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The goal of this talk, rather than recapping the history of the subject, is to sketch some key concepts which have emerged in my work with various collaborators, including Paul Hacking, Sean Keel, Maxim Kontsevich and Bernd Siebert, which have had applications outside of mirror symmetry, and even outside of geometry.

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Let $\sigma \subseteq M_{\mathbb{R}}$ be a lattice polytope.

Define a map

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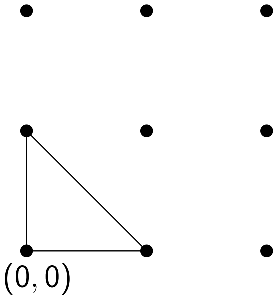
Examples. (1) Take σ to be the triangle:



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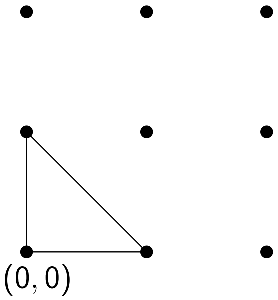

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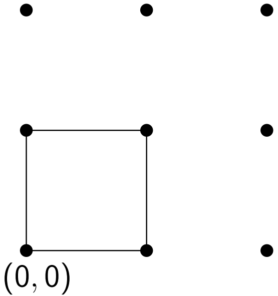

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If coordinates on $\mathbb{P}^3$ are X, Y, Z, W, then the above image satisfies the equation

$$XW = YZ,$$

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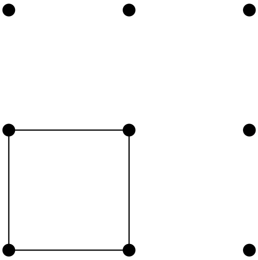

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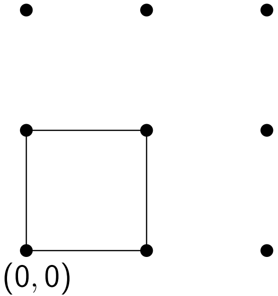

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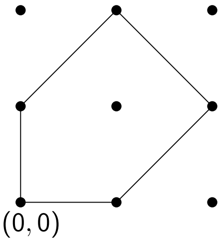

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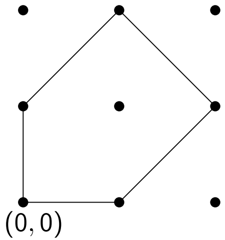

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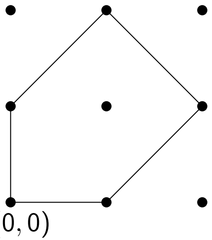

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Features:

- A projective toric variety $\mathbb{P}_\sigma$ is defined by binomial equations in projective space.
- A toric variety always contains a dense open set isomorphic to the algebraic torus T_N.
- The monomials z^m extend to sections of a line bundle $\mathcal{O}_{\mathbb{P}_\sigma}(1)$ on $\mathbb{P}_\sigma$. These are "canonical" sections.

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Definition

An *integral affine manifold* is a real n-dimensional manifold B with an atlas with transition functions in $\mathrm{Aff}(\mathbb{Z}^n)$.

An *integral affine manifold with singularities* is a real manifold B with an open subset $B_0 \subseteq B$, with $\Delta := B \setminus B_0$ of codimension ≥ 2, such that B_0 has the structure of an integral affine manifold. We also allow a variant with boundary, where locally at the boundary B is a lattice polytope.

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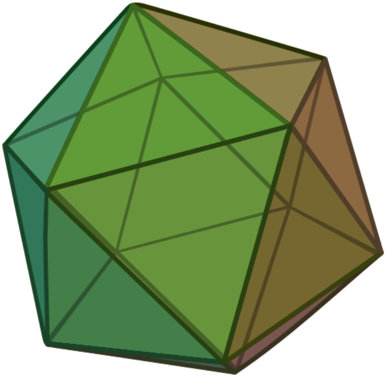
By flattening the edges of an icosahedron, we get a sphere with 12 singular points.



