theorists have presented mathematicians with an amazing piece of complex mathematics. We have been reverse engineering this mathematics ever since.

There are now many proposed constructions for mirror pairs:

Mirror symmetry: A very brief and biased history.

But do we have a definition of mirror symmetry?

Mirror symmetry: A very brief and biased history.

But do we have a definition of mirror symmetry?

Yes, we do.

Mirror symmetry: A very brief and biased history.

But do we have a definition of mirror symmetry?

Yes, we do.

Definition (Potter Stewart, 1964, Jacobellis vs. Ohio)

I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description, and perhaps I could never succeed in intelligibly doing so. But I know it when I see it...

Mirror symmetry: A very brief and biased history.

But do we have a definition of mirror symmetry?

Yes, we do.

Definition (Potter Stewart, 1964, Jacobellis vs. Ohio)

I shall not today attempt further to define the kinds of material I understand to be embraced within that shorthand description, and perhaps I could never succeed in intelligibly doing so. But I know it when I see it...

As in the legal world, we have agreed on tests for mirror symmetry: mirror symmetry at genus 0, homological mirror symmetry,....

Mirror symmetry: A very brief and biased history.

We would like to have a general construction.

Mirror symmetry: A very brief and biased history.

We would like to have a general construction.

The framework I will discuss is a program developed with Bernd Siebert, starting in 2001. In particular, I will talk about the construction of mirrors using *theta functions*, as will be described in forthcoming work with Paul Hacking, Sean Keel, and Bernd Siebert.

Mirror symmetry: A very brief and biased history.

We would like to have a general construction.

The framework I will discuss is a program developed with Bernd Siebert, starting in 2001. In particular, I will talk about the construction of mirrors using *theta functions*, as will be described in forthcoming work with Paul Hacking, Sean Keel, and Bernd Siebert.

Reading:

Fukaya, "Multivalued Morse theory, asymptotic analysis and mirror symmetry," (2001).

Kontsevich and Soibelman, "Affine structures and non-archimedean analytic spaces," (2004).

Mirror symmetry: A very brief and biased history.

G., Siebert, "From real affine geometry to complex geometry," (2007).

Carl, Pumperla, Siebert, "A tropical view on Landau-Ginzburg models," (2010).

G., Hacking, Keel, "Mirror symmetry for log Calabi-Yau surfaces I" (2011).

G., Siebert, "Theta functions and mirror symmetry" (2012).

G., Siebert, "Local mirror symmetry in the tropics" (2014).

Mirror symmetry: A very brief and biased history.

G., Siebert, "From real affine geometry to complex geometry," (2007).

Carl, Pumperla, Siebert, "A tropical view on Landau-Ginzburg models," (2010).

G., Hacking, Keel, "Mirror symmetry for log Calabi-Yau surfaces I" (2011).

G., Siebert, "Theta functions and mirror symmetry" (2012).

G., Siebert, "Local mirror symmetry in the tropics" (2014).

There will be a number of forthcoming papers developing the subject as discussed in the remainder of the talk.

The general setup

Three “standard” situations we would like to consider.

The general setup

Three "standard" situations we would like to consider.

I. (X, D) a *log Calabi-Yau pair with maximal boundary*.

The general setup

Three "standard" situations we would like to consider.

I. (X, D) a *log Calabi-Yau pair with maximal boundary*.
For simplicity in our situation, we will take this to mean a compact non-singular variety X along with a reduced normal crossings divisor D, such that $K_X + D = 0$.

The general setup

Three "standard" situations we would like to consider.

I. (X, D) a *log Calabi-Yau pair with maximal boundary*.
For simplicity in our situation, we will take this to mean a compact non-singular variety X along with a reduced normal crossings divisor D, such that $K_X + D = 0$.

"Maximal" means that D has a 0-dimensional stratum, i.e., an intersection of a subset of irreducible components of D is 0-dimensional.

The general setup

Three "standard" situations we would like to consider.

I. (X, D) a *log Calabi-Yau pair with maximal boundary*.
For simplicity in our situation, we will take this to mean a compact non-singular variety X along with a reduced normal crossings divisor D, such that $K_X + D = 0$.

"Maximal" means that D has a 0-dimensional stratum, i.e., an intersection of a subset of irreducible components of D is 0-dimensional.

