PUNCTURED LOGARITHMIC MAPS

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Abstract. We introduce a variant of stable logarithmic maps, which we call *punctured logarithmic maps*. They allow an extension of logarithmic Gromov-Witten theory in which marked points have a negative order of tangency with boundary divisors. These are constructed with several applications in mind. First, they appear naturally in a generalization of the Li-Ruan and Jun Li gluing formulas, with punctured invariants playing the role of relative invariants in these classical gluing formulae. Second, they provide key enumerative invariants for constructions in mirror symmetry.

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1. Introduction

Logarithmic Gromov-Witten theory, developed by the authors in [Che14], [AC14], [GS13], has proved a successful generalization of the notion of relative Gromov-Witten invariants developed in [LR01], [Li01], [Li02]. Relative Gromov-Witten invariants are invariants of pairs (X, D) where X is a non-singular variety and D is a smooth divisor on X; these invariants count curves with imposed orders of tangency with D at marked points. Logarithmic Gromov-Witten theory allows D instead to be normal crossings, or more generally, allows (X, D) to be a toroidal crossings variety.

One of the main intended applications of the theory considers degenerations $X \to B$ where B is a non-singular curve with a point $b_0 \in B$ such that $(X, D) \to (B, b_0)$ is a toroidal crossings morphism, i.e., is log smooth. Thus the fibre X_0 over b_0 may be quite singular, but nevertheless logarithmic Gromov-Witten theory makes sense on X_0 relative to b_0. One then wishes to describe the Gromov-Witten theory of the general fibre in terms of the logarithmic Gromov-Witten theory of the special fibre. In the case that X_0 is the normal crossings union of two divisors, this leads to the gluing formulae of [LR01] and [Li02], which have proved to be immensely useful tools in the Gromov-Witten toolkit. However, a practically useful generalization of this gluing formula has proved somewhat elusive.

In [ACGS17], we initiated a program generalizing these classical gluing formulae. Given a class of logarithmic curve for a log smooth target space X/B, we obtain a moduli space of stable log maps $\mathscr{M}(X/B, \beta)$ which fibres over B. The fibre of this map over 0 is $\mathscr{M}(X_0/b_0, \beta)$, and this was shown to have a "virtual irreducible decomposition" into components indexed by rigid tropical curves. However, there still remains a problem of describing these virtual irreducible components and calculating their virtual fundamental classes.

Put simply, the next problem which arises is as follows. Suppose given a stable log map $f : C/W \to X$, and suppose given a closed subscheme $\underline{C}'$ of the underlying scheme $\underline{C}$ of C which is a union of irreducible components of $\underline{C}$. As $\underline{f} : \underline{C} \to \underline{X}$ is required to be an ordinary stable map, $\underline{f}|_{\underline{C}'}$ is also an ordinary stable map provided we mark those non-singular points of $\underline{C}'$ which are double points of $\underline{C}$. However, if we restrict the log structure of C to $\underline{C}'$ to obtain a log scheme C' and a log morphism $f|_{C'} : C' \to X$, this morphism fails to be a stable log map for the very simple reason that $C' \to W$ is not a log smooth family: the log structure at those non-singular points of $\underline{C}'$ which were double points of $\underline{C}$ is not the standard one at marked points. Further, if we replaced the log structure at those points with the standard marked point log structure used in log Gromov-Witten theory, the morphism to X may not exist.

The solution presented here is to broaden the treatment of marked points to allow more interesting log structures. While we delay precise definitions until §2, we explain briefly how these new log structures differ from old-fashioned marked points. Consider a logarithmic curve $\pi : C \to W$ with $W = \mathrm{Spec}(Q \to \Bbbk)$ a logarithmic point. In ordinary log Gromov-Witten theory, the stalk of the ghost sheaf $\overline{\mathcal{M}}_C$ of C at a non-special point is Q, at a marked point is $Q\oplus\mathbb{N}$, and is more complicated at a node. In punctured theory, we allow more complicated choices of monoids at marked points, which we now call *punctured points* or *punctures*. At such a point, the stalk of the ghost sheaf is a fine (but not necessarily saturated) monoid $Q^\circ \subseteq Q \oplus \mathbb{Z}$ containing $Q \oplus \mathbb{N}$. The possible choices of Q° are somewhat restricted by the need that this be the stalk of a ghost sheaf of a log structure, but nevertheless this still allows a range of possible choices. We have chosen here to restrict the possible choices by imposing an additional condition which we call *pre-stability*, which only makes sense in the presence of a log morphism $f : C \to X$. Here, if $p \in C$ is a punctured point, we then obtain an induced morphism on stalks of ghost sheaves

$$\bar{f}^\flat : \overline{\mathcal{M}}_{X,f(p)} \to \overline{\mathcal{M}}_{C,p} = Q^\circ \subseteq Q \oplus \mathbb{Z}.$$

Pre-stability then is the condition that Q° is the submonoid of $Q \oplus \mathbb{Z}$ generated by $Q \oplus \mathbb{N}$ and the image of $\bar{f}^\flat$. Essentially, we are choosing the smallest possible monoid for which the morphism f exists.

Crucially, the composition of $\bar{f}^\flat$ with the projection $Q \oplus \mathbb{Z} \to \mathbb{Z}$ gives a homomorphism

$$u_p : \overline{\mathcal{M}}_{X,f(p)} \to \mathbb{Z}, \tag{1.1}$$

called the *contact order* of the punctured point. In ordinary log Gromov-Witten theory, this homomorphism would take values in $\mathbb{N}$ and record the order of tangency of the curve at the marked point with various boundary divisors. Thus in punctured theory, this is viewed as giving the possibility of negative contact order.

More specifically, suppose that the log structure on X arises from a normal crossings divisor $D = D_1 + \cdots + D_n$ of X, with D_i irreducible. If $f(p)$ lies in the intersection of irreducible components D_i for $i \in I$ an index set, then $\overline{\mathcal{M}}_{X,f(p)} = \mathbb{N}^I$, and for $i \in I$, the i^{th} component of u_p indicates the contact order of the map f with D_i at p. If this contact order is negative, then the irreducible component $\underline{C}'$ of $\underline{C}$ containing p should satisfy $f(\underline{C}') \subseteq D_i$, see Remark 2.19.

At this point the reader may reasonably wonder why such punctured invariants do not appear in the original Li-Ruan and Jun Li gluing formulae. In those theories, the Gromov-Witten theory of the central fibre $\underline{X}_0 = \underline{Y}_1 \cup \underline{Y}_2$ is described in terms of relative Gromov-Witten invariants of the pairs $(\underline{Y}_1, D)$, $(\underline{Y}_2, D)$ with $D = \underline{Y}_1 \cap \underline{Y}_2$. In fact, there are two log structures on $\underline{Y}_i$: the restriction of the log structure of X to Y_i, which we write as $Y_i^\dagger$, and the divisorial log structure coming from $D \subseteq \underline{Y}_i$, which we write as Y_i. There is a canonical morphism $Y_i^\dagger \to Y_i$ given by inclusion of log structures, hence inducing by composition a morphism $\mathscr{M}(Y_i^\dagger/b_0, \beta) \to \mathscr{M}(Y_i, \beta)$. One can show in this case that this induces an isomorphism of underlying stacks and obstruction theories (although the log structures are necessarily different). In particular, in the proof of the classical gluing formulas, given a stable log map $f : C \to X$ lying in one of the virtual irreducible components of $\mathscr{M}(X_0/b_0, \beta)$, there is a way of splitting $\underline{C} = \underline{C}_1 \cup \underline{C}_2$ so that $f|_{C_i}$ can be viewed as a morphism to $Y_i^\dagger$, and hence by composition with the morphism $Y_i^\dagger \to Y_i$, we obtain a stable log map. The moduli of stable log maps to Y_i is closely related to the Jun Li moduli space of stable relative maps to the pair $(\underline{Y}_i, D)$ and gives the same numerical invariants, see [AMW14].

On the other hand, in more complicated gluing situations, such as when the central fibre X_0 has a triple point, there can be stable log maps $f : C \to X_0$ with some components of C mapping to the triple point, and there is no reasonable way to view this component as mapping to a specific irreducible component of X_0. For an example of this, see the extended example of [ACGS17], §6.2, especially §6.2.4, in which the curve component C_4 may not be viewed as a relative curve in any irreducible component. Thus any reasonable generalization of the classical gluing formula will need to take into account some more complicated invariants.

This is the first reason that punctured invariants are useful to us. The second is that soon after discussions amongst the four of us began on this project in 2011, the last two authors of this paper realised that it was likely that such invariants were exactly what was necessary for describing holomorphic versions of certain tropical constructions in [GS11], [GHK15] which appear naturally in the Gross-Siebert mirror symmetry program. This has now led to a general mirror symmetry construction, announced in [GS18], in which certain punctured Gromov-Witten invariants are used to define the (homogeneous) coordinate ring of the mirror. The proofs of many of the announced results of [GS18] are now available in [GS19] and depend crucially on this paper. The notion of Gromov-Witten invariants with negative orders of tangency is absolutely essentially, while the proof of associativity of the product rule relies crucially on the gluing formalism developed in §5 of the current paper.

We note also that [GS18] also constituted an announcement of this paper, and followed an early draft of this preprint which was made public in 2016. However, the reader familiar with that draft or [GS18] will note that in fact the definition of a punctured point has changed. The original definition given in these older references dealt with the possible non-uniqueness of the log structure at punctured points by taking the limit over all possible punctured log structures, resulting in a non-finitely generated stalk of the ghost sheaf at a puncture. With more experience, we have found the formulation in this paper to be technically simpler, as fine log structures are better understood. However, despite the apparently different formulation, the theories are equivalent, and we give a brief discussion of this older theory in Remark 2.3.

We now turn to the structure of the paper, and outline novel features of the theory. §2 introduces the notion of a punctured log structure, specializing quickly to the case of a punctured point on a curve. This allows us to generalize the notion of stable log map to that of stable punctured map. Once the notion of punctured log structure is introduced, there are no surprises in the definition of a punctured log map. From there, much of the theory is developed analogously to that of ordinary log stable maps, with notions of combinatorial types of punctured maps and basic punctured maps precisely as in the usual case.

The first important difference between the punctured theory and the ordinary theory occurs in §2.5. There, we explain how any family $f : C/W \to X$ of punctured log maps induces a natural *idealized* log structure on W, in the sense of [Ogu18], III §1.3. Crucially, this structure encodes certain combinatorial obstructions to deforming punctured log curves which do not exist in the ordinary case. Intuitively, for example, suppose the target is a normal crossings pair (X, D) with $D = \sum D_i$ the decomposition into irreducible components. If C has an irreducible component $\underline{C}'$ containing a puncture with a negative contact order with some D_i, then we must have $f(\underline{C}') \subseteq D_i$, see Remark 2.19. Thus no deformation of this punctured map may deform the image of $\underline{C}'$ away from D_i, and in particular, if C has a node q with $q \in \underline{C}'$, this node may not be smoothed if the other branch $\underline{C}''$ of C containing q has $f(\underline{C}'') \not\subseteq D_i$. The idealized structure effectively encodes such purely combinatorial, local obstructions to deforming. As we shall see, this becomes especially important when one wishes to build a virtual fundamental class on moduli space of punctured maps.

The next subtlety involves defining families of contact orders. For an individual punctured map over a log point, we have the notion of contact order of (1.1). However, to obtain finite type moduli spaces, we need to impose contact orders at marked points, and as the point $f(p)$ varies in a connected family, we need to understand how contact orders vary. This turns out to be much more subtle than in the ordinary case. In §2.6, we explore this issue, leading to a classification of possible contact orders. However, at this point, contact orders are only well-behaved if $\overline{\mathcal{M}}_X$ is generated by its global sections, i.e., $\Gamma(X, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ is surjective for each $x \in X$. Otherwise, it is possible that even a connected family of contact orders may have an infinite number of irreducible components, making it difficult to prove that moduli spaces are of finite type. Thus this assumption is made in many places in this paper to obtain a good theory. Note this assumption always holds when the log structure on X arises from a normal crossings divisor.

In §2.7, we generalize the tropical point of view of [GS13], [ACGS16] to the punctured case, showing how to interpret various aspects of the theory tropically. In particular, under tropicalization, punctured points become line segments or unbounded rays. The vanishing locus of the puncturing ideal also has a simple tropical interpretation, see Remark 2.53.

We turn to §3. Following the point of view of [AW18], if given a target $X \to B$, one lets $\mathcal{A}_X$ be the relative Artin fan for $X \to B$ (see [ACM+15] for an exposition of Artin fans): this is equipped with a morphism $\mathcal{A}_X \to \mathcal{A}_B$ to the Artin fan of B. We set $\mathcal{X} := \mathcal{A}_X \times_{\mathcal{A}_B} B$. We define stacks $\mathscr{M}_{g,n}(X/B)$ and $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ of punctured maps to X, $\mathcal{X}$ respectively, with their basic log structure. Here one considers domain curves of genus g with n punctured points, and note that for $\mathscr{M}_{g,n}(X/B)$ we impose the condition of stability, i.e., that the underlying map of schemes is a stable map, but this cannot be imposed in the case of punctured maps to $\mathcal{X}$.

We also define the notion of a class β of punctured map to X/B, which includes the data of an underlying curve class, genus, number of punctured points, and contact orders at the punctures. This gives sub-moduli spaces $\mathscr{M}(X/B,\beta)$ and $\mathfrak{M}(\mathcal{X}/B,\beta)$, the latter moduli space forgetting the underlying curve class of β.

The main results of §3 are summarized by:

Theorem 1.1. *Suppose given a target $X \to B$ with X Zariski. Then:*

(1) The stack $\mathscr{M}_{g,n}(X/B)$ of stable punctured maps of genus g with n punctured points and target X/B is a logarithmic Deligne-Mumford stack locally of finite presentation.
(2) Let β be a class of punctured log curve, and suppose that $\overline{\mathcal{M}}_X$ is generated by global sections and $X \to B$ finite type. Then $\mathscr{M}(X/B,\beta) \to B$ is of finite type.
(3) The forgetful map $\mathscr{M}(X/B,\beta) \to \mathscr{M}(\underline{X}/\underline{B},\underline{\beta})$, where $\underline{\beta}$ just remembers the class of underlying curve, the genus, and number of marked points, satisfies the weak valuative criterion for properness.

These three items are Theorems 3.1, 3.7 and 3.12 respectively. In particular, in the case that X is proper over B and $\overline{\mathcal{M}}_X$ is generated by global sections, $\mathscr{M}(X/B,\beta)$ is in fact a proper Deligne-Mumford stack over B.

The proofs of these are essentially the same as in the ordinary log Gromov-Witten case, and we only note when additional care must be taken at the punctures.

§4 then develops the relative obstruction theory for $\mathscr{M}(X/B) \to \mathfrak{M}(\mathcal{X}/B)$. Again, the punctures do not play any particular role here, but some care is taken in the development of the theory to allow for a clean gluing statement later in the paper. The main results, from Proposition 4.2 and Theorem 4.5, are:

Theorem 1.2. *Suppose $X \to B$ is log smooth. Then:*

(1) There is a perfect relative obstruction theory for $\mathscr{M}_{g,n}(X/B) \to \mathfrak{M}_{g,n}(\mathcal{X}/B)$.
(2) The natural forgetful morphism $\mathfrak{M}_{g,n}(\mathcal{X}/B) \to \mathbf{M}_{g,n} \times B$ is idealized log smooth, where $\mathbf{M}_{g,n}$ denotes the Artin stack of pre-stable log curves with the basic log structure. Here $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ is idealized via its puncturing log ideal, while $\mathbf{M}_{g,n} \times B$ carries the empty log ideal.