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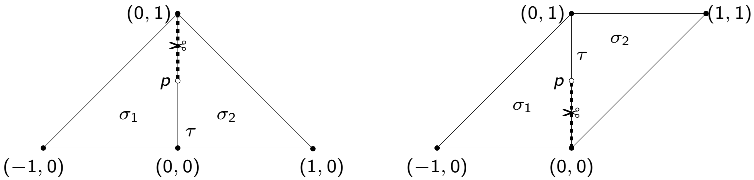

The diagram shows the affine embeddings of two charts, obtained by cutting the union of two triangles as indicated in the two figures. Note that the vertical line segment is an invariant direction, being a straight line in both charts.

Following coordinates around a counterclockwise loop, one gets a coordinate transformation

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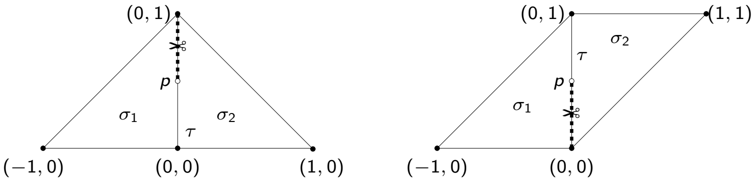

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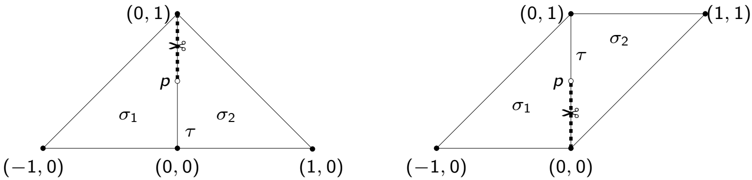

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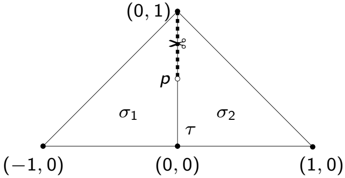

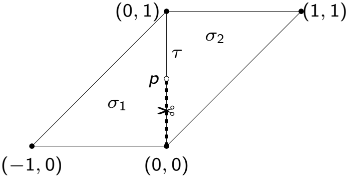

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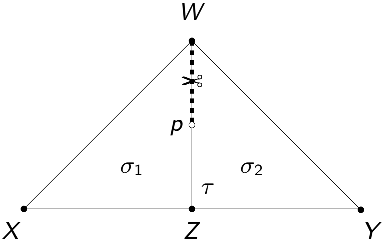
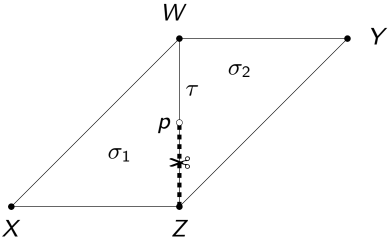
Question: What is the equation of the surface corresponding to this picture?



$XY = Z^2$



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Which equation is the correct one?

Correct answer: choose both choices for the product XY, i.e.,

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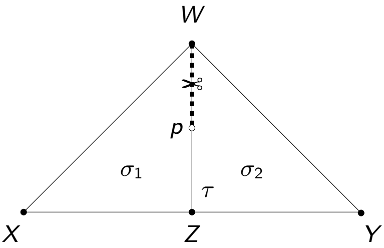

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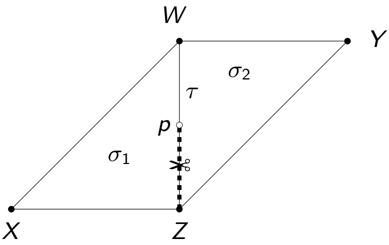