(More generally: allow singularities of the minimal model program, or toroidal crossings boundary)

The general setup

II. $X \to \operatorname{Spec} \mathbb{C}[[t]]$ *is a maximally unipotent degeneration of Calabi-Yau varieties.* This is a flat morphism, with generic fibre X_η a non-singular Calabi-Yau manifold. For simplicity, we assume this is a normal crossings degeneration and relatively minimal ($K_{X/\operatorname{Spec}\mathbb{C}[[t]]} = 0$).

The general setup

II. $X \to \operatorname{Spec} \mathbb{C}[[t]]$ *is a maximally unipotent degeneration of Calabi-Yau varieties.* This is a flat morphism, with generic fibre X_η a non-singular Calabi-Yau manifold. For simplicity, we assume this is a normal crossings degeneration and relatively minimal ($K_{X/\operatorname{Spec} \mathbb{C}[[t]]} = 0$).

(More generally: allow singularities of the minimal model program, Hacon-Xu and Birkar, or toric degenerations, G.-Siebert)

The general setup

III. *The hybrid situation:* A flat family of pairs $(X, D) \to \operatorname{Spec} \mathbb{C}[[t]]$, *a maximal degeneration of log Calabi-Yau varieties.* For simplicity, we will assume that D and the morphism are normal crossings, and the family is relatively minimal $(K_{X/\operatorname{Spec} \mathbb{C}[[t]]} + D = 0)$.

The general setup

III. *The hybrid situation:* A flat family of pairs $(X, D) \to \operatorname{Spec} \mathbb{C}[[t]]$, *a maximal degeneration of log Calabi-Yau varieties.* For simplicity, we will assume that D and the morphism are normal crossings, and the family is relatively minimal ($K_{X/\operatorname{Spec} \mathbb{C}[[t]]} + D = 0$).

The advantage of this is it allows a generic fibre (X_η, D_η) where D_η is not maximal.

The general setup

III. *The hybrid situation:* A flat family of pairs $(X, D) \to \operatorname{Spec} \mathbb{C}[[t]]$, *a maximal degeneration of log Calabi-Yau varieties.* For simplicity, we will assume that D and the morphism are normal crossings, and the family is relatively minimal $(K_{X/\operatorname{Spec}\mathbb{C}[[t]]} + D = 0)$.

The advantage of this is it allows a generic fibre (X_η, D_η) where D_η is not maximal.

(More generally: allow singularities of the minimal model program, or toric degenerations.)

The general setup

I will focus on the log Calabi-Yau case.

The general setup

I will focus on the log Calabi-Yau case.

Fix (X, D), and a saturated finitely generated submonoid $P \subset H_2(X, \mathbb{Z})$ such that:

The general setup

I will focus on the log Calabi-Yau case.

Fix (X, D), and a saturated finitely generated submonoid $P \subset H_2(X, \mathbb{Z})$ such that:

- P contains the classes of all effective curves.

The general setup

I will focus on the log Calabi-Yau case.

Fix (X, D), and a saturated finitely generated submonoid $P \subset H_2(X, \mathbb{Z})$ such that:

- P contains the classes of all effective curves.
- $p, -p \in P$ if and only if $p = 0$.

The general setup

Let k be a field of characteristic zero, and let

$$k[P] := \bigoplus_{p} kz^p$$

denote the monoid ring defined by P.

The general setup

Let k be a field of characteristic zero, and let

$$k[P] := \bigoplus_{p} kz^p$$

denote the monoid ring defined by P.

This has a maximal monomial ideal

$$\mathfrak{m} = \langle z^p \mid p \in P \setminus \{0\} \rangle$$

The general setup

Let k be a field of characteristic zero, and let

$$k[P] := \bigoplus_{p} kz^p$$

denote the monoid ring defined by P.

This has a maximal monomial ideal

$$\mathfrak{m} = \langle z^p \,|\, p \in P \setminus \{0\} \rangle$$

Let $\widehat{k[P]}$ denote the completion of $k[P]$ with respect to $\mathfrak{m}$.

The general setup

Let k be a field of characteristic zero, and let

$$k[P] := \bigoplus_p kz^p$$

denote the monoid ring defined by P.