The second statement is very important. In the ordinary stable log map case, [AW18] showed that $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ was in fact log étale over $\mathbf{M}_{g,n} \times B$, and hence is log smooth over B. For example, if $B = \operatorname{Spec} \Bbbk$, then this tells us that smooth locally, $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ looks like a toric variety. On the other hand, if we are considering punctures, then $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ is only idealized log smooth over B. Again, if

$B = \operatorname{Spec} \Bbbk$, this means that smooth locally, $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ looks like a scheme defined by a monomial ideal in a toric variety. While idealized log smoothness means that it is easy to control the local structure of $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ from a combinatorial point of view, it need not be equi-dimensional, see Example 4.7. This means that there is not a virtual fundamental class in general, and in any particular situation where we wish to extract numbers, one must apply virtual pull-back to a suitably chosen cycle on $\mathfrak{M}(\mathcal{X}/B,\beta)$. This depends on the particular application one may have in mind. However, it is very natural to consider virtual pull-backs of strata of $\mathfrak{M}_{g,n}(\mathcal{X}/B)$ selected out by certain tropical data. For example, such has been done in the proposed construction of the canonical scattering diagram in [GS18].

§5 begins the exploration of gluing using punctured maps. We first justify the original motivation of punctured curves: splitting a stable log map at a node produces a curve with two punctures. We then reverse the procedure, explaining how to glue curves. Unfortunately, this is rather more technical than one might hope. For ordinary stable maps, suppose given two families of stable curves $\underline{f}_i : \underline{C}_i/\underline{W}_i \to \underline{X}$, along with marked points $x_i : \underline{W}_i \to \underline{C}_i$. Suppose further we wish to glue these two families by identifying x_1 and x_2. Of course there are evaluation maps $\underline{\mathrm{ev}}_i = \underline{f}_i \circ x_i$ at x_i, which we may use to form a fibre product $\underline{W}_1 \times_{\underline{X}} \underline{W}_2$ parameterizing the glued family.

If instead we had two families of punctured curves $f_i : C_i/W_i \to X$, with punctures x_i, to be able to glue we first need the contact orders at x_1 and x_2 to be the negative of each other in an appropriate sense. We define the notion of *opposite contact orders* in Definition 2.47 to make this precise. Unfortunately, one does not in any event have evaluation maps $\mathrm{ev}_i : W_i \to X$, as the log structure on W_i and the log structure on $x_i(\underline{W}_i)$ don't agree. However, if we define $\widetilde{W}_i$ to be the saturation of the log scheme $(\underline{W}_i, x_i^*\mathcal{M}_{C_i})$, then there is an evaluation map $\mathrm{ev}_i : \widetilde{W}_i \to X$. This allows us to form the product $\widetilde{W} := \widetilde{W}_1 \times_X \widetilde{W}_2$ in the category of fs log schemes. Again, $\widetilde{W}$ is not quite the right thing: it does in fact parameterize the glued family, but it does not carry the basic log structure, even if W_1 and W_2 do. Instead, one can show that there is a sub-log structure of $\widetilde{W}$ which gives the glued family.

The precise statement in all generality is Theorem 5.12. We do not give the statement in the introduction, as it requires a rather detailed setup.

After having constructed the glued moduli spaces, the remaining question we address is compatibility of gluing with the relative obstruction theories constructed in this paper. The culmination of this are Theorems 5.15 and 5.17. Again, these statements are quite technical, and even worse, at this point are quite difficult to use. It is worth emphasizing one of the basic sources of this difficulty is the fact that the underlying spaces of fibre products in the fs log category do not agree with the fibre products of underlying spaces. This means that naive attempts to make use of Fulton-style intersection theory are bound to fail. Nevertheless, despite this difficulty, [GS19] has managed to apply the gluing techniques introduced here to a quite general situation. We anticipate that a great deal of future work will be devoted to making Theorems 5.15 and 5.17 broadly usable in practice. Indeed, a sequel paper to this one will explore these gluing techniques further, showing how to adapt gluing in the degeneration situation.

We end this introduction to discuss related work. First, our approach owes a great deal to Brett Parker's program of exploded manifolds, [Par11]. We have often found ourselves trying to translate Parker's results in the category of exploded manifolds into the category of log schemes. Indeed, some of the original versions of the definition of punctured invariants, as well as the approach to gluing, arose after discussions with Parker,

After the earlier manuscript version of this paper was distributed, Mohammed Teherani [Teh17], in developing a symplectic analogue of stable log maps, found that punctures were naturally described in the theory. Even more recently, [FWY18] used rubber invariants to define negative contact order Gromov-Witten invariants relative to a smooth divisor. While it is not yet clear what the precise relationship between these invariants and those of this paper are, very likely they can be defined as the virtual pull-back of certain cycles in $\mathfrak{M}_{g,n}(\mathcal{X}/B)$.

Besides the immediate applications of punctures already mentioned above, punctures also have been used by Hülya Argüz in [Arg17] to build a logarithmic analogue of certain tropical objects in the Tate elliptic curve related to Floer theory.

Finally, we also mention recent work of Dhruv Ranganathan [Ran19] taking a different point of view on gluing in log Gromov-Witten theory using an approach closer in spirit to the expanded degeneration picture of Jun Li.

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1.2. **Convention.** All logarithmic schemes and stacks are defined over an algebraically closed field $\Bbbk$ of characteristic 0. We follow the convention that if X is a log scheme or stack, then $\underline{X}$ is the underlying scheme or stack. We almost always write $\mathcal{M}_X$ for the sheaf of monoids on X and $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ for the structure map. If P is a monoid, we write $P^\vee := \mathrm{Hom}(P, \mathbb{N})$ and $P^* = \mathrm{Hom}(P, \mathbb{Z})$.

2. Punctured maps

2.1. **Definitions.**

2.1.1. *Puncturing.*

Definition 2.1. Let $Y = (\underline{Y}, \mathcal{M}_Y)$ be a fine and saturated logarithmic scheme with a decomposition $\mathcal{M}_Y = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. Denote $\mathcal{E} := \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$ and $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{O}^\times$. A *puncturing* of Y along $\mathcal{P} \subset \mathcal{M}_Y$ is a sub-sheaf of monoids

$$\mathcal{M}_{Y^\circ} \subset \mathcal{E} = \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$$

containing $\mathcal{M}_Y$ with a structure map $\alpha_{\mathcal{M}_{Y^\circ}} : \mathcal{M}_{Y^\circ} \to \mathcal{O}_Y$ such that

(1) The inclusion $\mathrm{p}^\flat : \mathcal{M}_Y \to \mathcal{M}_{Y^\circ}$ is a morphism of fine logarithmic structures on $\underline{Y}$.
(2) For any geometric point $\bar{x}$ of $\underline{Y}$ let $s_{\bar{x}} \in \mathcal{M}_{Y^\circ, \bar{x}}$ be such that $s_{\bar{x}} \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}_{\bar{x}}$. Representing $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \in \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}_{\bar{x}}$, we have $\alpha_{\mathcal{M}_{Y^\circ}}(s_{\bar{x}}) = \alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ in $\mathcal{O}_{Y,\bar{x}}$.

Denote by $Y^\circ = (\underline{Y}, \mathcal{M}_{Y^\circ})$. We will also call the induced morphism of logarithmic schemes $\mathrm{p} : Y^\circ \to Y$ a *puncturing* of Y along $\mathcal{P}$, or call Y° a *puncturing* of Y along $\mathcal{P}$.

We say the puncturing is *trivial* if p is an isomorphism.

Remark 2.2. In all examples in this paper, the condition $\alpha_{\mathcal{M}}(m_{\bar{x}}) = 0$ is redundant. Indeed, suppose $\mathcal{P}$ is a $DF(1)$ log structure, i.e., there is a surjective sheaf homomorphism $\underline{\mathbb{N}} \to \overline{\mathcal{P}}$. For $s_{\bar{x}} = (m_{\bar{x}}, p_{\bar{x}}) \notin \mathcal{M}_{\bar{x}} \oplus_{\mathcal{O}^\times} \mathcal{P}$, suppose $\alpha_{\mathcal{M}_{Y^\circ}}(s_{\bar{x}}) = 0$. Note that the $DF(1)$ assumption implies that $p_{\bar{x}}^{-1} \in \mathcal{P}_{\bar{x}}$, so $\alpha_{\mathcal{M}}(m_{\bar{x}}) = \alpha_Y(m_{\bar{x}}, 1) = \alpha_{\mathcal{M}_{Y^\circ}}(s_{\bar{x}} \cdot p_{\bar{x}}^{-1}) = 0$. More generally, the same argument works if $\mathcal{P}$ is valuative.

Remark 2.3. Puncturings $\mathcal{M}^\circ$ of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$ are not unique. In a widely distributed early version of this manuscript as well as in [GS18], we found it instructive to work with a uniquely defined object $\mathcal{M}^{\mathcal{P}}$ we call here the *final puncturing.* It may be defined as the direct limit

$$\mathcal{M}^{\mathcal{P}} := \varinjlim_{\mathcal{M}^\circ \in \Lambda} \mathcal{M}^\circ,$$

over the collection Λ of all puncturings of $\mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}$. This exists in the category of quasi-coherent, not necessarily coherent, logarithmic structures. It has the advantage of being independent of any choice. Its disadvantage, apart from not being finitely generated, is in that its behavior under base change is subtle.

2.1.2. *Pre-stable punctured log structures.* In case a puncturing is equipped with a morphism to another fs log scheme, there is a canonical choice of puncturing. The following proposition follows immediately from the definitions.

Proposition 2.4. *Let X be an fs log scheme, and Y as in Definition 2.1, with a choice of puncturing Y° and a morphism $f : Y^\circ \to X$. Let $\widetilde{Y}^\circ$ denote the puncturing of Y given by the subsheaf of $\mathcal{M}_{Y^\circ}$ generated by $\mathcal{M}_Y$ and $f^\flat(f^*\mathcal{M}_X)$. Then*

(1) We have $\mathcal{M}_{\widetilde{Y}^\circ}$ is a sub-logarithmic structure of $\mathcal{M}_{Y^\circ}$.

(2) There is a factorization

$$\begin{tikzcd} Y^\circ \arrow[rr, "f"] \arrow[dr] & & X. \\ & \widetilde{Y}^\circ \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

(3) Given $Y_1^\circ \to Y_2^\circ \to Y$ with both Y_1°, Y_2° puncturings of Y, then $\widetilde{Y}_1^\circ = \widetilde{Y}_2^\circ$.

Definition 2.5. A morphism $f : Y^\circ \to X$ from a puncturing of a log scheme Y is said to be *pre-stable* if the induced morphism $Y^\circ \to \widetilde{Y}^\circ$ in the above proposition is the identity. In particular, one has $f = \tilde{f}$.

Corollary 2.6. *A morphism $f : Y^\circ \to X$ is pre-stable if and only if the induced morphism of sheaves of monoids $f^*\overline{\mathcal{M}}_X \oplus \overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_{Y^\circ}$ is surjective.*

2.1.3. *Pull-backs of puncturings.*

Proposition 2.7. *Let X and Y be fs log schemes with log structures $\mathcal{M}_X$ and $\mathcal{M}_Y$, and suppose given a morphism $g : X \to Y$. Suppose also given a log structure $\mathcal{P}_Y$ on $\underline{Y}$ and an induced log structure $\mathcal{P}_X := g^*\mathcal{P}_Y$ on $\underline{X}$. Set*

$$X' = (\underline{X}, \mathcal{M}_X \oplus_{\mathcal{O}_X^\times} \mathcal{P}_X), \quad Y' = (\underline{Y}, \mathcal{M}_Y \oplus_{\mathcal{O}_Y^\times} \mathcal{P}_Y).$$

Further, let Y° be a puncturing of Y' along $\mathcal{P}_Y$. Then there is a diagram

$$\begin{tikzcd} X^\circ \arrow[r, "g^\circ"] \arrow[d] & Y^\circ \arrow[d] \\ X' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

with all squares Cartesian in the category of underlying schemes, the lower square Cartesian in the category of fs log schemes, and the top square Cartesian in the category of fine log schemes. Furthermore, X° is a puncturing of X' along $\mathcal{P}_X$.

Proof. We define X° to be the fibre product $X' \times_{Y'} Y^\circ$ in the fine log category. The bottom square is obviously Cartesian in all categories. Thus it is sufficient to show (1) the upper square is Cartesian in the ordinary category, i.e., the underlying map of $X^\circ \to X'$ is the identity and (2) X° is a puncturing of X'.

Note that the fibre product $X' \times_{Y'} Y^\circ$ in the category of log schemes is defined as $\big(\underline{X}, \mathcal{M} := \mathcal{M}_{X'} \oplus_{g^*\mathcal{M}_{Y'}} g^*\mathcal{M}_{Y^\circ}\big)$. This push-out need not, in general, be integral, so we must integralize. Note there is a canonical isomorphism

$$\mathcal{M}^{\mathrm{gp}} = \mathcal{M}^{\mathrm{gp}}_{X'} \oplus_{g^*\mathcal{M}^{\mathrm{gp}}_{Y'}} g^*\mathcal{M}^{\mathrm{gp}}_{Y^\circ} \cong \mathcal{M}^{\mathrm{gp}}_{X'}$$

given by $(s_1, s_2) \mapsto s_1 \cdot (g')^\flat(s_2)$, where $(g')^\flat : g^*\mathcal{M}^{\mathrm{gp}}_{Y'} \to \mathcal{M}^{\mathrm{gp}}_{X'}$ is induced by g'. The integralization $\mathcal{M}^{\mathrm{int}}$ of $\mathcal{M}$ is then the image of $\mathcal{M}$ in $\mathcal{M}^{\mathrm{gp}}$, which thus can be described as the subsheaf of $\mathcal{M}_{X'}^{\mathrm{gp}}$ generated by $\mathcal{M}_{X'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$. Note $\mathcal{M}_{X'}$ and $(g')^\flat(g^*\mathcal{M}_{Y^\circ})$ both lie in $\mathcal{M}_X \oplus_{\mathcal{O}_X^\times} \mathcal{P}_X^{\mathrm{gp}}$, so we can replace $\mathcal{M}^{\mathrm{gp}}$ with this subsheaf of $\mathcal{M}^{\mathrm{gp}}$ in describing $\mathcal{M}^{\mathrm{int}}$.

It is now sufficient to show that we can define a structure map $\alpha : \mathcal{M}^{\mathrm{int}} \to \mathcal{O}_X$ compatible with the structure maps $\alpha_{X'} : \mathcal{M}_{X'} \to \mathcal{O}_X$ and $\alpha_{Y^\circ} : g^*\mathcal{M}_{Y^\circ} \to \mathcal{O}_X$. If $s \in \mathcal{M}^{\mathrm{int}}$ is of the form $s_1 \cdot (g')^\flat(s_2)$ for $s_1 \in \mathcal{M}_{X'}$ and $s_2 \in g^*\mathcal{M}_{Y^\circ}$, then we define $\alpha(s) = \alpha_{X'}(s_1) \cdot \alpha_{Y^\circ}(s_2)$. We need to show this is well-defined. If $s_2 \in g^*\mathcal{M}_{Y'}$, then $(g')^\flat(s_2) \in \mathcal{M}_{X'}$, and thus as g' is a log morphism,

$$\alpha(s) = \alpha_{X'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = \alpha_{X'}(s_1)\alpha_{X'}((g')^\flat(s_2)) = \alpha_{X'}(s).$$

In particular, $\alpha(s)$ only depends on s, and not on the particular representation of s as a product, provided that $s_2 \in g^*\mathcal{M}_{Y'}$.