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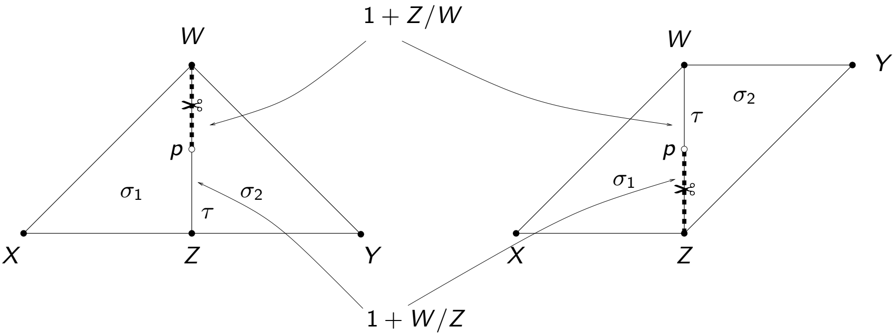
Example. In our running example, we use two walls, rays emanating from the singularity.



If we want to move X across the upper ray into the right-hand triangle in order to compare it with Y, we need to apply a wall-crossing automorphism

$$X \mapsto X(1 + Z/W), \quad Z \mapsto Z, \quad W \mapsto W,$$

while if we cross the lower ray, we apply

$$X \mapsto X(1 + W/Z), \quad Z \mapsto Z, \quad W \mapsto W.$$

Thus if we use the left-hand diagram, let us calculate $\vartheta_X \vartheta_Y$, where we use ϑ_X, ϑ_Y to distinguish these from the "raw" X, Y. We transport X over to the right, use $\vartheta_X = X(1 + W/Z)$, $\vartheta_Y = Y$, so

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Similarly, if we use the right-hand diagram and transport X through the upper ray, so $\vartheta_X = X(1 + Z/W)$, we still get

$$\vartheta_X \vartheta_Y = X(1 + Z/W)Y = WZ + Z^2,$$

and the equation stays the same.

Thus we get a projective variety which comes along with four canonical functions

$$\vartheta_X, \vartheta_Y, \vartheta_W = W, \vartheta_Z = Z.$$

We no longer have a binomial equation.

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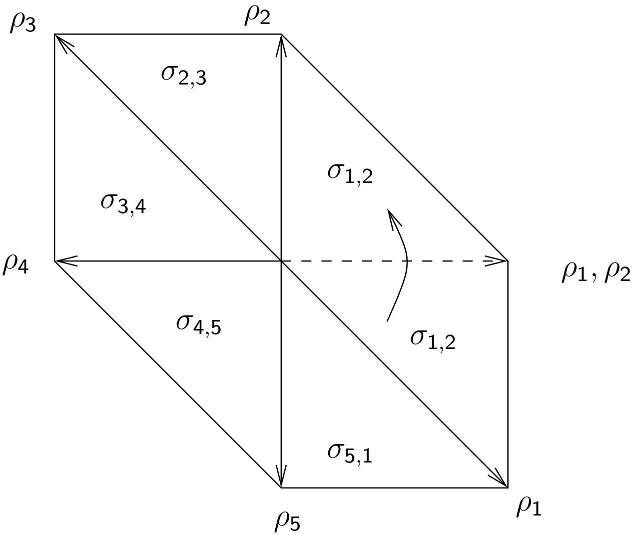

ρ_3
ρ_2
$\sigma_{2,3}$
$\sigma_{1,2}$
$\sigma_{3,4}$
ρ_4
ρ_1, ρ_2
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ρ_1
ρ_5

In this case we need a scattering diagram consisting of the rays $\rho_1, \ldots, \rho_5$, with function $1 + W/X_i$ attached to the ray ρ_i. Here $X_1, \ldots, X_5$ correspond to the vertices and W to the interior point.