This has a maximal monomial ideal

$$\mathfrak{m} = \langle z^p \mid p \in P \setminus \{0\} \rangle$$

Let $\widehat{k[P]}$ denote the completion of $k[P]$ with respect to $\mathfrak{m}$.
Goal: Produce a "mirror family" $\check{\mathfrak{X}} \to \mathrm{Spf}\, \widehat{k[P]}$.

The construction of the mirror to (X, D)

Let (B, Σ) be the *dual intersection complex* of the pair (X, D). Here B is a topological space and Σ is a decomposition of B into cones.

The construction of the mirror to (X, D)

Let (B, Σ) be the *dual intersection complex* of the pair (X, D). Here B is a topological space and Σ is a decomposition of B into cones.

If $D = \bigcup_{i=1}^{p} D_i$ is the decomposition of D into irreducible components, each cone of Σ can be viewed as a subset of $\mathbb{R}^p$, with basis $e_1, \ldots, e_p$.

The construction of the mirror to (X, D)

Let (B, Σ) be the *dual intersection complex* of the pair (X, D). Here B is a topological space and Σ is a decomposition of B into cones.

If $D = \bigcup_{i=1}^{p} D_i$ is the decomposition of D into irreducible components, each cone of Σ can be viewed as a subset of $\mathbb{R}^p$, with basis $e_1, \ldots, e_p$.

Then Σ contains a cone $\sum_{j=1}^{q} \mathbb{R}_{\geq 0} e_{i_j}$ if and only if $D_{i_1} \cap \cdots \cap D_{i_q} \neq \emptyset$.

(For simplicity, we assume such intersections are always irreducible.)

Let (B, Σ) be the *dual intersection complex* of the pair (X, D). Here B is a topological space and Σ is a decomposition of B into cones.

If $D = \bigcup_{i=1}^{p} D_i$ is the decomposition of D into irreducible components, each cone of Σ can be viewed as a subset of $\mathbb{R}^p$, with basis $e_1, \ldots, e_p$.

Then Σ contains a cone $\sum_{j=1}^{q} \mathbb{R}_{\geq 0} e_{i_j}$ if and only if $D_{i_1} \cap \cdots \cap D_{i_q} \neq \emptyset$.

(For simplicity, we assume such intersections are always irreducible.)

Then

$$B = \bigcup_{\sigma \in \Sigma} \sigma.$$

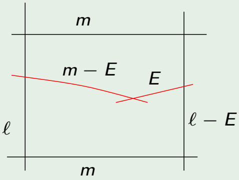
The construction of the mirror to (X, D)

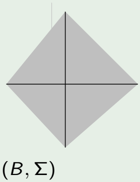
Example (Running example)

Consider $\mathbb{P}^1 \times \mathbb{P}^1$, with toric boundary

$$\bar{D} = (\{0, \infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{0, \infty\}).$$

Let $p : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up at a non-singular point of $\bar{D}$, and let D be the proper transform of $\bar{D}$.





We can take P to be generated by the classes $\ell - E$, $m - E$ and E.

The construction of the mirror to (X, D)

Set

$$B(\mathbb{Z}) = B \cap \mathbb{Z}^p,$$

the integral points of B.

Set

$$B(\mathbb{Z}) = B \cap \mathbb{Z}^p,$$

the integral points of B.
For any $n \geq 0$, we shall construct an $A_n := k[P]/\mathfrak{m}^{n+1}$-algebra

$$R_n := \bigoplus_{p \in B(\mathbb{Z})} A_n \vartheta_p.$$

The construction of the mirror to (X, D)

Set

$$B(\mathbb{Z}) = B \cap \mathbb{Z}^p,$$

the integral points of B.
For any $n \geq 0$, we shall construct an $A_n := k[P]/\mathfrak{m}^{n+1}$-algebra

$$R_n := \bigoplus_{p \in B(\mathbb{Z})} A_n \vartheta_p.$$

This is easy for $n = 0$. We define

$$\vartheta_p \cdot \vartheta_q = \begin{cases} \vartheta_{p+q} & \text{if } p, q \text{ lie in the same cone of } B; \\ 0 & \text{otherwise.} \end{cases}$$

The construction of the mirror to (X, D)