On the other hand, if $s_2 \in (g^*\mathcal{M}_{Y^\circ}) \setminus (g^*\mathcal{M}_{Y'})$, then $\alpha_{Y^\circ}(s_2) = 0$ by definition of a puncturing. So in this case $\alpha(s) = 0$. Hence to check that α is well-defined, it is enough to show that if $s = s_1 \cdot (g')^\flat(s_2) = s_1' \cdot (g')^\flat(s_2')$ with $s_2 \in g^*\mathcal{M}_{Y'}$ but $s_2' \notin g^*\mathcal{M}_{Y'}$, then $\alpha_{X'}(s_1) \cdot \alpha_{Y^\circ}(s_2) = 0$. Writing $s_i = (m_i, p_i)$, $s_i' = (m_i', p_i')$ using the descriptions $\mathcal{M}_{X'} = \mathcal{M}_X \oplus_{\mathcal{O}_X^\times} \mathcal{P}_X$ and $g^*\mathcal{M}_{Y^\circ} \subset g^*\mathcal{M}_Y \oplus_{\mathcal{O}_X^\times} \mathcal{P}_X^{\mathrm{gp}}$, we note that we must have $m_1 g^\flat(m_2) = m_1' g^\flat(m_2')$. As $s_2' \notin g^*\mathcal{M}_{Y'}$, by condition 2.1(2) we necessarily have $\alpha_Y(m_2') = 0$. Hence $\alpha_X(m_1' g^\flat(m_2')) = 0$, so $\alpha_X(m_1 g^\flat(m_2)) = 0$. We deduce that $\alpha_{X'}(s_1(g')^\flat(s_2)) = 0$, as desired. This shows α is well-defined.

Finally, it is clear from the above description that X° is a puncturing. ♠

Definition 2.8. In the situation of Proposition 2.7, we say that X° is the *pull-back of the puncturing* Y°.

Proposition 2.9. *Consider the situation of Proposition 2.7, and suppose in addition given a pre-stable morphism $f : Y^\circ \to Z$. Then the composition $f \circ g^\circ : X^\circ \to Z$ is also pre-stable.*

Proof. This follows immediately from the definition of pre-stability and the construction of X° in the proof of Proposition 2.7. ♠

2.1.4. *Punctured curves.* Essentially throughout the paper, we will only be interested in puncturing along logarithmic structures from designated marked points of logarithmic curves. Let $\pi : C \to W$ be a logarithmic curve in the sense of [Kat00, Ols07]:

1. (1) The underlying morphism $\underline{\pi}$ is a family of usual prestable curves with disjoint sections $p_1, \ldots, p_k$ of $\underline{\pi}$.
2. (2) π is a proper logarithmically smooth and integral morphism of fine and saturated logarithmic schemes.
3. (3) If $\underline{U} \subset \underline{C}$ is the non-critical locus of $\underline{\pi}$ then $\overline{\mathcal{M}}_C|_{\underline{U}} \cong \underline{\pi}^*\overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^k p_{i*}\mathbb{N}_W$.

We write $\alpha_C : \mathcal{M}_C \to \mathcal{O}_C$ for the structure map of the logarithmic structure on C. We call a geometric point of C *special* if it is either a marked or a nodal point.

Definition 2.10. A *punctured curve* over a fine and saturated logarithmic scheme W is given by the following data:

$$\big(C^\circ \xrightarrow{\mathrm{p}} C \xrightarrow{\pi} W, \mathbf{p} = (p_1, \ldots, p_n)\big) \tag{2.1}$$

where

(1) $C \to W$ is a logarithmic curve with a set of disjoint sections $\{p_1, \ldots, p_n\}$.
(2) $C^\circ \to C$ is a puncturing of C along $\mathcal{P}$, where $\mathcal{P}$ is the divisorial logarithmic structure on $\underline{C}$ induced by the divisor $\bigcup_{i=1}^n p_i(\underline{W})$.

When there is no danger of confusion, we may call $C^\circ \to W$ a punctured curve. Sections in $\mathbf{p}$ are called *punctured points*, or simply *punctures*. We also say C° is a puncturing of C along the punctured points $\mathbf{p}$.

If locally around a punctured point p_i the puncturing is trivial, we say that the punctured point is a *marked point*. In this case, the theory will agree with the treatment of marked points in [Che14],[AC14],[GS13].

Examples 2.11. (1) Let $W = \operatorname{Spec} \Bbbk$ be the point with the trivial logarithmic structure, and C be a non-singular curve over W. Choose a point $p \in C$ and a puncturing $\mathcal{M}_{C^\circ}$ of C at p. Then $\mathcal{M}_{C^\circ} = \mathcal{P}$, as $\mathcal{M}_{C^\circ} \subset \mathcal{P}^{\mathrm{gp}}$ can have no sections s with $\alpha_{C^\circ}(s) = 0$. Thus, in this case the only puncturing $C^\circ \to C$ is the trivial one.

(2) Let $W = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$ be the standard logarithmic point, and C be a non-singular curve over W, so that $\mathcal{M}_W = \mathcal{O}_W^\times \oplus \underline{\mathbb{N}}$, where $\underline{\mathbb{N}}$ denotes the constant sheaf on C with stalk $\mathbb{N}$. Again choose a puncture $p \in C$. Let $\mathcal{M}_{C^\circ} \subset \pi^*\mathcal{M}_W \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$ be a puncturing. Let s be a local section of $\mathcal{M}_{C^\circ}$ near p. Write $s = \big((\varphi, n), t^m\big)$ with $\varphi \in \mathcal{O}_C^\times$, $n \in \mathbb{N}$. If $m < 0$, then Condition (2) of Definition 2.1 implies that $\alpha_{\pi^*(\mathcal{M}_W)}(\varphi, n) = 0$, so we must have $n > 0$. Thus we see that

$$\overline{\mathcal{M}}_{C^\circ, p} \subset \{(n, m) \in \mathbb{N} \oplus \mathbb{Z} \,|\, m \geq 0 \text{ if } n = 0\}.$$

Conversely, any fine submonoid of the right-hand-side of the above inclusion which contains $\mathbb{N} \oplus \mathbb{N}$ can be realised as the stalk of the ghost sheaf at p for a puncturing. Note the monoid on the right-hand side is not finitely generated, and is the stalk of the ghost sheaf of the final puncturing, see Remark 2.3.

(3) Let $\underline{W} = \operatorname{Spec} \Bbbk[\epsilon]/(\epsilon^{k+1})$, and let W be given by the chart $\mathbb{N} \to \Bbbk[\epsilon]/(\epsilon^{k+1})$, $1 \mapsto \epsilon$. Let C_0 be a non-singular curve over $\operatorname{Spec} \Bbbk$ with the trivial logarithmic structure, and let $C = W \times C_0$. Choose a section $p : W \to C$, with image locally defined by an equation $t = 0$. Again Condition (2) of Definition 2.1 implies that a section s of a puncturing $\mathcal{M}_{C^\circ}$ near p takes the form $\big((\varphi, n), t^m\big)$ where $\varphi \in \mathcal{O}_C^\times$, and $0 \leq n \leq k$ implies $m \geq 0$. In particular,

$$\overline{\mathcal{M}}_{C^\circ, p} \subset \{(n, m) \in \mathbb{N} \oplus \mathbb{Z} \,|\, m \geq 0 \text{ if } n \leq k\},$$

and any fine submonoid of the right-hand side containing $\mathbb{N} \oplus \mathbb{N}$ can be realised as the stalk of the ghost sheaf at p of a puncturing.

2.1.5. *Pull-backs of punctured curves.* Consider a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism of fine and saturated logarithmic schemes $h : T \to W$. Denote by $(C_T \to T, \mathbf{p}_T)$ the pull-back of the log curve $C \to W$ via $T \to W$. By Proposition 2.7, we obtain a commutative diagram

$$\begin{tikzcd} C_T^\circ \arrow[r] \arrow[d, "\mathrm{p}_T"'] & C^\circ \arrow[d, "\mathrm{p}"] \\ C_T \arrow[r] \arrow[d, "\pi_T"'] & C \arrow[d, "\pi"] \\ T \arrow[r, "h"] & W \end{tikzcd}$$

where the bottom square is cartesian in the fine and saturated category, and the square on the top is cartesian in the fine category, and such that C_T° is a puncturing of the curve C_T along p_T.

Definition 2.12. We call $C_T^\circ \to T$ the *pull-back* of the punctured curve $C^\circ \to W$ along $T \to W$.

2.1.6. *Punctured maps.* We now fix a morphism of fine and saturated logarithmic schemes $X \to B$.

Definition 2.13. A *punctured map to a family* $X \to B$ over a fine and saturated logarithmic scheme W consists of a punctured curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism f fitting into a commutative diagram

$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ W \arrow[r] & B \end{tikzcd}$$

Such a punctured map is denoted by $(C^\circ \to W, \mathbf{p}, f)$.

The *pull-back* of a punctured map $(C^\circ \to W, \mathbf{p}, f)$ along a morphism of fine and saturated logarithmic schemes $T \to W$ is the punctured map $(C_T^\circ \to T, \mathbf{p}_T, f_T)$ consisting of the pull-back $C_T^\circ \to T$ of the punctured curve $C \to W$ and the pull-back f_T of f.

When there is no danger of confusion, we may write $f : C^\circ \to X/B$ or $f : C^\circ \to X$ for the punctured map.

Definition 2.14. A punctured map $(C^\circ \to W, \mathbf{p}, f)$ is called *pre-stable* if $f : C^\circ \to X$ is pre-stable in the sense of Definition 2.5.

A pre-stable punctured map is called *stable* if its underlying map is stable in the usual sense.

Proposition 2.15. *Let $(C^\circ \to W, \mathbf{p}, f)$ be a punctured map over W.*

(1) The locus of points of W with pre-stable fibers forms an open sub-scheme of $\underline{W}$.

(2) If $f : C^\circ \to X$ is pre-stable, then the pull-back $f_T : C^\circ \to X$ along any morphism of fine and saturated logarithmic schemes $T \to W$ is also pre-stable.

Proof. The punctured map $f : C^\circ \to X$ induces a morphism of fine logarithmic structures

$$f^\flat \oplus \mathrm{p}^\flat : f^*\mathcal{M}_X \oplus_{\mathcal{O}_C^\times} \mathcal{M}_C \to \mathcal{M}_{C^\circ}.$$

The pre-stability of f is equivalent to the condition that $f^\flat \oplus \mathrm{p}^\flat$ is surjective by Corollary 2.6. Statement (1) can be proved by applying Lemma 2.16 below to the neighborhood of each puncture. Statement (2) follows immediately from Proposition 2.9. ♠

Lemma 2.16. *Let $\underline{Y}$ be a scheme, and $\psi^\flat : \mathcal{M} \to \mathcal{N}$ be a morphism of fine log structures on $\underline{Y}$. Assume that $\overline{\mathcal{M}}^{\mathrm{gp}}$ and $\overline{\mathcal{N}}^{\mathrm{gp}}$ are torsion-free. Then the locus $\underline{Y}' \subset \underline{Y}$ over which $\psi^\flat$ is surjective forms an open subscheme of $\underline{Y}$.*

Proof. Note that the surjectivity of $\psi^\flat$ can be checked on the level of ghost sheaves. Since the statement is local on $\underline{Y}$, shrinking $\underline{Y}$, we may assume that there are global charts $\phi_{\mathcal{M}} : \overline{\mathcal{M}}_y \to \mathcal{M}$ and $\phi_{\mathcal{N}} : \overline{\mathcal{N}}_y \to \mathcal{N}$ for some point $y \in \underline{Y}$; indeed as $\overline{\mathcal{M}}_y$ and $\overline{\mathcal{N}}_y$ are torsion free, [Ogu18, II, Proposition 2.3.7] applies. Consider another point $t \in \underline{Y}$ specializing to y. Denote by

$$E = \{e \in \overline{\mathcal{M}}_y \mid \alpha_{\mathcal{M}} \circ \phi_{\mathcal{M}}(e)|_t \in \mathcal{O}^*_{\underline{Y},t}\} \quad \text{and} \quad F = \{e \in \overline{\mathcal{N}}_y \mid \alpha_{\mathcal{N}} \circ \phi_{\mathcal{N}}(e)|_t \in \mathcal{O}^*_{\underline{Y},t}\}.$$

Denote by $E^{-1}\overline{\mathcal{M}}_y \subset \overline{\mathcal{M}}_y^{\mathrm{gp}}$ (respectively $E^{-1}\overline{\mathcal{N}}_y \subset \overline{\mathcal{N}}_y^{\mathrm{gp}}$) the submonoid generated by E^{gp} and $\overline{\mathcal{M}}_y$ (respectively F^{gp} and $\overline{\mathcal{N}}_y$). We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E^{\mathrm{gp}} \arrow[r] \arrow[d] & E^{-1}\overline{\mathcal{M}}_y \arrow[r] \arrow[d, "(\overline{\psi'})^\flat_y"] & \overline{\mathcal{M}}_t \arrow[r] \arrow[d, "\overline{\psi}^\flat_t"] & 0 \\
0 \arrow[r] & F^{\mathrm{gp}} \arrow[r] & F^{-1}\overline{\mathcal{N}}_y \arrow[r] & \overline{\mathcal{N}}_t \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are induced by $\psi^\flat$, and the two horizontal sequences are exact by [Ols03, Lemma 3.5(i)]. The surjectivity of $\psi^\flat$ at y implies the surjectivity of $\overline{\psi}^\flat_y$. It follows that $E^{\mathrm{gp}} \to F^{\mathrm{gp}}$ is surjective, hence $(\overline{\psi'})^\flat_y$ is surjective, and so is $\overline{\psi}^\flat_t$. This proves the statement. ♠

Example 2.17. The intuition behind punctured curves is that it allows points with negative orders of tangency to divisors. To see this explicitly, let $\underline{X}$ be a surface, $\underline{D} \subseteq \underline{X}$ a non-singular rational curve with self-intersection -1 inducing the divisorial log structure X on $\underline{X}$. Let $C \to W$ be the punctured curve of Examples 2.11, (2), with $C \cong \mathbb{P}^1$. Let $\underline{f} : \underline{C} \to \underline{X}$ be an isomorphism of $\underline{C}$ with $\underline{D}$. This can be enhanced to a punctured map $C^\circ \to X$ as follows.

We first define $\bar{f}^\flat : \underline{f}^*\overline{\mathcal{M}}_X = \underline{\mathbb{N}} \to \overline{\mathcal{M}}_{C^\circ} \subseteq \overline{\mathcal{E}} = \underline{\mathbb{N}} \oplus \mathbb{Z}_p$ by $1 \mapsto (1, -1)$, where $\mathbb{Z}_p$ denotes the sky-scraper sheaf at p with stalk $\mathbb{Z}$. Note that $1 \in \Gamma(X, \overline{\mathcal{M}}_X)$ yields the $\mathcal{O}_X^\times$-torsor contained in $\mathcal{M}_X$ corresponding to the line bundle $\mathcal{O}_X(-D)$, and thus $1 \in \Gamma(C, \underline{f}^*\overline{\mathcal{M}}_X)$ yields the $\mathcal{O}_C^\times$-torsor corresponding to $\mathcal{O}_C(1)$, using $-D^2 = 1$. On the other hand, note that the torsor contained in $\mathcal{M}_{C^\circ}$ corresponding to $(1,0)$ is the torsor of $\mathcal{O}_C$ and the torsor corresponding to $(0,1)$ is the torsor of the ideal $\mathcal{O}_C(-p)$. Hence $(1,-1) \in \Gamma(C, \overline{\mathcal{M}}_{C^\circ})$ corresponds to $\mathcal{O}_C(1)$. Choosing an isomorphism of torsors then lifts the map $\bar{f}^\flat$ to a map $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ inducing a morphism $f : C^\circ \to X$.

Note this morphism does not lift to $C' \to W' = \operatorname{Spec}(\Bbbk[\epsilon]/(\epsilon^2))$ as in Examples 2.11, (3), since we can't even lift $\bar{f}^\flat$ at the level of ghost sheaves. Indeed, $(1,-1)$ is not a section of the ghost sheaf of $(C')^\circ$.

2.2. **Combinatorial Types.** Now assume the target $X \to B$ has the logarithmic structure $\mathcal{M}_X$ defined in the Zariski site, and $B = \operatorname{Spec} \Bbbk$ with the trivial logarithmic structure. The combinatorial structure of punctured maps is similar to the case of logarithmic maps in [GS13, AC14, Che14] except at the punctured points. We explain the combinatorial structure below.