There exist canonical sections $\vartheta_{X_1}, \ldots, \vartheta_{X_5}, \vartheta_W$ of a line bundle on the corresponding variety. These satisfy the relations

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So far:

- Used scattering diagrams to transform monomials between different chambers to fix inconsistency caused by singularities of the affine structure.
- Each chamber in this picture corresponds to a copy of an algebraic torus, and walls tell us how to glue tori.
- We were lucky that transforming monomials gave well-defined functions on each torus, which then could be used to embed into projective space.
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Theorem

(G.-Siebert, 2007, G.-Hacking-Keel-Siebert, forthcoming) Suppose given a compact integral affine manifold with singularities B*, a decomposition* $\mathscr{P}$ *of* B *into lattice polytopes, and a "multi-valued convex piecewise linear function with integral slopes." Given a "consistent" scattering diagram, there exists a projective family*

$$\mathcal{X} \to \mathrm{Spec}\, k[[t]]$$

such that the special fibre satisfies

$$\mathcal{X}_0 = \bigcup_{\sigma \in \mathscr{P}_{\max}} \mathbb{P}_\sigma$$

and the generic fibre is irreducible. Furthermore, $\mathcal{X}$ *carries a relatively ample line bundle* $\mathcal{O}_{\mathcal{X}}(1)$ *with a canonical basis of sections*

$$\{\vartheta_p \,|\, p \in B \text{ a point with integral coordinates}\}.$$

Examples. (1) Going back to the simplest example of an affine manifold with singularities, take $\mathscr{P}$ to consist of the two triangles. Then with suitable choice of PL function, we get the degeneration

$$\vartheta_X \vartheta_Y = t\vartheta_Z(\vartheta_W + \vartheta_Z).$$

For $t = 0$, we get the union of two planes.

(2) $B = \mathbb{R}^n/\Gamma$ where $\Gamma \subseteq \mathbb{Z}^n$ is a lattice. Choose $\mathscr{P}$ and a PL function. The scattering diagram can be taken to be empty. Then our construction reproduces the degeneration of abelian varieties $\mathcal{X} \to \operatorname{Spec} k[[t]]$ constructed by Mumford, and our theta functions coincide with classical theta functions.

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(2) $B = \mathbb{R}^n/\Gamma$ where $\Gamma \subseteq \mathbb{Z}^n$ is a lattice. Choose $\mathscr{P}$ and a PL function. The scattering diagram can be taken to be empty. Then our construction reproduces the degeneration of abelian varieties $\mathcal{X} \to \operatorname{Spec} k[[t]]$ constructed by Mumford, and our theta functions coincide with classical theta functions.

The problem in general is to construct "consistent" scattering diagrams, which can be quite difficult.

This problem was solved for relatively nice singularities (generalizations of our simple two-dimensional example) in G.-Siebert (2007) following a 2004 solution to the problem by Kontsevich and Soibelman.

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Let Y be a non-singular rational projective surface and let D be a cycle of $\mathbb{P}^1$'s in Y representing the anti-canonical class.

Then we can construct an integral affine manifold with one singularity B homemorphic to $\mathbb{R}^2$, with the singularity at the origin, from this data, a kind of "generalized fan."

If D is contractible, this affine manifold is constructed as follows, for some T.

Let $T \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{Tr}\, T > 2$.

Then T has two real distinct eigenvalues. Let $\sigma \subseteq \mathbb{R}^2$ be a cone generated by two eigenvectors.

We take $B_0 = \mathrm{Int}(\sigma)/T^{\mathbb{Z}}$ and $B = B_0 \cup \{0\}$.

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GHK gives a construction of a consistent scattering diagram using Gromov-Witten theory (curve counts) of Y!

In the above case, the scattering diagram is extremely complicated: every ray of rational slope appears, with a formal power series attached.

Again, in the above case, we can extend the family analytically so that it includes a smoothing of a cusp singularity, resolving a conjecture of Looijenga about deformation theory of cusp singularities, telling us precisely when a cusp singularity is smoothable.