Set

$$B(\mathbb{Z}) = B \cap \mathbb{Z}^p,$$

the integral points of B.
For any $n \geq 0$, we shall construct an $A_n := k[P]/\mathfrak{m}^{n+1}$-algebra

$$R_n := \bigoplus_{p \in B(\mathbb{Z})} A_n \vartheta_p.$$

This is easy for $n = 0$. We define

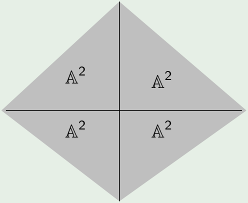
$$\vartheta_p \cdot \vartheta_q = \begin{cases} \vartheta_{p+q} & \text{if } p, q \text{ lie in the same cone of } B; \\ 0 & \text{otherwise.} \end{cases}$$

$\check{X}_n := \operatorname{Spec} R_n \to \operatorname{Spec} A_n$ will be our n-th order family.

The construction of the mirror to (X, D)

Note that for $n = 0$, Spec R_n is just a union of affine spaces glued together as dictated by the combinatorics of B.

Example (Running example)

$\check{X}_0$:  $\subseteq \mathbb{A}^4$

The construction of the mirror to (X, D)

For $n > 0$, the construction can be viewed on three levels:

The construction of the mirror to (X, D)

For $n > 0$, the construction can be viewed on three levels:

1. Construct an n-th order deformation $\check{X}_n^\circ$ of an open subset $\check{X}_0^\circ$ of $\check{X}_0$ obtained by deleting all codimension ≥ 2 strata of $\check{X}_0$. There are many such deformations, but the correct one is controlled by the Gromov-Witten theory of the pair (X, D).

The construction of the mirror to (X, D)

For $n > 0$, the construction can be viewed on three levels:

1. Construct an n-th order deformation $\check{X}_n^\circ$ of an open subset $\check{X}_0^\circ$ of $\check{X}_0$ obtained by deleting all codimension ≥ 2 strata of $\check{X}_0$. There are many such deformations, but the correct one is controlled by the Gromov-Witten theory of the pair (X, D).
2. We then take
$$\check{X}_n = \operatorname{Spec} \Gamma(\check{X}_n^\circ, \mathcal{O}_{\check{X}_n^\circ}).$$
For this to be a (partial) compactification of $\check{X}_n^\circ$, there must be enough regular functions. These are the *theta functions*, constructed using a logarithmic analogue of Maslov index two disks. These will be the ϑ_p's.

The construction of the mirror to (X, D)

For $n > 0$, the construction can be viewed on three levels:

1. Construct an n-th order deformation $\check{X}_n^\circ$ of an open subset $\check{X}_0^\circ$ of $\check{X}_0$ obtained by deleting all codimension ≥ 2 strata of $\check{X}_0$. There are many such deformations, but the correct one is controlled by the Gromov-Witten theory of the pair (X, D).
2. We then take
$$\check{X}_n = \operatorname{Spec} \Gamma(\check{X}_n^\circ, \mathcal{O}_{\check{X}_n^\circ}).$$
For this to be a (partial) compactification of $\check{X}_n^\circ$, there must be enough regular functions. These are the *theta functions*, constructed using a logarithmic analogue of Maslov index two disks. These will be the ϑ_p's.
3. ...

The construction of the mirror to (X, D)

1. ...
2. ...
3. Finally, the multiplication rule for theta functions can be described in terms of a logarithmic analogue of pairs of pants. We can avoid the first two steps by simply defining the multiplication rule in terms of (X, D), but we lose some refined information visible in the first two steps.

The construction of the mirror to (X, D)

I will focus largely on the third point, as this can be done with minimal technical baggage.

The construction of the mirror to (X, D)

I will focus largely on the third point, as this can be done with minimal technical baggage.

This construction should be related to symplectic cohomology, see e.g., forthcoming work of Ganatra-Pomerleano for direct comparisons in some very special cases.

Logarithmic Gromov-Witten invariants

Logarithmic Gromov-Witten invariants were developed by G.-Siebert (2011), Chen (2010), Abramovich-Chen (2011).

Logarithmic Gromov-Witten invariants

Logarithmic Gromov-Witten invariants were developed by G.-Siebert (2011), Chen (2010), Abramovich-Chen (2011).