2.2.1. *Induced maps of monoids.* Suppose given a punctured map $(\pi : C^\circ \to W, \mathbf{p}, f : C^\circ \to X)$ over W. We write $\mathcal{M} := \underline{f}^*\mathcal{M}_X$. Taking the corresponding morphisms of sheaves of monoids, we have

$$(\overline{\mathcal{M}}_W, \overline{\mathcal{M}}_C, \psi : \underline{\pi}^*\overline{\mathcal{M}}_W \to \overline{\mathcal{M}}_C, \varphi : \overline{\mathcal{M}} \to \overline{\mathcal{E}}) \tag{2.2}$$

where $\psi = \bar{\pi}^\flat$ and φ is given by the composition $\overline{\mathcal{M}} \to \overline{\mathcal{M}}_{C^\circ} \subset \overline{\mathcal{E}} := \overline{\mathcal{M}}_C \oplus_{\overline{\mathcal{P}}} \overline{\mathcal{P}}^{\mathrm{gp}}$.

2.2.2. *The ghost sheaf category* $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$. Just as in [GS13], we may focus on the combinatorial structure, and define the category $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ as follows.

Let $\underline{C} \to \underline{W}$ be a family of underlying pre-stable curves with markings $\mathbf{p}$, and let $\underline{f} : \underline{C} \to \underline{X}$ be a morphism, hence $\mathcal{M} := \underline{f}^*\mathcal{M}_X$. An object of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ is abstractly a collection of data (2.2) such that

(1) The data $(\overline{\mathcal{M}}_W, \overline{\mathcal{M}}_C, \psi : \underline{\pi}^*\overline{\mathcal{M}}_W \to \overline{\mathcal{M}}_C)$ come from a log curve $C \to W$.
(2) The pair of morphisms (ψ, φ) satisfies the descriptions in Sections 2.2.3 and 2.2.4 below over each geometric fiber.

A morphism of objects in $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$

$$(\overline{\mathcal{M}}_{W,1}, \overline{\mathcal{M}}_{C,1}, \psi_1, \varphi_1) \to (\overline{\mathcal{M}}_{W,2}, \overline{\mathcal{M}}_{C,2}, \psi_2, \varphi_2)$$

is given by a pair of local homomorphisms[1] $\overline{\mathcal{M}}_{W,1} \to \overline{\mathcal{M}}_{W,2}$ and $\overline{\mathcal{M}}_{C,1} \to \overline{\mathcal{M}}_{C,2}$ with the obvious compatibilities with ψ_i and φ_i, $i = 1,2$.

Note that the descriptions in Sections 2.2.3 and 2.2.4 below are only necessary conditions for an object (2.2) to be induced from a punctured map. The category $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ is, roughly speaking, the collection of objects of the form (2.2) which satisfy these necessary conditions. Similarly as in [GS13], Discussion 1.8, these descriptions determine an object of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$.

We next describe the pair (ψ, φ) over geometric fibers, and assume $W = \operatorname{Spec}(Q \to \Bbbk)$.

[1] A homomorphism of monoids $\varphi : P \to Q$ is *local* if $\varphi^{-1}(Q^\times) = P^\times$.

2.2.3. *The structure of ψ.* The morphism ψ is an isomorphism when restricted to the complement of the special (nodal or punctured) points of C°. The sheaf $\overline{\mathcal{E}}$ has stalks $Q \oplus \mathbb{Z}$ and $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ at punctured points and nodal points respectively. The fibred sum in the nodal point case is determined by a map

$$\mathbb{N} \to Q, \quad 1 \mapsto \rho_q \tag{2.3}$$

and the diagonal map $\mathbb{N} \to \mathbb{N}^2$ (see Def. 1.5 of [GS13]). The map ψ at these special points is given by the inclusions of Q into the first components of the direct sums $Q \oplus \mathbb{Z}$ and $Q \oplus_{\mathbb{N}} \mathbb{N}^2$.

2.2.4. *The structure of φ.* For any point $y \in C^\circ$ and its algebraic closure $\bar{y} \to y$, the morphism φ induces a well-defined morphism $\varphi_{\bar{y}} : P_y \to \overline{\mathcal{M}}_{C^\circ, \bar{y}} \subset \mathcal{E}_{\bar{y}}$ for

$$P_y := \overline{\mathcal{M}}_y$$

as the logarithmic structure $\mathcal{M}$ is Zariski. Away from the punctured points, the description of φ is identical to the case of stable logarithmic maps. Following Discussion 1.8 of [GS13], we have the following behavior at points on C°:

(i) $y = \eta$ is a generic point, giving a local homomorphism of monoids

$$\varphi_{\bar{\eta}} : P_\eta \longrightarrow Q.$$

(ii) $y = p$ is a punctured point. We have u_p the composition

$$u_p : P_p \xrightarrow{\varphi_{\bar{p}}} Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}. \tag{2.4}$$

The element $u_p \in P_p^*$ is called the *contact order* at the puncture p.

(iii) $y = q$ is a node contained in the closures of η_1, η_2. If $\chi_i : P_q \to P_{\eta_i}$ are the generization maps, there exists a homomorphism

$$u_q : P_q \to \mathbb{Z},$$

called the *contact order at q*, such that

$$\varphi_{\bar{\eta}_2}\big(\chi_2(m)\big) - \varphi_{\bar{\eta}_1}\big(\chi_1(m)\big) = u_q(m) \cdot \rho_q, \tag{2.5}$$

with $\rho_q \neq 0$ given in Equation (2.3), see [GS13], (1.8). These data completely determines the local homomorphism $\varphi_{\bar{q}} : P_q \to Q \oplus_{\mathbb{N}} \mathbb{N}^2$.

The choice of ordering η_1, η_2 for the branches of C containing a node is called an *orientation* of the node. We note that reversing the orientation of a node q (by interchanging η_1 and η_2) results in reversing the sign of u_q.

Remark 2.18. If $u_p \in P_p^\vee$, i.e., takes values only in $\mathbb{N} \subset \mathbb{Z}$, then a punctured point behaves precisely like marked points as previously considered in [Che14, AC14, GS13]. Indeed, in this case pre-stability implies that $\mathcal{M}_C$ and $\mathcal{M}_{C^\circ}$ agree along p. Thus there is no need to distinguish between punctured points and marked points previously considered in the above references. However, we will use the convention that a punctured point with contact order $u_p \in P_p^\vee$ is called a *marked point*.

Remark 2.19. Let $f : C^\circ/W \to X$ be a punctured map with $W = \operatorname{Spec}(Q \to \Bbbk)$. Suppose $p \in C$ is a punctured point which is not a marked point, and let C' be the irreducible component containing p, with generic point η. Then, intuitively, C' has negative order of tangency with certain strata in X, and this forces C' to be contained in those strata.

Explicitly, if $\delta \in P_p$ with $u_p(\delta) < 0$, then we must have $\mathrm{pr}_1 \circ \varphi_p(\delta) \neq 0$, as there is no element of $\overline{\mathcal{M}}_{C^\circ,p} \subset Q \oplus \mathbb{Z}$ of the form $(0, n)$ with $n < 0$. Thus if $\chi : P_p \to P_\eta$ denotes the generization map, we must have $u_p^{-1}(\mathbb{Z}_{<0}) \cap \chi^{-1}(0) = \emptyset$. This restricts the strata in which $f(C')$ can lie.

For example, if $X = (\underline{X}, D)$ for a simple normal crossings divisor D with irreducible components $D_1, \ldots, D_n$, then $P_p = \bigoplus_{i:f(p)\in D_i} \mathbb{N}$. The value u_p on the generator of P_p corresponding to D_i is the contact order with D_i. Then $f(C')$ must lie in the intersection of D_i that have negative contact order at p.

2.2.5. *Dual graphs and combinatorial types.* To describe the combinatorial structure of nodal curves and their maps, a graph G will consist of a set of vertices $V(G)$, a set of edges $E(G)$ and a separate set of *legs* or *half-edges* $L(G)$, with appropriate incidence relations between vertices and edges, and between vertices and half-edges.

Let $G_{\underline{C}}$ be the dual intersection graph of the underlying curve $\underline{C}$. This is the graph which has a vertex v_η for each generic point η of $\underline{C}$, an edge E_q joining v_{η_1}, v_{η_2} for each node q contained in the closure of η_1 and η_2, and where E_q is a loop if q is a double point in an irreducible component of $\underline{C}$. Note that an ordering of the two branches of $\underline{C}$ at a node gives rise to an orientation on the corresponding edge. Finally, $G_{\underline{C}}$ has a leg L_y with endpoint v_η for each punctured point y contained in the closure of η.

Definition 2.20. Given an object in $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ of the form (2.2) over a geometric point $\underline{W}$, its *combinatorial type* is a pair $(G_{\underline{C}}, \mathbf{u} = \{u_p\} \cup \{u_q\})$ such that

(1) $G_{\underline{C}}$ is the dual intersection graph.
(2) $u_p \in P_p^*$ is the contact order corresponding to each punctured point in $\mathbf{p}$.
(3) $u_q \in P_p^*$ is the contact order corresponding to each oriented node of $\underline{C}$.

Given a combinatorial type $(G_{\underline{C}}, \mathbf{u})$, denote by $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$ the full subcategory of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ with objects of type $(G_{\underline{C}}, \mathbf{u})$. Note that the dual intersection graph $G_{\underline{C}}$ is determined by the underlying curve $\underline{C}$.

The *combinatorial type* of a punctured map over a logarithmic point is the combinatorial type of its associated object in $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$.

2.2.6. *Generization of combinatorial types.* We consider a punctured map $(C^\circ \to W, \mathbf{p}, f)$ over an arbitrary fine and saturated logarithmic scheme W.

Lemma 2.21. *Let $(G, \mathbf{u})$ and $(G', \mathbf{u}')$ be the combinatorial types of the punctured map at two geometric points $\bar{w} \to \underline{W}$, $\bar{w}' \to \underline{W}$ with $\bar{w} \in \mathrm{cl}(\bar{w}')$. For $y \in \underline{C}_{\bar{w}}$, $y' \in \underline{C}_{\bar{w}'}$ with $y \in \mathrm{cl}(y')$, let $\chi_{y',y} : P_y \to P_{y'}$ be the generization map of the stalks* *of* $\underline{f}^*\overline{\mathcal{M}}_X$*. Then if* y, y' *are punctured or nodal, we have*

$$u_y = u_{y'} \circ \chi_{y',y}.$$

Proof. The proof is exactly as in [GS13], Lemma 1.11, with punctures being treated like marked points in that proof. ♠

2.3. **Basicness.**

2.3.1. *Construction of the monoid.* We follow Construction 1.16 of [GS13]. Suppose given $\underline{C} \to \underline{W}$ with $\underline{W}$ a geometric point. Let $(G, \mathbf{u})$ be a combinatorial type for $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ and assume $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$ is non-empty.

Consider the following monoid

$$N := \prod_\eta P_\eta \times \prod_q \mathbb{N} \tag{2.6}$$

where η runs through all the generic points of irreducible components, and q runs through the nodes of $\underline{C}$. For a node $q \in \underline{C}$ and two generic points η_1, η_2 corresponding to the two branches meeting at q, denote by $\chi_{\eta_i,q} : P_q \to P_{\eta_i}$ the two generization maps. For each $m \in P_q$, let

$$a_q(m) := \big((\ldots, \chi_{\eta_1,q}(m), \ldots, -\chi_{\eta_2,q}(m), \ldots), (\ldots, u_q(m), \ldots)\big) \in N^{\mathrm{gp}}$$

be the element with all vanishing entries except the indicated ones at places η_1, η_2 and q. Let $R \subset N^{\mathrm{gp}}$ be the subgroup generated by $a_q(m)$ for all nodes $q \in \underline{C}$ and $m \in P_q$, and R^{sat} be its saturation in N^{gp}. The natural map

$$N^{\mathrm{gp}}\big/R \to N^{\mathrm{gp}}\big/R^{\mathrm{sat}}$$

is the quotient by the torsion subgroup of $N^{\mathrm{gp}}\big/R$. Hence $N^{\mathrm{gp}}\big/R^{\mathrm{sat}}$ is torsion free.

Denote by $N\big/R^{\mathrm{sat}}$ the image monoid of the following composition

$$N \hookrightarrow N^{\mathrm{gp}} \to N^{\mathrm{gp}}\big/R^{\mathrm{sat}}.$$

Define the *basic monoid* Q to be the saturation of $N\big/R^{\mathrm{sat}}$ in $N^{\mathrm{gp}}\big/R^{\mathrm{sat}}$. By definition, the monoid Q is fine and saturated.

The inclusions of the various factors define homomorphisms

$$\begin{aligned} \varphi_{\bar\eta} : P_\eta \to \prod_\eta P_\eta \times \prod_q \mathbb{N} \to Q, \\ \mathbb{N} \to \prod_\eta P_\eta \times \prod_q \mathbb{N} \to Q, \qquad 1 \mapsto \rho_q, \end{aligned} \tag{2.7}$$

The element $a_q(m)$ is precisely the difference of the two sides of (2.5), so (2.5) holds for these choices of $\varphi_{\bar\eta}$ and ρ_q with the given u_q. Thus the data Q, ρ_q, and $\varphi_{\bar\eta}$ define a distinguished *basic object* $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ of $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u})$, except that we don't know that $Q^\times = 0$, so that all relevant morphisms are local.

2.3.2. *Basic families.*

Proposition 2.22. *If* $\underline{\mathrm{GS}}(\overline{\mathcal{M}}, \mathbf{u}) \neq \emptyset$*, then it has as an initial object the basic object* $(Q, \overline{\mathcal{M}}_C, \psi, \varphi)$ *from Section 2.3.1.*

Proof. This is identical to [GS13], Proposition 1.19. ♠

Definition 2.23. A pre-stable punctured map $(C/W, \mathbf{p}, f)$ is called *basic* if for any geometric point $\bar{w} \to \underline{W}$ the induced object of $\underline{\mathrm{GS}}(\underline{f}^*_{\bar{w}}\overline{\mathcal{M}}_X, \mathbf{u})$ is initial, i.e. the basic object. Here $\mathbf{u}$ is given by the combinatorial type of $(C/W, \mathbf{p}, f)$ at $\bar{w}$.

Proposition 2.24. *Let* $(C/W, \mathbf{p}, f)$ *be a pre-stable punctured map. Then*

$$\Omega := \{\bar{w} \in |\underline{W}| \,\big|\, \{\bar{w}\} \times_{\underline{W}} (C/W, \mathbf{p}, f) \text{ is basic}\}$$

is an open subset of $|\underline{W}|$*.*

Proof. This is identical to [GS13], Proposition 1.22. ♠

Proposition 2.25. *Any pre-stable punctured map to the target* X *arises as the pull-back from a basic pre-stable punctured map with the same underlying ordinary pre-stable map. Both the basic pre-stable punctured map and the morphism are unique up to a unique isomorphism.*

Proof. The proof is similar to [GS13], Proposition 1.24, however some care must be taken at the punctures. Let $(C \to W, \mathbf{p}, f)$ be a pre-stable punctured map. For each geometric point $\bar{w} \in W$, one obtains a combinatorial type $(G_{\bar{w}}, \mathbf{u}_{\bar{w}})$ by restriction, and these types are compatible under generization by Lemma 2.21. Following the argument of Lemma 1.23 of [GS13], one has an initial object of the full subcategory $\underline{\mathrm{GS}}\big(\overline{\mathcal{M}}, (\mathbf{u}_{\bar{w}})\big)$ of objects of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ that have type $\mathbf{u}_{\bar{w}}$ over the geometric point $\bar{w}$. Write this universal object as $(\overline{\mathcal{M}}_{W,\mathrm{bas}}, \overline{\mathcal{M}}_{C,\mathrm{bas}}, \psi_{\mathrm{bas}}, \varphi_{\mathrm{bas}})$.