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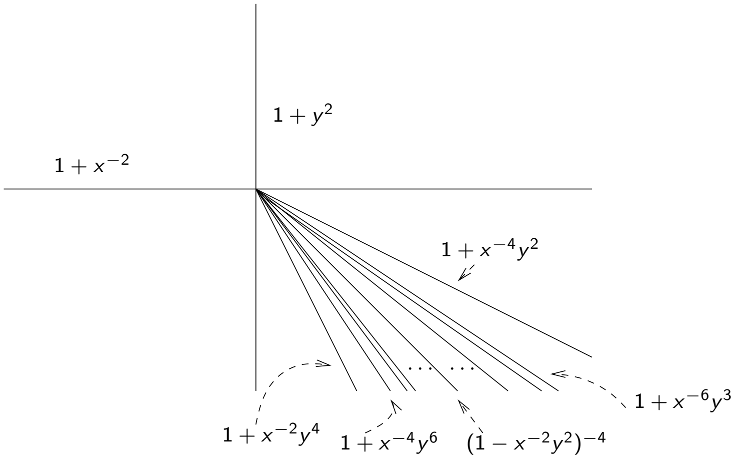
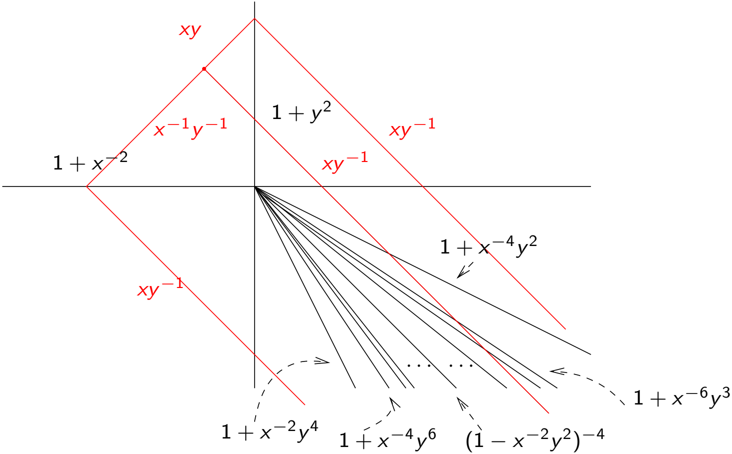
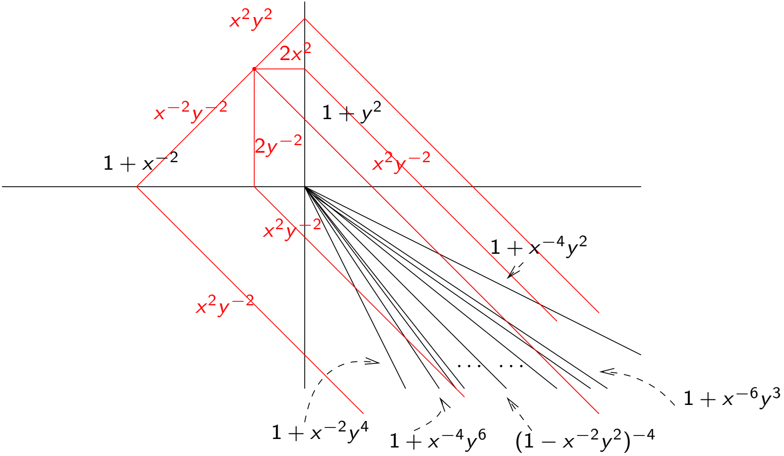
The basic idea behind the construction of theta functions comes from tropical geometry.

Here is an example of a consistent scattering diagram in $B = \mathbb{R}^2$ arising from cluster algebras:





$$\vartheta_{(1,-1)} = xy^{-1}(1+x^{-2}+y^2).$$



$$\vartheta_{(2,-2)} = x^2y^{-2}(1+2x^{-2}+2y^2+x^{-4}+y^{-4}) = \vartheta_{1,-1}^2 - 2.$$

In fact, there is a scattering diagram associated to every cluster algebra (of geometric type).

This allows us (G.-Hacking-Keel-Kontsevich) to prove some of the major conjectures in the theory of cluster algebras, including positivity of the Laurent phenomenon. This follows from the fact that cluster variables are theta functions, and the corresponding scattering diagrams all have walls with attached functions having positive coefficients.

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