Without going into any technical detail, log GW invariants allow the counting of a kind of stable map from marked curves

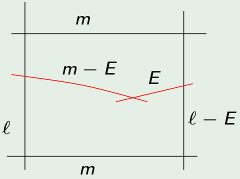
$$(C, p_1, \ldots, p_n) \to X$$

with orders of tangency with components of D specified at each p_i. This generalizes relative invariants of Li-Ruan, Ionel-Parker, and Jun Li.

Logarithmic Gromov-Witten invariants

For example, the crucial data for constructing the correct deformation of $\check{X}_0^\circ$ involves counts of "$\mathbb{A}^1$-curves." These are maps $(C, p) \to X$ with C a rational curve and some non-trivial specified tangency condition at p.

Example (Running example)



The two red curves are both $\mathbb{A}^1$-curves. In addition, multiple covers of each of these totally ramified over the intersection points with D also occur.

Logarithmic Gromov-Witten invariants

While this allows the completion of Step 1, it is insufficient to complete steps 2 or 3.

Logarithmic Gromov-Witten invariants

While this allows the completion of Step 1, it is insufficient to complete steps 2 or 3.

In G.-Hacking-Keel (2011), covering the case of surfaces, we were able to apply the main result of G.-Pandharipande-Siebert (2009) which gives an alternative description of these counts, and combine this with a result of Carl-Pumperla-Siebert (2010) in order to carry out Steps 2 and 3 at a tropical level.

Logarithmic Gromov-Witten invariants

While this allows the completion of Step 1, it is insufficient to complete steps 2 or 3.

In G.-Hacking-Keel (2011), covering the case of surfaces, we were able to apply the main result of G.-Pandharipande-Siebert (2009) which gives an alternative description of these counts, and combine this with a result of Carl-Pumperla-Siebert (2010) in order to carry out Steps 2 and 3 at a tropical level.

In higher dimension, we need *punctured invariants*, to be defined in forthcoming work of Abramovich-Chen-G.-Siebert.

Logarithmic Gromov-Witten invariants

While this allows the completion of Step 1, it is insufficient to complete steps 2 or 3.

In G.-Hacking-Keel (2011), covering the case of surfaces, we were able to apply the main result of G.-Pandharipande-Siebert (2009) which gives an alternative description of these counts, and combine this with a result of Carl-Pumperla-Siebert (2010) in order to carry out Steps 2 and 3 at a tropical level.

In higher dimension, we need *punctured invariants*, to be defined in forthcoming work of Abramovich-Chen-G.-Siebert.

Intuitively, we allow "negative orders of tangency at points."

Logarithmic Gromov-Witten invariants

For example, suppose $(C, p_1, \ldots, p_n)$ is a non-singular marked curve with assigned orders of tangency $d_i \in \mathbb{Z}$, $i = 1, \ldots, n$, and (X, D) is a pair with D a smooth divisor.

Logarithmic Gromov-Witten invariants

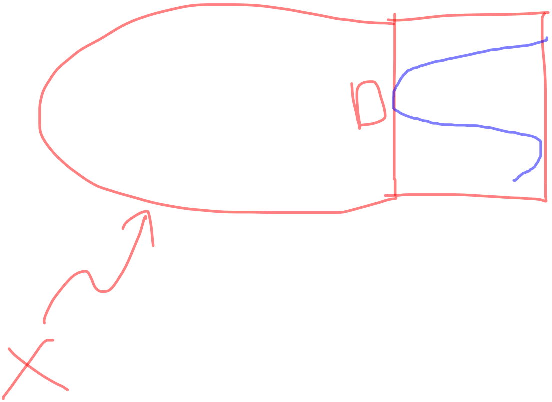
For example, suppose $(C, p_1, \ldots, p_n)$ is a non-singular marked curve with assigned orders of tangency $d_i \in \mathbb{Z}$, $i = 1, \ldots, n$, and (X, D) is a pair with D a smooth divisor.

If any of the d_i is negative, the only allowable "punctured maps" $f : C \to X$ have image contained in D. The log structure carries additional data, which in this case is a non-zero *meromorphic* section of $f^* N_{D/X}$, defined up to scaling, non-vanishing except at the p_i, with the order of zero at p_i given by d_i (pole if $d_i < 0$).