On the other hand, write the object of $\underline{\mathrm{GS}}(\overline{\mathcal{M}})$ determined by the given pre-stable punctured map as $(\overline{\mathcal{M}}_W, \overline{\mathcal{M}}_C, \psi, \varphi)$. Recall the notation $\varphi : \overline{\mathcal{M}} \to \overline{\mathcal{E}}$ and $\varphi_{\mathrm{bas}} : \overline{\mathcal{M}} \to \overline{\mathcal{E}}_{\mathrm{bas}}$ from (2.2). Furthermore, as we have a map $\overline{\mathcal{E}}_{\mathrm{bas}} \to \overline{\mathcal{E}}$, the basic object being the initial object in the category, we then define $\overline{\mathcal{M}}_{C^\circ,\mathrm{bas}} \subset \overline{\mathcal{E}}_{\mathrm{bas}}$ to be the fine sub-sheaf generated by the image of $\overline{\mathcal{M}}_{C,\mathrm{bas}} \oplus \overline{\mathcal{M}} \to \overline{\mathcal{E}}_{\mathrm{bas}}$.

We observe that the composition $\overline{\mathcal{M}}_{C^\circ,\mathrm{bas}} \hookrightarrow \overline{\mathcal{E}}_{\mathrm{bas}} \to \overline{\mathcal{E}}$ factors through $\overline{\mathcal{M}}_{C^\circ}$. Indeed, the composition $\overline{\mathcal{M}}_{C,\mathrm{bas}} \to \overline{\mathcal{E}}_{\mathrm{bas}} \to \overline{\mathcal{E}}$ factors through $\overline{\mathcal{M}}_C \subset \overline{\mathcal{M}}_{C^\circ}$, and the composition $\overline{\mathcal{M}} \to \overline{\mathcal{E}}_{\mathrm{bas}} \to \overline{\mathcal{E}}$ factors through $\overline{\mathcal{M}} \to \overline{\mathcal{M}}_{C^\circ} \to \overline{\mathcal{E}}$.

As in the proof of [GS13], Proposition 1.24, we can now define

$$\begin{aligned}
\mathcal{M}_{W,\mathrm{bas}} &= \mathcal{M}_W \times_{\overline{\mathcal{M}}_W} \overline{\mathcal{M}}_{W,\mathrm{bas}}, \\
\mathcal{M}_{C,\mathrm{bas}} &= \mathcal{M}_C \times_{\overline{\mathcal{M}}_C} \overline{\mathcal{M}}_{C,\mathrm{bas}}, \\
\mathcal{M}_{C^\circ,\mathrm{bas}} &= \mathcal{M}_{C^\circ} \times_{\overline{\mathcal{M}}_{C^\circ}} \overline{\mathcal{M}}_{C^\circ,\mathrm{bas}}.
\end{aligned}$$

Each of these is a log structure with the structure map being the composition of the projection to the first factor followed by the structure map for that log structure.

The inclusion $\mathcal{M}_C \to \mathcal{M}_{C^\circ}$ induces an inclusion of log structures $\mathcal{M}_{C,\mathrm{bas}} \to \mathcal{M}_{C^\circ,\mathrm{bas}}$. Furthermore for any local section s of $\mathcal{M}_{C^\circ,\mathrm{bas}}$, if $s \notin \mathcal{M}_{C,\mathrm{bas}}$ then the image of s via $\mathcal{M}_{C^\circ,\mathrm{bas}} \to \mathcal{M}_{C^\circ}$ is not contained in $\mathcal{M}_C$, hence $\alpha_{\mathcal{M}_{C^\circ,\mathrm{bas}}}(s) = 0 \in \mathcal{O}_{\underline{C}}$. Thus, using Remark 2.2, we have a puncturing $C^\circ_{\mathrm{bas}} = (\underline{C}, \mathcal{M}_{C^\circ,\mathrm{bas}}) \to (\underline{C}, \mathcal{M}_{C,\mathrm{bas}})$ along the sections of $\mathbf{p}$. As in the proof of [GS13], Proposition 1.24, this now allows us to define a basic punctured map $f_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} \to X$ over $W_{\mathrm{bas}} = (\underline{W}, \mathcal{M}_{W,\mathrm{bas}})$. Since $\overline{\mathcal{M}}_{C^\circ,\mathrm{bas}}$ is generated by the image of $\overline{\mathcal{M}}_{C,\mathrm{bas}} \oplus \overline{\mathcal{M}} \to \overline{\mathcal{E}}_{\mathrm{bas}}$, the map f_{bas} is pre-stable.

Denote by $f_{\mathrm{bas},W} : C^\circ_{\mathrm{bas},W} \to X$ the pull-back of the punctured map f_{bas} via $W \to W_{\mathrm{bas}}$. Since f_{bas} is pre-stable, $f_{\mathrm{bas},W}$ is also pre-stable. Observe that the morphism $\mathcal{M} \to \mathcal{M}_{C^\circ}$ factors through $\mathcal{M}_{C^\circ_{\mathrm{bas}}} \to \mathcal{M}_{C^\circ}$, hence the pre-stable punctured map $f : C^\circ \to Y$ factors through $f_{\mathrm{bas},W}$. By Proposition 2.4, the two punctured maps f and $f_{\mathrm{bas},W}$ are isomorphic. Thus, f is the pull-back of the basic map f_{bas}. ♠

Proposition 2.26. *An automorphism $\varphi : C^\circ/W \to C^\circ/W$ of a basic pre-stable punctured map $(C^\circ/W, \mathbf{p}, f)$ with $\underline{\varphi} = \mathrm{id}_{\underline{C}^\circ}$ is trivial.*

Proof. This is identical to [GS13], Proposition 1.25. ♠

2.4. **Family of targets.** More generally, we consider a relative target $X \to B$ with $\mathcal{M}_X$ defined in the Zariski site.

Definition 2.27. A pre-stable punctured map to the family $X \to B$ is called *basic* if the induced pre-stable punctured map to the target X is basic.

Proposition 2.28. *Any pre-stable punctured map to the family $X \to B$ arises as the pull-back from a basic pre-stable punctured map to $X \to B$ with the same underlying ordinary pre-stable map. Both the basic pre-stable punctured map and the morphism are unique up to a unique isomorphism.*

Proof. Consider a pre-stable punctured map $(C^\circ \to W, \mathbf{p}, f)$ to the family $X \to B$. Forgetting the morphism to B, denote by $f_{\mathrm{bas}} : C^\circ_{\mathrm{bas}} \to X$ the corresponding basic punctured map over W_{bas} as in Proposition 2.25. We have a canonical commutative diagram of solid arrows as follows:

$$\begin{tikzcd}
C^\circ \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & C^\circ_{\mathrm{bas}} \arrow[r, "f_{\mathrm{bas}}"'] \arrow[d] & X \arrow[d] \\
W \arrow[r] \arrow[rr, bend right, "h"'] & W_{\mathrm{bas}} \arrow[r, dashed, "h_{\mathrm{bas}}"] & B.
\end{tikzcd}$$

We will show that there is a canonical dashed arrow h_{bas} making the above diagram commutative, hence the desired basic punctured map to $X \to B$.

Since the underlying morphism $\underline{W} \to \underline{W}_{\mathrm{bas}}$ is an isomorphism, pulling back to $\underline{W}$ it suffices to show that the morphism $h^*\mathcal{M}_B \to \mathcal{M}_W$ factors through $\mathcal{M}_{W_{\mathrm{bas}}}$ canonically.

Pulling back to $\underline{C}$, we observe that the composition $\overline{\mathcal{M}}_B|_{\underline{C}} \to \overline{\mathcal{M}}_X|_{\underline{C}} \to \overline{\mathcal{M}}_{C^\circ}$ factors through $\overline{\mathcal{M}}_W|_{\underline{C}}$. Thus, the contact order of any elements in $\overline{\mathcal{M}}_B$ is trivial at all nodes and punctures. Consequently, the composition $\overline{\mathcal{M}}_B|_{\underline{C}} \to \overline{\mathcal{M}}_X|_{\underline{C}} \to \overline{\mathcal{M}}_{C^\circ_{\mathrm{bas}}}$ factors through $\overline{\mathcal{M}}_{W_{\mathrm{bas}}}|_{\underline{C}}$. Since ghost sheaves are constructible, and the fiber of $\underline{C} \to \underline{W}$ is connected, the morphism $\overline{\mathcal{M}}_B|_{\underline{C}} \to \overline{\mathcal{M}}_{W_{\mathrm{bas}}}|_{\underline{C}}$ descends to a morphism $\bar{h}^\flat_{\mathrm{bas}} : \overline{\mathcal{M}}_B|_{\underline{W}} \to \overline{\mathcal{M}}_{W_{\mathrm{bas}}}$.

To see that $\bar{h}^\flat_{\mathrm{bas}}$ lifts to a morphism of logarithmic structures, notice that any global section δ of $\mathcal{M}_B$, viewed as a global section of $\mathcal{M}_B|_{\underline{C}}$, maps to a global section $f^\flat_{\mathrm{bas}}(\delta)$ of $\mathcal{M}_{W_{\mathrm{bas}}}|_{\underline{C}} \subset \mathcal{M}_{C^\circ_{\mathrm{bas}}}$. As the family $\underline{C} \to \underline{W}$ is proper, $f^\flat_{\mathrm{bas}}(\delta)$ is constant over each fiber of $\underline{C} \to \underline{W}$, hence descends to a section $h^\flat_{\mathrm{bas}}(\delta)$ of $\mathcal{M}_{W_{\mathrm{bas}}}$. This defines the desired morphism $h^\flat_{\mathrm{bas}}$. ♠

2.5. **Puncturing log-ideals.** The punctured points which are not marked points impose extra important constraints we now describe. This is a key new feature of the theory. Given a monoid Q and a punctured log map over $W = \mathrm{Spec}(Q \to \Bbbk)$ with puncture p contained in a component with generic point η, consider the commutative diagram

$$\begin{array}{ccc} P_p & \xrightarrow{\varphi_p} & Q \oplus \mathbb{Z} \\ {\scriptstyle\chi_{\eta,p}}\downarrow & & \downarrow{\scriptstyle\chi'_{\eta,p}} \\ P_\eta & \xrightarrow{\varphi_{\bar\eta}} & Q \end{array} \tag{2.8}$$

where vertical arrows are generization maps. The morphism $\chi'_{\eta,p}$ is the projection to the first factor. Then φ_p is given by

$$m \mapsto \varphi_p(m) = \big(\varphi_{\bar\eta} \circ \chi_{\eta,p}(m), u_p(m)\big) \in \overline{\mathcal{M}}_{C^\circ,p} \subset Q \oplus \mathbb{Z}. \tag{2.9}$$

Suppose that $u_p(m) < 0$. By (2) of Definition 2.1, any lifting of $\varphi_p(m)$ to $\mathcal{M}_{C^\circ}$ has its image in $\mathcal{O}_{\underline{C}}$ vanishing in a neighborhood of p. By the commutativity of the above diagram, we thus have $\varphi_{\bar\eta} \circ \chi_{\eta,p}(m) \neq 0$. We summarize the constraint of φ around p as follows:

$$\text{for any } m \in P_p \text{ such that } u_p(m) < 0, \text{ we have } \varphi_{\bar\eta} \circ \chi_{\eta,p}(m) \neq 0. \tag{2.10}$$

Denote by $K \subset Q$ the ideal

$$\langle \varphi_{\bar\eta} \circ \chi_{\eta,p}(m) \mid \text{there is a puncture } p \text{ and } m \in P_p \text{ such that } u_p(m) < 0 \rangle \tag{2.11}$$

and call it the *puncturing ideal* of the punctured map.

Given a fine log scheme W, a sheaf of ideals $\overline{\mathcal{K}} \subset \overline{\mathcal{M}}_W$ is called *coherent* if for any $w, w' \in W$ with $w \in \mathrm{cl}(w')$, the generization map $\overline{\mathcal{K}}_{\bar w} \to \overline{\mathcal{K}}_{\bar w'}$ is surjective.

Lemma 2.29. *The fiber-wise constructed ideal in Equation (2.11) glues to a coherent sheaf of ideals $\overline{\mathcal{K}}_W \subset \overline{\mathcal{M}}_W$.*

Proof. It suffices to verify the construction in (2.11) is compatible with generization. More explicitly, consider two geometric points $\bar w \to \underline{W}$, $\bar w' \to \underline{W}$ with $\bar w \in \mathrm{cl}(\bar w')$. Denote by $K_{\bar w}$ and $K_{\bar w'}$ the two puncturing ideals associated to the corresponding fibers of punctured maps. We need to show that the image of the composition $K_{\bar{w}} \to \overline{\mathcal{M}}_{W,\bar{w}} \to \overline{\mathcal{M}}_{W,\bar{w}'}$ generates $K_{\bar{w}'}$.

Take a punctured point $\bar{p}$ over $\bar{w}$ and a punctured point $\bar{p}'$ over $\bar{w}'$ such that $p \in \mathrm{cl}(p')$. Write $Q_{\bar{w}} = \overline{\mathcal{M}}_{W,\bar{w}}$ and $Q_{\bar{w}'} = \overline{\mathcal{M}}_{W,\bar{w}'}$. Lemma 2.21 implies the following commutative diagram

$$\begin{tikzcd}
u_p^{-1}(\mathbb{Z}_{<0}) \arrow[r, hook] \arrow[d] & P_p \arrow[r] \arrow[d, "\chi_{p'p}"] & Q_{\bar{w}} \oplus \mathbb{Z} \arrow[r] \arrow[d] & Q_{\bar{w}} \arrow[d] \\
u_{p'}^{-1}(\mathbb{Z}_{<0}) \arrow[r, hook] & P_{p'} \arrow[r] & Q_{\bar{w}'} \oplus \mathbb{Z} \arrow[r] & Q_{\bar{w}'}
\end{tikzcd}$$

where all the vertical arrows are generization maps. By the construction in (2.11) the puncturing ideals $K_{\bar{w}}$ and $K_{\bar{w}'}$ are generated by the images of $u_p^{-1}(\mathbb{Z}_{<0}) \to Q_{\bar{w}}$ and $u_{p'}^{-1}(\mathbb{Z}_{<0}) \to Q_{\bar{w}'}$ respectively for all punctures. It remains to show that $u_p^{-1}(\mathbb{Z}_{<0}) \to u_{p'}^{-1}(\mathbb{Z}_{<0})$ is surjective.

Consider the sub-monoid $F = \chi_{p'p}^{-1}(0) \subset P_p$. Denote by $F^{-1}P_{p'} \subset P_p^{\mathrm{gp}}$ the submonoid generated by P_p and F^{gp}. Then the quotient $h : F^{-1}P_p \to P_{p'}$ by F^{gp} yields an isomorphism $F^{-1}P_p/F^{\mathrm{gp}} \cong P_{p'}$. For any $a' \in u_{p'}(\mathbb{Z}_{<0})$, choose $a \in F^{-1}P_{p'}$ such that $h(a) = a'$. Note that if $a \in P_p$ then $a \in u_p^{-1}(\mathbb{Z}_{<0})$ by Lemma 2.21. Suppose $a \notin P_p$. Then it is of the form $a = b - c$ for some $b \in P_p$ and $c \in F$. Observe that $u_p(F) = 0$. Thus, we have $b \in u_p^{-1}(\mathbb{Z}_{<0})$ and $h(b) = a'$. This proves the desired surjectivity. ♠

Recall (see e.g., [Ogu18], III §1.3) that a *log-ideal* $\mathcal{K}$ over a fine log scheme W is a sheaf of ideals $\mathcal{K}$ of $\mathcal{M}_W$. A log-ideal $\mathcal{K}$ over W is called *coherent* if for any points $x, y \in W$ with $x \in \mathrm{cl}(y)$, the generization map $\mathcal{K}_{\bar{x}} \to \mathcal{K}_{\bar{y}}$ is surjective.