Logarithmic Gromov-Witten invariants

From Jun Li's expanded degeneration point of view, these curves can be viewed as follows:

From Jun Li's expanded degeneration point of view, these curves can be viewed as follows:



The construction of the mirror to (X, D)

We need to define the structure constants for the algebra:

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in B(\mathbb{Z})} \alpha_{pqr} \vartheta_r,$$

with $\alpha_{pqr} \in A_n$.

We can view $p \in B(\mathbb{Z})$ as representing a tangency condition. If p lies in the interior of a cone

$$\sigma = \sum_{j=1}^{q} \mathbb{R}_{\geq 0} e_{i_j},$$

we can write

$$p = \sum_{j=1}^{q} n_j e_{i_j},$$

$n_j > 0$.

The construction of the mirror to (X, D)

We can view $p \in B(\mathbb{Z})$ as representing a tangency condition. If p lies in the interior of a cone

$$\sigma = \sum_{j=1}^{q} \mathbb{R}_{\geq 0} e_{i_j},$$

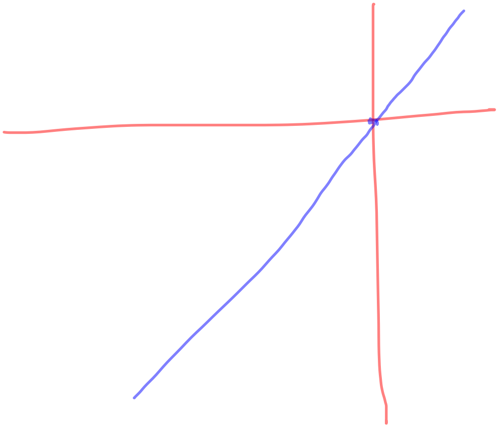
we can write

$$p = \sum_{j=1}^{q} n_j e_{i_j},$$

$n_j > 0$.
We can interpret this as a tangency condition at a point on a curve which is tangent to the divisor D_{i_j} to order n_j.

The construction of the mirror to (X, D)

E.g., $p = e_1 + e_2$:



The construction of the mirror to (X, D)

We define

$$\alpha_{pqr} = \sum_{\beta \in H_2(X,\mathbb{Z})} N^{\beta}_{pqr} z^{\beta}$$

where N^{β}_{pqr} is the count of three-pointed stable punctured curves representing the homology class β

$$f : (C, x_p, x_q, x_r) \to (X, D)$$

with tangency conditions at x_p and x_q specified as above by $p, q \in B(\mathbb{Z})$.

r, however, is interpreted as a punctured point, and if $r = \sum_j n_j e_{i_j}$ with $n_j > 0$ for all j, then we require tangency order $-n_j$ at x_r. Furthermore, we fix a point

$$x \in \bigcap_j D_{i_j}$$

and require x_r to map to x.

The construction of the mirror to (X, D)

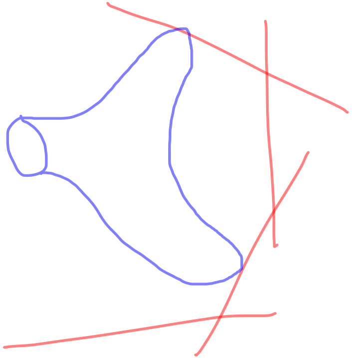
r, however, is interpreted as a punctured point, and if $r = \sum_j n_j e_{i_j}$ with $n_j > 0$ for all j, then we require tangency order $-n_j$ at x_r. Furthermore, we fix a point

$$x \in \bigcap_j D_{i_j}$$

and require x_r to map to x.
This problem has virtual dimension zero.

The construction of the mirror to (X, D)

Intuition: we are counting holomorphic disks with boundary on a fibre of the SYZ fibration which look like:



The construction of the mirror to (X, D)

Theorem (Forthcoming)

The above structure constants define a commutative A_n-algebra structure on R_n, lifting the given algebra structure on R_0.

The construction of the mirror to (X, D)

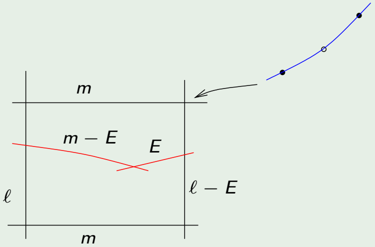
Example

Returning to the running example of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point, let $p_1, \ldots, p_4$ be the points of $B(\mathbb{Z})$ which are generators of the rays corresponding to the four boundary divisors, starting with $\ell - E$ and proceeding counterclockwise.