Given a morphism of fine log schemes $h : T \to W$ and a log-ideal $\mathcal{K}$ over W, the *pull-back* $f^\bullet\mathcal{K}$ is the log-ideal over T generated by the image of $f^{-1}\mathcal{K} \to f^{-1}\mathcal{M}_W \to \mathcal{M}_T$. The pull-back $\bar{f}^\bullet(\overline{\mathcal{K}})$ of an ideal $\overline{\mathcal{K}} \subset \overline{\mathcal{M}}_W$ is defined similarly.

Observe that if $\mathcal{K}$ (respectively $\overline{\mathcal{K}}$) is coherent, the pull-back $f^\bullet\mathcal{K}$ (respectively $\bar{f}^\bullet\overline{\mathcal{K}}$) is coherent as well.

For any pre-stable punctured map $f : C^\circ \to X$ over W, let $\overline{\mathcal{K}}_W \subset \overline{\mathcal{M}}_W$ be the coherent sheaf of ideals introduced in Lemma 2.29. Consider the log-ideal $\mathcal{K}_W := \mathcal{M}_W \times_{\overline{\mathcal{M}}} \overline{\mathcal{K}}_W \subset \mathcal{M}_W$. The coherence of $\overline{\mathcal{K}}_W$ implies the coherence of $\mathcal{K}_W$.

Definition 2.30. The coherent log-ideal $\mathcal{K}_W$ is called the *puncturing log-ideal* associated to the pre-stable punctured map $f : C^\circ \to X$ over W. The log-ideal $\mathcal{K}_W$ is said to be *basic* if f is basic.

Proposition 2.31. *Let $f : C^\circ \to X$ be any pre-stable punctured map over W, and $f_T : C_T^\circ \to X$ be the pull-back of f via $h : T \to W$. Then $f^\bullet\mathcal{K}_W = \mathcal{K}_T$. In particular, $\mathcal{K}_W$ is the pull-back of the corresponding basic puncturing log-ideal.*

Proof. It suffices to show that $\bar{h}^\bullet\overline{\mathcal{K}}_W = \overline{\mathcal{K}}_T$, and it suffices to check this at each geometric point. We may assume that both $\underline{T}$ and $\underline{W}$ are geometric points. For each punctured point p, let $u_p : P_p \to \mathbb{Z}$ be the contact order at p. We have the following commutative diagram:

$$\begin{tikzcd}
u_p^{-1}(\mathbb{Z}_{<0}) \arrow[r, hook] \arrow[d, "="] & P_p \arrow[r, "\bar{f}^\flat"] \arrow[d, "="] & \overline{\mathcal{M}}_W \oplus \mathbb{Z} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_W \arrow[d, "\bar{h}^\flat"] \\
u_p^{-1}(\mathbb{Z}_{<0}) \arrow[r, hook] & P_p \arrow[r, "\bar{f}_T^\flat"] & \overline{\mathcal{M}}_T \oplus \mathbb{Z} \arrow[r] & \overline{\mathcal{M}}_T.
\end{tikzcd}$$

The puncturing ideals $\overline{\mathcal{K}}_W$ and $\overline{\mathcal{K}}_T$ are generated by the images of compositions of the top and bottom arrows respectively for all punctures, see (2.11), and hence the statement follows. ♠

The puncturing log-ideal is a new phenomenon for punctured maps compared to log maps which puts extra constraints as follows.

Theorem 2.32. *Let $f : C^\circ \to X$ be any pre-stable punctured map over W, and $\mathcal{K}_W$ be the corresponding puncturing log-ideal. Then we have $\alpha_{\mathcal{M}_W}(\mathcal{K}_W) = 0$.*

Proof. Since the statement can be checked locally on W, shrinking W we may assume given a chart $\gamma : \overline{\mathcal{M}}_{W,w} \to \mathcal{M}_W$ for some geometric point $w \in W$. Since $\mathcal{K}_W$ is coherent, it is generated by $\gamma(\overline{\mathcal{K}}_{W,w})$. It suffices to show that $\alpha_{\mathcal{M}_W} \circ \gamma(\overline{\mathcal{K}}_{W,w}) = 0$.

Let $p \in C^\circ_w$ be a puncture with contact order u_p. For any $\bar{\delta} \in u_p^{-1}(\mathbb{Z}_{<0}) \subset P_p$, denote by $\bar{e}_{\bar{\delta}}$ its image via the following composition

$$P_p \xrightarrow{\bar{f}^\flat} \overline{\mathcal{M}}_{W,w} \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_1} \overline{\mathcal{M}}_{W,w}.$$

Let $e_{\bar{\delta}}$ be a local section of $\mathcal{M}_W$ over $\bar{e}_{\bar{\delta}}$. It suffices to note that by Definition 2.1, (2), $\alpha_{\mathcal{M}_W}(e_{\bar{\delta}}) = 0$. ♠

This demonstrates that the base of a family of punctured maps is naturally an *idealized log scheme* (or stack). We recall from [Ogu18, III Def. 1.3]:

Definition 2.33. An idealized log scheme is a log scheme $(X, \mathcal{M}_X)$ equipped with a sheaf of ideals $\mathcal{K}_X \subseteq \mathcal{M}_X$ such that $\mathcal{K}_X \subseteq \alpha_X^{-1}(0)$. A morphism of idealized log schemes $f : X \to Y$ is a morphism of log schemes such that $f^\flat$ maps $f^{-1}(\mathcal{K}_Y)$ into $\mathcal{K}_X$.

Corollary 2.34. *In the situation of Theorem 2.32, the triple $(W, \mathcal{M}_W, \mathcal{K}_W)$ is a coherent idealized log scheme.*

2.6. **Contact orders.** For a target X, consider the following étale sheaves over $\underline{X}$:

$$\overline{\mathcal{M}}_X^\vee = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{N}) \quad \text{and} \quad \overline{\mathcal{M}}_X^* = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathbb{Z}) \cong \mathcal{H}om(\overline{\mathcal{M}}_X^{\mathrm{gp}}, \mathbb{Z}).$$

Definition 2.35. A *family of contact orders of* X consists of a strict morphism $Z \to X$ and a section $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}_Z^*)$ satisfying the following condition. Let

$u : \mathcal{M}_Z \to \overline{\mathcal{M}}_Z \xrightarrow{\mathbf{u}} \mathbb{Z}$ be the composite homomorphism associated to $\mathbf{u}$. Then the map $\alpha : \mathcal{M}_Z \to \mathcal{O}_Z$ sends $u^{-1}(\mathbb{Z} \smallsetminus \{0\})$ to 0.

We call the ideal $\mathcal{I}_{\mathbf{u}} \subset \mathcal{M}_Z$ generated by $u^{-1}(\mathbb{Z} \smallsetminus \{0\})$ the *contact ideal* associated to $\mathbf{u}$, and denote by $\overline{\mathcal{I}}_{\mathbf{u}}$ the corresponding *ghost contact ideal* in $\overline{\mathcal{M}}_Z$.

The family of contact orders is said to be *connected* if Z is connected.

For simplicity, we will refer to $\mathbf{u}$ as the contact order when there is no confusion about the strict morphism $Z \to X$. Given a family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}^*_Z)$ of X, the *pull-back* of $\mathbf{u}$ along a strict morphism $Z' \to Z$ defines a family of contact orders $\mathbf{u}' \in \Gamma(Z', \overline{\mathcal{M}}^*_{Z'})$.

Example 2.36. To motivate this definition, consider a punctured map $f : C^\circ \to X$ over W, and a section $p \in \mathbf{p}$. Take $\underline{Z} := \underline{W}$, and give $\underline{Z}$ the log structure given by pull-back of $\mathcal{M}_X$ via $\underline{f} \circ p$, so that $Z \to X$ is strict. Let $\mathbf{u}$ be the following composition

$$\overline{\mathcal{M}}_Z \xrightarrow{\bar{f}^\flat} p^*\overline{\mathcal{M}}_{C^\circ} \subset \overline{\mathcal{M}}_W \oplus \mathbb{Z} \longrightarrow \mathbb{Z}. \tag{2.12}$$

where the last arrow is the projection to the second factor.

We claim that $\mathbf{u}$ defines a family of contact orders of X. Indeed, let $\delta \in \mathcal{M}_Z$ and represent $f^\flat(\delta) = (e_\delta, \sigma^{u_p(\delta)})$, where σ is the element of $\mathcal{M}_C$ corresponding to a local defining equation of the section p.

If $u_p(\delta) > 0$ then

$$\alpha_Z(\delta) = p^*\alpha_C(f^\flat(\delta)) = p^*\alpha_C(e_\delta) \cdot p^*\alpha_C(\sigma^{u_p(\delta)}) = 0$$

since $p^*\alpha_C(\sigma) = 0$.

If $u_p(\delta) < 0$ then $f^\flat(\delta) \notin \mathcal{M}_C$ and hence, by Definition 2.1 (2) we have $\alpha_Z(\delta) = 0$.

2.6.1. *Family of contact orders of Artin cones.* Let $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}^*_Z)$ be a family of contact orders of X. For any strict morphism $X \to Y$, $\mathbf{u}$ is naturally a family of contact orders of Y via the composition $Z \to X \to Y$. Thus we may parameterize contact orders of the Artin fan $\mathcal{A}_X$ instead of X. We first study the local case.

Consider a fine saturated sharp monoid σ and the Artin cone

$$\mathcal{A}_\sigma = [\operatorname{Spec}(\sigma^\vee \to \Bbbk[\sigma^\vee]) / \operatorname{Spec}(\Bbbk[\sigma^*])]. \tag{2.13}$$

Choose an integral vector $u \in \sigma^{\mathrm{gp}}$. Let I_u be the ideal of $\sigma^\vee$ generated by $u^{-1}(\mathbb{Z} \smallsetminus \{0\})$. This generates a $\Bbbk[\sigma^*]$-invariant ideal in $\Bbbk[\sigma^\vee]$, defining a closed substack $\mathcal{Z}_{u,\sigma} \subset \mathcal{A}_\sigma$. We proceed to construct a family of contact orders parametrized by $\mathcal{Z}_{u,\sigma}$.

For each face $\tau \prec \sigma$, denote by $\mathcal{Z}_{\tau\prec\sigma} \subset \mathcal{A}_\sigma$ the locally closed sub-stack where the fiber of $\overline{\mathcal{M}}^\vee_{\mathcal{Z}_{\tau\prec\sigma}}$ is identified with τ.

Lemma 2.37. *We have* $(\mathcal{Z}_{u,\sigma})_{\mathrm{red}} = \bigcup_{\tau^{\mathrm{gp}} \ni u} \mathcal{Z}_{\tau\prec\sigma} \subset \mathcal{A}_\sigma$.

Proof. Working with monoid ideals, we want to show that $\sqrt{I_u}$ coincides with the monoid ideal $I(\cup_{\tau^{\mathrm{gp}}\ni u}\mathcal{Z}_{\tau\prec\sigma})$ of monomials vanishing on the union $\cup_{\tau^{\mathrm{gp}}\ni u}\mathcal{Z}_{\tau\prec\sigma}$.

Note that $I(\mathcal{Z}_{\tau\prec\sigma}) = \sigma^\vee \smallsetminus (\tau^\perp \cap \sigma^\vee)$. The ideal $\sqrt{I_u}$ defines some union of strata and we identify those strata $\mathcal{Z}_{\tau\prec\sigma}$ on which it vanishes. If $u \notin \tau^{\mathrm{gp}}$ there is an element $p \in \tau^\perp \cap \sigma^\vee$ such that $u(p) \neq 0$. Therefore $p \in I_u$ but the monomial z^p does not vanish at the generic point of $\mathcal{Z}_{\tau\prec\sigma}$. If $u \in \tau^{\mathrm{gp}}$, and if $p \in u^{-1}(\mathbb{Z} \smallsetminus \{0\})$, then $p \notin \tau^\perp \cap \sigma^\vee$, hence z^p vanishes along $\mathcal{Z}_{\tau\prec\sigma}$. ♠

As $\overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma}}$ and $\mathbb{Z}$ are constructible, we have

$$\Gamma(\mathcal{Z}_{u,\sigma}, \overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}) = \Gamma((\mathcal{Z}_{u,\sigma})_{\mathrm{red}}, \overline{\mathcal{M}}^*_{(\mathcal{Z}_{u,\sigma})_{\mathrm{red}}}).$$

We define an element $\mathbf{u}_{u,\sigma}$ of this group by defining it on stalks in a manner compatible with generization. For a point $z \in \mathcal{Z}_{\tau\prec\sigma}$ the condition $u \in \tau^{\mathrm{gp}}$ guarantees that $u : \sigma^\vee \to \mathbb{Z}$ descends to $u : \overline{\mathcal{M}}_{\mathcal{Z}_{u,\sigma},z} = (\sigma^\vee + \tau^\perp)/\tau^\perp \to \mathbb{Z}$. Being induced by the same element u, this is compatible with generization. Note that the scheme $\mathcal{Z}_{u,\sigma}$ was defined in such a way so that $\alpha_{\mathcal{Z}_{u,\sigma}}(\mathcal{I}_{\mathbf{u}_{u,\sigma}}) = 0$, so that $\mathcal{Z}_{u,\sigma}$ acquires the structure of an idealized log stack.

Thus u defines a family of contact orders of $\mathcal{A}_\sigma$

$$\mathbf{u}_{u,\sigma} \in \Gamma(\mathcal{Z}_{u,\sigma}, \overline{\mathcal{M}}^*_{\mathcal{Z}_{u,\sigma}}). \tag{2.14}$$

It is connected since the most degenerate stratum $\mathcal{Z}_{\sigma\prec\sigma}$ is contained in the closure of $\mathcal{Z}_{\tau\prec\sigma}$ for any face τ.

Lemma 2.38. *For any connected family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}^*_Z)$ of $\mathcal{A}_\sigma$, there exists a unique $u \in \sigma^{\mathrm{gp}}$ such that $\psi : Z \to \mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_{u,\sigma}$, and $\mathbf{u}_{u,\sigma}$ pulls back to $\mathbf{u}$.*

Proof. The global chart $\sigma^\vee \to \overline{\mathcal{M}}_{\mathcal{A}_\sigma}$ over $\mathcal{A}_\sigma$ pulls back to a global chart $\sigma^\vee \to \overline{\mathcal{M}}_Z$ over Z. The composition $\sigma^\vee \longrightarrow \overline{\mathcal{M}}_Z \xrightarrow{\mathbf{u}} \mathbb{Z}$ defines an integral vector $u \in \sigma^{\mathrm{gp}}$. Consider the sheaf of monoid ideals $\mathcal{J}_u \subset \mathcal{M}_{\mathcal{A}_\sigma}$ generated by I_u. By definition of the contact ideal $\mathcal{I}_{\mathbf{u}}$ we have $\mathcal{I}_{\mathbf{u}} = \psi^\bullet \mathcal{J}_u$. Since $\alpha_Z(\mathcal{I}_{\mathbf{u}}) = 0$ we have the factorization $Z \to \mathcal{Z}_{u,\sigma} = V(\alpha_{\mathcal{A}_\sigma}(\mathcal{J}_u))$ of ψ, with $\mathbf{u}$ the pull-back of $\mathbf{u}_{u,\sigma}$. ♠

We can now assemble all the $\mathcal{Z}_{u,\sigma}$ by defining

$$\mathcal{Z}_\sigma = \coprod_{u\in\sigma^{\mathrm{gp}}} \mathcal{Z}_{u,\sigma},$$

and write $\psi_\sigma : \mathcal{Z}_\sigma \to \mathcal{A}_\sigma$ for the morphism which restricts to the closed immersion $\mathcal{Z}_{u,\sigma} \hookrightarrow \mathcal{A}_\sigma$ on each connected component $\mathcal{Z}_{u,\sigma}$ of $\mathcal{Z}_\sigma$. Then the $\mathbf{u}_{u,\sigma}$ yield a section $\mathbf{u}_\sigma \in \Gamma(\mathcal{Z}_\sigma, \overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$, giving the universal family of contact orders over $\mathcal{A}_\sigma$, as follows immediately from Lemma 2.38 by restricting to connected components.

Lemma 2.39. *For any family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}^*_Z)$ of $\mathcal{A}_\sigma$, $\psi : Z \to \mathcal{A}_\sigma$ factors uniquely through $\mathcal{Z}_\sigma$, and $\mathbf{u}_\sigma$ pulls back to $\mathbf{u}$.*

Lemma 2.40. *If τ is a face of σ, viewing $\mathcal{A}_\tau$ naturally as an open substack of $\mathcal{A}_\sigma$ we then have $\mathcal{Z}_\tau \cong \psi_\sigma^{-1}(\mathcal{A}_\tau)$, and the section $\mathbf{u}_\sigma \in \Gamma(\mathcal{Z}_\sigma, \overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$ pulls back to the section $\mathbf{u}_\tau \in \Gamma(\mathcal{Z}_\tau, \overline{\mathcal{M}}^*_{\mathcal{Z}_\tau})$.*

Proof. Note that the open immersion $\mathcal{A}_\tau \subseteq \mathcal{A}_\sigma$ is induced by the open immersion of toric varieties $\operatorname{Spec} \Bbbk[\sigma^\vee + \tau^\perp] \subseteq \operatorname{Spec} \Bbbk[\sigma^\vee]$. From Lemma 2.37, it follows that $\mathcal{Z}_{u,\sigma} \cap \psi_\sigma^{-1}(\mathcal{A}_\tau)$ is non-empty if and only if $u \in \tau^{\mathrm{gp}}$. If $u \in \tau^{\mathrm{gp}}$, then $\mathcal{Z}_{u,\tau}$ is defined by the monoid ideal in $\sigma^\vee + \tau^\perp$ generated by $u^{-1}(\mathbb{Z} \smallsetminus \{0\})$, and this coincides with the extension of the monoid ideal in $\sigma^\vee$ defining $\mathcal{Z}_{u,\sigma}$. Thus in this case $\mathcal{Z}_{u,\tau} = \psi_\sigma^{-1}(\mathcal{A}_\tau) \cap \mathcal{Z}_{u,\sigma}$, giving the first claim.

Let $u \in \sigma^{\mathrm{gp}}$ be the vector corresponding to a component $\mathbf{u}_{u,\sigma}$ of $\mathbf{u}$. Observe that if $u \notin \tau^{\mathrm{gp}}$, then the image of $\mathcal{Z}_{u,\sigma} \to \mathcal{A}_\sigma$ avoids $\mathcal{A}_\tau \subseteq \mathcal{A}_\sigma$. Furthermore, if $u \in \tau^{\mathrm{gp}}$, then $\mathbf{u}_{u,\sigma}$ pulls back to $\mathbf{u}_{u,\tau}$ by the construction of (2.14). Therefore, $\mathbf{u}_\sigma \in \Gamma(\mathcal{Z}_\sigma, \overline{\mathcal{M}}^*_{\mathcal{Z}_\sigma})$ pulls back to the section $\mathbf{u}_\tau \in \Gamma(\mathcal{Z}_\tau, \overline{\mathcal{M}}^*_{\mathcal{Z}_\tau})$. ♠

2.6.2. *Family of contact orders of Zariski Artin fans.* We now consider the case of an Artin fan $\mathcal{A}_X$. Recall that $\mathcal{A}_X$ has an étale cover by Artin cones, and was constructed in [ACMW17, Proposition 3.1.1], as a colimit of Artin cones $\mathcal{A}_\sigma$, viewed as sheaves over Log.

Definition 2.41. We say that the Artin fan $\mathcal{A}_X$ is *Zariski* if it admits a Zariski cover by Artin cones.

It was shown, for example, in [ACGS17, Lemma 2.2.4], that if X is log smooth over $\Bbbk$ then $\mathcal{A}_X$ is Zariski.

Over a Zariski Artin fan, one can construct $\mathcal{Z}$ as the colimit of the $\mathcal{Z}_\sigma$ viewed as sheaves over $\mathcal{A}_X$. Indeed, $\mathcal{Z}$ is obtained by gluing together the local model $\mathcal{Z}_\sigma$ for each Zariski open $\mathcal{A}_\sigma \subset \mathcal{A}_X$ via the canonical identification given by Lemma 2.40.[2]

We then have

Proposition 2.42. *There is a section $\mathbf{u}_\mathcal{X} \in \Gamma(\mathcal{Z}, \overline{\mathcal{M}}^*_\mathcal{Z})$ making $\mathcal{Z}$ into a family of contact orders for $\mathcal{A}_X$. This family of contact orders is universal in the sense that for any family of contact orders $\mathbf{u} \in \Gamma(Z, \overline{\mathcal{M}}^*_Z)$ of $\mathcal{A}_X$, $\psi : Z \to \mathcal{A}_X$, there is a unique factorization of ψ through $\mathcal{Z} \to \mathcal{A}_X$ such that $\mathbf{u}$ is the pull-back of $\mathbf{u}_\mathcal{X}$.*

Proof. If $\mathcal{A}_\sigma \to \mathcal{A}_X$ is a Zariski open set, then by the construction of $\mathcal{Z}$,

$$\mathcal{Z} \times_{\mathcal{A}_X} \mathcal{A}_\sigma = \mathcal{Z}_\sigma.$$

By Lemma 2.40, the sections $\mathbf{u}_\sigma$ glue to give a section $\mathbf{u}_\mathcal{X} \in \Gamma(\mathcal{Z}, \overline{\mathcal{M}}^*_\mathcal{Z})$, yielding a family of contact orders in $\mathcal{A}_X$.

Consider a family of contact orders $Z \to \mathcal{A}_X$, $\mathbf{u}$. To show the desired factorization, it suffices to prove the existence and uniqueness locally on each Zariski open subset $\mathcal{A}_\sigma \to \mathcal{A}_X$, which follows from Lemma 2.39. ♠

Definition 2.43. A *connected contact order* for X is a choice of connected component of $\mathcal{Z}$.

[2]It should be possible to carry this process out for more general Artin fans. However, given how rarely one needs more general Artin fans in practice, it did not seem to be worth the extra technical baggage to carry this out.

We end this discussion with a couple of properties of the space $\mathscr{Z}$ of contact orders.

Proposition 2.44. *Suppose that the Artin fan $\mathcal{A}_X$ of X is Zariski. There is a one-to-one correspondence between irreducible components of $\mathscr{Z}$ and pairs (u, σ) where $\sigma \in \Sigma(X)$ is a minimal cone such that $u \in \sigma^{\mathrm{gp}}$.*

Proof. Since we are interested in classifying irreducible components of contact orders, we may assume $\mathcal{A}_X = \mathcal{A}_\sigma$. Then the statement follows from the description of $\mathscr{Z}_{u,\sigma}$ in Lemma 2.37. ♠

Remark 2.45. Note that if $u \in \sigma$ or $-u \in \sigma$, then $\mathscr{Z}_{u,\sigma}$ is already irreducible, being the closure of the stratum $\mathscr{Z}_{\tau \prec \sigma}$ where $\tau \subset \sigma$ is the minimal face containing u. Further, the ideal generated by $u^{-1}(\mathbb{Z} \setminus \{0\})$ is precisely $\sigma^\vee \setminus \tau^\perp$, so that $\mathscr{Z}_{u,\sigma}$ is reduced. In the case that $u \in \sigma$, this is the case of contact orders associated to ordinary marked points, as developed in [Che14],[AC14],[GS13]. The situation for more general contact orders associated to punctured points may be more complex, and in addition, even in the Zariski case, there may be monodromy.

For example, consider the three-dimensional toric variety Y (not of finite type) defined by a fan consisting of the collection of three-dimensional cones

$$\Sigma^{[3]} = \{\mathbb{R}_{\geq 0}(n, 0, 1) + \mathbb{R}_{\geq 0}(n+1, 0, 1) + \mathbb{R}_{\geq 0}(n, 1, 1) + \mathbb{R}_{\geq 0}(n+1, 1, 1) \,|\, n \in \mathbb{Z}\}$$

and their faces. Projection onto the third coordinate yields a toric morphism $Y \to \mathbb{A}^1$. After a base-change $\widehat{Y} = Y \times_{\mathbb{A}^1} \operatorname{Spec} \Bbbk[[t]] \to \operatorname{Spec} \Bbbk[[t]]$, one may divide out $\widehat{Y}$ by the action of $\mathbb{Z}$ defined as follows. This action is generated by an automorphism of $\widehat{Y}$ induced by an automorphism of Y defined over $\mathbb{A}^1$. This automorphism is given torically via the linear transformation $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{pmatrix} 1 & 0 & \ell \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where ℓ is a fixed positive integer. We then define $X = \widehat{Y}/\mathbb{Z}$, with log structure induced by the toric log structure on Y. Then $X \to \operatorname{Spec} \Bbbk[[t]]$ is a degeneration of the total space of a $\mathbb{G}_m$-torsor over an elliptic curve, the torsor corresponding to a 2-torsion element of the Picard group of the elliptic curve. As long as $\ell \geq 2$, X has a Zariski log structure. Further, $\Sigma(X)$ is a cone over a Möbius strip composed of ℓ squares. If one takes $u = (0, 1, 0) \in \sigma^{\mathrm{gp}}$ for any three-dimensional cone in $\Sigma(X)$, then the twist in the Möbius strip identifies u with $-u$. The connected contact order corresponding to such a u is then a cycle of 2ℓ copies of $\mathbb{P}^1$, mapping 2 to 1 to the cycle of ℓ copies of $\mathbb{P}^1$ contained in the central fibre of $X \to \operatorname{Spec} \Bbbk[[t]]$.

In fact, there exist examples where this kind of monodromy (even in a Zariski log smooth situation) produces connected contact orders which have an infinite number of components, and then one does not expect well-behaved moduli spaces. Thus, additional hypotheses are usually needed to obtain good control of these spaces. For example:

Proposition 2.46. *Suppose $\overline{\mathcal{M}}_X$ is generated by its global sections. Then every connected component of contact orders of $\mathcal{A}_X$ has finitely many irreducible components.*

Proof. Denote by $\sigma^\vee = \Gamma(\underline{X}, \overline{\mathcal{M}}_X)$. Suppose $\mathbf{u} \in \Gamma(\mathcal{Z}, \overline{\mathcal{M}}^*_{\mathcal{Z}})$ is a connected component of contact orders of $\mathcal{A}_X$. Denote the composition $\sigma^\vee \longrightarrow \overline{\mathcal{M}}_{\mathcal{Z}} \xrightarrow{\mathbf{u}} \mathbb{Z}$ by v. As $\overline{\mathcal{M}}_X$ is globally generated, for each irreducible component of $\mathcal{Z}$, its corresponding vector u as in Proposition 2.44 is uniquely determined by v. By Proposition 2.44 again $\mathcal{Z}$ has finitely many irreducible components, as $\Sigma(X)$ has finitely many cones. ♠

2.6.3. *Opposite contact orders.* When we proceed to gluing punctured log curves, we may only glue punctures p and p' to form nodes when $u_p = -u_{p'}$. It is useful to formalize this as follows.

If σ is a fine saturated sharp monoid, $u \in \sigma^{\mathrm{gp}}$, then $\mathcal{Z}_{u,\sigma} = \mathcal{Z}_{-u,\sigma}$ as closed substacks of $\mathcal{A}_\sigma$, as they are defined by the same ideal. Thus there is a natural involution

$$\mathrm{opp} : \mathcal{Z}_\sigma \to \mathcal{Z}_\sigma$$

defined over $\mathcal{A}_\sigma$ taking $\mathcal{Z}_{u,\sigma}$ to $\mathcal{Z}_{-u,\sigma}$ for any $u \in \sigma^{\mathrm{gp}}$. If $\mathcal{A}_X$ is a Zariski Artin fan, we can then patch this involution over each $\mathcal{A}_\sigma \subset \mathcal{A}_X$ to obtain an involution

$$\mathrm{opp} : \mathcal{Z} \to \mathcal{Z}.$$

Definition 2.47. We say two connected contact orders $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{Z}$ are *opposite* if $\mathrm{opp}(\mathcal{Z}_1) = \mathcal{Z}_2$.

Remark 2.48. A connected contact order can be opposite to itself, as is the case in the example given in Remark 2.45. However, an irreducible contact order is only opposite to itself if the contact order is trivial, i.e., $u = 0$.

2.7. **The tropical interpretation.** The construction of the basic monoid in [GS13] was motivated by a description of the dual of the basic monoid as a moduli space of tropical curves. The tropical interpretation of a stable log map over the standard log point is described in [GS13, §1.4], and the tropical interpretation of the basic monoid is given in [GS13, Remark 1.18]. This is expanded on at length in [ACGS17, §2.1.4]. Here we discuss briefly how punctures affect this interpretation.

Recall from [GS13, Appendix B], or more generally [ACGS17, §2.1.4], the tropicalization functor. In [ACGS17, §2.1.4], we associate to any Deligne-Mumford fs log stack X a generalized polyhedral complex $\Sigma(X)$. For η the generic point of a stratum of X, we have an associated cone

$$\sigma_\eta = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar\eta}, \mathbb{R}_{\geq 0}),$$

(a cone is viewed as also carrying an integral structure from the lattice $\overline{\mathcal{M}}^*_{X,\bar\eta}$). Then $\Sigma(X)$ is the cone complex presented by a diagram of these cones with morphisms between them the inclusions of faces dual to generization maps. In particular, a stable log map $(C/W, \mathbf{p}, f)$ gives rise via functoriality of Σ to the diagram

$$\begin{tikzcd} \Sigma(C) \arrow[r, "\Sigma(f)"] \arrow[d, "\Sigma(\pi)"'] & \Sigma(X) \arrow[d] \\ \Sigma(W) \arrow[r] & \Sigma(S) \end{tikzcd} \tag{2.15}$$

This is then interpreted as a family of tropical curves, with each fibre of $\Sigma(\pi)$ being a graph, and $\Sigma(f)$ restricted to a fibre defines a map to $\Sigma(X)$. In the case that $W = \mathrm{Spec}(Q \to \kappa)$ is a log point, the basicness of $(C/W, \mathbf{p}, f)$ is then seen to be a kind of universality of this family of tropical curves.

The same approach works in the punctured case: all we need to do is modify the treatment of a marked point in [ACGS17, §2.5.4(iii)] to punctured points as follows:

(iii′) If $p \in C$ is a punctured point, then we describe $\sigma_p = \mathrm{Hom}(\overline{\mathcal{M}}_{C^\circ,p}, \mathbb{R}_{\geq 0})$ as follows. Let $P_p = \overline{\mathcal{M}}_{X,f(p)}$ and $Q = \overline{\mathcal{M}}_{W,\pi(p)}$. Denote the dual of the homomorphism $\bar{f}^\flat : P_p \to \overline{\mathcal{M}}_{C^\circ,p}$ by $(\bar{f}^\flat)^t$. Then

$$\sigma_p = \big(((\bar{f}^\flat)^t)^{-1}(P_p^\vee) \cap (Q \oplus \mathbb{N})^\vee\big)_{\mathbb{R}}.$$

Indeed, the stalk of the ghost sheaf at p of the prestable punctured logarithmic structure is the smallest fine submonoid of $Q \oplus \mathbb{Z}$ containing both $Q \oplus \mathbb{N}$ and $\bar{f}^\flat P_p$.

The map $\Sigma(\pi) : \sigma_p \to Q^\vee_{\mathbb{R}}$ is the projection. Its fiber over an element $q \in Q^\vee_{\mathbb{R}}$ is

$$\big\{n \geq 0 \,|\, (\bar{f}^\flat)^t(q, 0) + n \cdot u_p \in (P_p^\vee)_{\mathbb{R}}\big\}.$$

Here $\Sigma(f)(q, 0)$ is the image of the vertex corresponding to the irreducible component containing p, and pre-stability means that this is either a ray, when $u_p \in P_p^\vee$, namely p is a marked point, or a segment whose image extends as far as possible in the cone $(P_p^\vee)_{\mathbb{R}}$, if p is genuinely a puncture.