The construction of the mirror to (X, D)

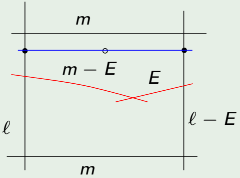
Example

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \vartheta_{p_1+p_2}.$$



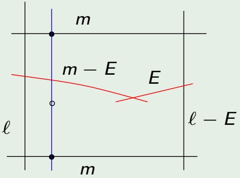
Example

$$\vartheta_{p_1} \cdot \vartheta_{p_3} = z^m \vartheta_0 = z^m.$$



Example

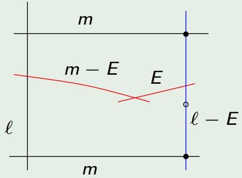
$$\vartheta_{p_2} \cdot \vartheta_{p_4} = z^{\ell} + z^{\ell - E} \vartheta_{p_1}$$



The construction of the mirror to (X, D)

Example

$$\vartheta_{p_2} \cdot \vartheta_{p_4} = z^{\ell} + z^{\ell - E} \vartheta_{p_1}$$



Example

In this example, the construction works over the non-completed ring

$$A = k[P] = k[z^{\ell - E}, z^{m-E}, z^E].$$

The construction of the mirror to (X, D)

Example

In this example, the construction works over the non-completed ring

$$A = k[P] = k[z^{\ell - E}, z^{m-E}, z^E].$$

We then have

$$R = \frac{A[\vartheta_{p_1}, \vartheta_{p_2}, \vartheta_{p_3}, \vartheta_{p_4}]}{(\vartheta_{p_1}\vartheta_{p_3} - z^m, \vartheta_{p_2}\vartheta_{p_4} - z^{\ell} - z^{\ell - E}\vartheta_{p_1})}.$$

The construction of the mirror to (X, D)

Example

In this example, the construction works over the non-completed ring

$$A = k[P] = k[z^{\ell - E}, z^{m-E}, z^E].$$

We then have

$$R = \frac{A[\vartheta_{p_1}, \vartheta_{p_2}, \vartheta_{p_3}, \vartheta_{p_4}]}{(\vartheta_{p_1}\vartheta_{p_3} - z^m, \vartheta_{p_2}\vartheta_{p_4} - z^{\ell} - z^{\ell - E}\vartheta_{p_1})}.$$

This gives the family of mirrors.

Generalizations

- A similar construction works for degenerations of Calabi-Yau manifolds $X \to \operatorname{Spec} k[[t]]$, essentially by working with the pair (X, X_0). We then get the *homogeneous* coordinate ring of the mirror.

Generalizations

- A similar construction works for degenerations of Calabi-Yau manifolds $X \to \operatorname{Spec} k[[t]]$, essentially by working with the pair (X, X_0). We then get the *homogeneous* coordinate ring of the mirror.
- We can also start, in this case, with a DLT relatively minimal model and an snc resolution, embedding the dual intersection complex of the DLT model in the dual intersection complex of the snc resolution, with image being the Kontsevich-Soibelman skeleton (Nicaise-Xu). This allows us to get away from the snc assumption.

Questions

- Independence of model?

Questions

- Independence of model?
- When is the generic fibre of the mirror smooth?

Questions

- Independence of model?
- When is the generic fibre of the mirror smooth?
- Actual computations and comparison with previous mirror constructions.

Questions

- Independence of model?
- When is the generic fibre of the mirror smooth?
- Actual computations and comparison with previous mirror constructions.
- Homological mirror symmetry?

Questions

- Independence of model?
- When is the generic fibre of the mirror smooth?
- Actual computations and comparison with previous mirror constructions.
- Homological mirror symmetry?
- Genus 0 (or higher genus) mirror symmetry?

Questions

- Independence of model?
- When is the generic fibre of the mirror smooth?
- Actual computations and comparison with previous mirror constructions.
- Homological mirror symmetry?
- Genus 0 (or higher genus) mirror symmetry?
- ...