Note the fibres $\Sigma(\pi)^{-1}(x)$ of $\Sigma(\pi)$ for $x \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$ can be identified with the dual graph $G_{\underline{C}}$ of $\underline{C}$, with the proviso that the legs of $G_{\underline{C}}$ corresponding to punctured points are either closed line segments or rays. If x instead lies in the boundary of $Q^\vee_{\mathbb{R}}$, $\Sigma(\pi)^{-1}(x)$ is obtained from $G_{\underline{C}}$ by contracting some edges and legs of $G_{\underline{C}}$ whose lengths have gone to zero.

Note the language of tropical curves from [ACGS16], Definitions 2.5.2 and 2.5.3 can be easily adapted to the current setting, as follows. We consider connected graphs G with sets of vertices $V(G)$, edges $E(G)$ and legs $L(G)$. However, unlike in the marked point case, a leg may be a compact interval or a ray. In either case, a leg has only one endpoint in $V(G)$. A tropical curve $\Gamma = (G, \mathbf{g}, \ell)$ of combinatorial type $(G, \mathbf{g})$ is the choice of a genus function $\mathbf{g} : V(G) \to \mathbb{N}$ and a length function $\ell : E(G) \to \mathbb{R}_{>0}$.

We summarize the definition of a tropical curve in $\Sigma(X)$ as given in [ACGS16, Def. 2.5.3], with the slight modification from punctures. We recall from [ACGS16, §2.1] that a cone $\sigma \in \Sigma(X)$ is equipped with a lattice of integral tangent vectors N_σ. A tropical curve in $\Sigma(X)$ is then data of (1) a tropical curve Γ; (2) a map

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X),$$

thinking of $\Sigma(X)$ as a set of cones; (3) a bijection between $L(G)$ and a marking set; (4) for each edge $E_q \in E(G)$ with an orientation a *weight vector* $u_q \in N_{\boldsymbol{\sigma}(E_q)}$ (the lattice of integral tangent vectors to the cone $\boldsymbol{\sigma}(E_q)$); (5) for each $E_x \in L(G)$ an element $u_x \in N_{\boldsymbol{\sigma}(E_x)}$; (6) a continuous map $f : |\Gamma| \to |\Sigma(X)|$. This data satisfies conditions enumerated in [ACGS16], Definition 2.5.3, with one modification due to punctures: if $E_x \in L(G)$ is a leg with vertex v, it holds that $f(\mathrm{Int}(E_x)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_x))$ and f maps E_x affine linearly to the ray or line segment

$$(f(v) + \mathbb{R}_{\geq 0} u_x) \cap \boldsymbol{\sigma}(E_x) \subset N_{\boldsymbol{\sigma}(E_x)} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.16}$$

In other words, a leg E_x with vertex v associated to a punctured point is mapped to the longest possible line segment in the cone $\boldsymbol{\sigma}(E_x)$ with one endpoint $f(v)$ with tangent direction u_x. Thus if u_x lies in the cone $\boldsymbol{\sigma}(E_x)$, in fact this line segment is a ray, which is the case more clasically for marked points.

We also recall that if v_1, v_2 are vertices of an edge E_q from v_1 to v_2, then $f(\mathrm{Int}(E_q)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_q))$,

$$f(v_2) - f(v_1) = \ell(E_q) u_q, \tag{2.17}$$

and f maps E_q affine linearly to the line segement joining $f(v_1)$ and $f(v_2)$.

A *combinatorial type* of tropical map to $\Sigma(X)$ is all of the above data except for the contiunous map f and the length function ℓ.

2.7.1. *The balancing condition.* The above discussion fits well with the tropical balancing condition at vertices of the dual graph of C°. In fact, the statement [GS13, Proposition 1.15] holds unchanged. There is no balancing condition at the endpoint of the segment described above. As we will need the balancing condition to prove boundedness, we review this statement here.

Suppose given a stable punctured map $(C/W, \mathbf{p}, f)$ with $W = \mathrm{Spec}(\mathbb{N} \to \kappa)$ the standard log point over a field. Let $g : \tilde{D} \to C$ be the normalization of an irreducible component D with generic point η of C. One then obtains, with $\overline{\mathcal{M}} = \underline{f}^* \overline{\mathcal{M}}_X$, composed maps

$$\tau_\eta^X : \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}) \longrightarrow \mathrm{Pic}\, \tilde{D} \xrightarrow{\deg} \mathbb{Z}$$
$$\tau_\eta^C : \Gamma(\tilde{D}, g^* \overline{\mathcal{M}}_{C^\circ}) \longrightarrow \mathrm{Pic}\, \tilde{D} \xrightarrow{\deg} \mathbb{Z}$$

with the first map on each line given by taking a section of the ghost sheaf to the corresponding $\mathcal{O}_{\tilde{D}}^\times$-torsor. These are compatible: the pull-back of $f^\flat$ to $\tilde{D}$,

$\varphi : g^*\mathcal{M} \to g^*\mathcal{M}_{C^\circ}$, induces $\bar\varphi : g^*\overline{\mathcal{M}} \to g^*\overline{\mathcal{M}}_{C^\circ}$, and hence a commutative diagram

$$\begin{tikzcd} \Gamma(\tilde D, g^*\overline{\mathcal{M}}) \arrow[r, "\bar\varphi"] \arrow[dr, "\tau_\eta^X"'] & \Gamma(\tilde D, g^*\overline{\mathcal{M}}_{C^\circ}) \arrow[d, "\tau_\eta^C"] \\ & \mathbb{Z} \end{tikzcd}$$

The map τ_η^X is given by $\underline{f}$ and $\mathcal{M}$, so is dependent on the geometry of $f : C^\circ \to X$; however if $\underline{f}$ contracts $\underline{D}$, then $\tau_\eta^X = 0$. On the other hand, τ_η^C is determined using the notation in [GS13, §1.4]. Explicitly, for each point $q \in D$ over a node of $\underline{C}$ we have $\overline{\mathcal{M}}_{C^\circ,\bar q} = S_{e_q}$, the submonoid of $\mathbb{N}^2$ generated by $(0, e_q)$, $(e_q, 0)$ and $(1,1)$. The generization map $\chi_q : \overline{\mathcal{M}}_{C^\circ,\bar q} \to \overline{\mathcal{M}}_{C^\circ,\bar\eta} = \mathbb{N}$ is given by projection to the second coordinate: $\chi_q(a,b) = b$. We then have

$$\Gamma(\tilde D, g^*\overline{\mathcal{M}}_{C^\circ}) \quad\subseteq\quad \Gamma(\tilde D, g^*\overline{\mathcal{E}}),$$

where

$$\Gamma(\tilde D, g^*\overline{\mathcal{E}}) \quad=\quad \left\{ (n_q)_{q\in\tilde D} \;\middle|\; \begin{array}{c} n_q \in S_{e_q} \text{ and } \chi_q(n_q) = \chi_{q'}(n_{q'}) \\ \text{for } q, q' \in \tilde D \end{array} \right\} \oplus \bigoplus_{p\in\tilde D} \mathbb{Z},$$

We then obtain, with proof identical to that of [GS13, Lemma 1.14]:

Lemma 2.49. $\tau_\eta^C\big(((a_q, b)_{q\in\tilde D}, (n_p)_{p\in\tilde D})\big) = -\sum_{p\in\tilde D} n_p + \sum_{q\in\tilde D} \frac{b-a_q}{e_q}$.

The equation $\tau_\eta^X = \tau_\eta^C \circ \varphi$ is a formula in $N_D := \Gamma(\tilde D, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$, which is described in [GS13, Equations (1.12), (1.13)] as follows. Let $\Sigma \subset \tilde D$ be the set of special points p, q in $\tilde D$, that is mapping to a special point of D. Then

$$N_D = \varinjlim_{x\in\tilde D} P_x^* = \left(\bigoplus_{x\in\Sigma} P_x^*\right) \Big/ \sim$$

where for any $a \in P_\eta^*$ and any $x, x' \in \Sigma$,

$$(0,\ldots,0,\iota_{x,\eta}(a),0,\ldots,0) \sim (0,\ldots,0,\iota_{x',\eta}(a),0,\ldots,0).$$

Here $\iota_{x,\eta} : P_\eta^* \to P_x^*$ is the dual of generization, and the non-zero entries lie in the position indexed by x and x' respectively.

We then have, exactly as in [GS13, Proposition 1.15], the balancing condiiton:

Proposition 2.50. *Suppose $(C/W, \mathbf{p}, f)$ is a stable punctured map to X/S with $W = \operatorname{Spec}(\mathbb{N} \to \kappa)$ a standard log point. Let $D \subset \underline{C}$ be an irreducible component with generic point η and $\Sigma \subset \tilde D$ the preimage of the set of special points. If $\tau_\eta^X \in \Gamma(\tilde D, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$ is represented by $(\tau_x)_{x\in\Sigma}$, then*

$$(u_x)_{x\in\Sigma} + (\tau_x)_{x\in\Sigma} = 0$$

in $N_D = \Gamma(\tilde D, g^*\overline{\mathcal{M}}^{\mathrm{gp}})^*$.

The following is an encapsulation of balancing which gives easy to use restrictions on curve classes realized by punctured maps with given contact orders.

Proposition 2.51. *Suppose given a punctured curve $(C/W,\mathbf{p},f)$ with W a log point, $\mathbf{p}=\{p_1,\ldots,p_n\}$. For $s\in\Gamma(X,\overline{\mathcal{M}}_X^{\mathrm{gp}})$, denote by $\mathcal{L}_s^\times$ the corresponding torsor, i.e., the inverse image of s under the homomorphism $\mathcal{M}_X^{\mathrm{gp}}\to\overline{\mathcal{M}}_X^{\mathrm{gp}}$, and write $\mathcal{L}_s$ for the corresponding line bundle. Further, the stalk of s at $f(p_i)$ lies in $P_{p_i}^{\mathrm{gp}}$ and hence defines a homomorphism $P_p^*\to\mathbb{Z}$, which we write as $\langle\cdot,s\rangle$. Then*

$$\deg \underline{f}^*\mathcal{L}_s=-\sum_{i=1}^n\langle u_{p_i},s\rangle.$$

Proof. First, by making a base-change, we can assume W is the standard log point. Note $\underline{f}^*\mathcal{L}_s$ must be the line bundle $\mathcal{L}_{\bar f^\flat(s)}$ associated to the torsor corresponding to $\bar f^\flat(s)$.

Now the value of the total degree of $\mathcal{L}_{\bar f^\flat(s)}$ can be calculated using Lemma 2.49 and details of the proof of [GS13, Proposition 1.15]. Let $\tilde D$ be the normalization of an irreducible component of C with generic point η, $g:\tilde D\to C$ the obvious map. Then

$$\begin{aligned}\deg(f\circ g)^*\mathcal{L}_s&=\deg g^*\mathcal{L}_{\bar f^\flat(s)}\\&=\tau_\eta^C(\varphi(s))\\&=\sum_{q\in\tilde D}\frac{1}{e_q}\big(\langle V_\eta,s\rangle-\langle V_{\eta_q},s\rangle\big)-\sum_{x_i\in\tilde D}\langle u_{x_i},s\rangle,\end{aligned}$$

in the notation of [GS13, Propositions 1.14,1.15], and the last equality coming from the proof of [GS13, Proposition 1.15]. Summing over all irreducible components, the left-hand side becomes $\deg\underline{f}^*\mathcal{L}_s$ and on the right-hand side, all the contributions from the nodes cancel, giving

$$\deg\underline{f}^*(\mathcal{L}_s)=-\sum_i\langle u_{x_i},s\rangle,$$

as desired. ♠

2.7.2. *The puncturing ideal.* We end this subsection by giving a tropical interpretation for the puncturing ideal $\mathcal{K}_W$ associated to a punctured map.

Proposition 2.52. *Suppose given a sharp toric monoid Q, and a collection of sharp toric monoids $P_{p_1},\ldots,P_{p_r}$ along with monoid homomorphisms $\varphi_{p_i}:P_{p_i}\to Q\oplus\mathbb{Z}$ with $u_{p_i}:=\mathrm{pr}_2\circ\varphi_{p_i}$. Let $\mathrm{ev}_i:=(\mathrm{pr}_1\circ\varphi_{p_i})^t:Q_{\mathbb{R}}^\vee\to(P_{p_i})_{\mathbb{R}}^\vee$. Let the ideal $I\subset Q$ be the monoid ideal*

$$I=\langle \mathrm{pr}_1\circ\varphi_{p_i}(m)\,|\,\textit{there is an } i \textit{ such that } m\in P_{p_i} \textit{ and } u_{p_i}(m)<0\rangle.$$

For σ a face of the cone $Q_{\mathbb{R}}^\vee$, let $Z_\sigma=\operatorname{Spec}\Bbbk[\sigma^\perp\cap Q]$ be the closed toric stratum of $\operatorname{Spec}\Bbbk[Q]$ corresponding to σ. Then there is a decomposition

$$\operatorname{Spec}\Bbbk[Q]/\sqrt{I}=\bigcup_\sigma Z_\sigma$$

where the union is over all faces σ of $Q_{\mathbb{R}}^\vee$ such that if $x\in\operatorname{Int}(\sigma)$, then $\mathrm{ev}_i(x)+\epsilon u_{p_i}\in(P_{p_i})_{\mathbb{R}}^\vee$ for $\epsilon>0$ sufficiently small and $1\le i\le r$.

Proof. Let $I_{p_i} \subset Q$ be the monoid ideal

$$I_{p_i} = \langle \mathrm{pr}_1 \circ \varphi_{p_i}(m) \,|\, m \in P_{p_i} \text{ satisfies } u_{p_i}(m) < 0 \rangle.$$

Of course $V(I) = \bigcap_i V(I_{p_i})$. We first show that if σ satisfies the given condition, then $Z_\sigma \subseteq V(I_{p_i})$ for each i. The monomial ideal defining Z_σ is $Q \smallsetminus (\sigma^\perp \cap Q)$, so it is enough to show that $\sigma^\perp \cap I_{p_i} = \emptyset$. Choose an $x \in \mathrm{Int}(\sigma)$. Let $q \in I_{p_i}$ be a generator of I_{p_i}, i.e., there exists an $m \in P_{p_i}$ such that $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ and $u_{p_i}(m) < 0$. Since $m \in P_{p_i}$ and $\mathrm{ev}_i(x) + \epsilon u_{p_i} \in (P_{p_i})^\vee_{\mathbb{R}}$ for some $\epsilon > 0$, we have

$$0 \leq \langle \mathrm{ev}_i(x) + \epsilon u_{p_i}, m \rangle.$$

Thus $\langle u_{p_i}, m \rangle < 0$ implies $\langle \mathrm{ev}_i(x), m \rangle > 0$, or $\langle x, \mathrm{pr}_1(\varphi_{p_i}(m)) \rangle = \langle x, q \rangle > 0$, as desired.

Conversely, suppose that $Z_\sigma \subseteq V(I)$ for some face σ of $Q^\vee_{\mathbb{R}}$, but there exists an i and some $x \in \mathrm{Int}(\sigma)$ such that $\mathrm{ev}_i(x) + \epsilon u_{p_i} \notin (P_{p_i})^\vee_{\mathbb{R}}$ for any $\epsilon > 0$. Then there exists an $m \in P_{p_i}$ such that $\langle \mathrm{ev}_i(x) + \epsilon u_{p_i}, m \rangle < 0$ for all $\epsilon > 0$. Since $\langle \mathrm{ev}_i(x), m \rangle \geq 0$, we must have $\langle \mathrm{ev}_i(x), m \rangle = 0$ and $u_{p_i}(m) < 0$. Thus $q = \mathrm{pr}_1(\varphi_{p_i}(m))$ lies in I_{p_i}. We have $\langle x, q \rangle = \langle \mathrm{ev}_i(x), m \rangle = 0$, so $q \in \sigma^\perp$. In particular, z^q does not vanish on Z_σ, contradicting $Z_\sigma \subseteq V(I)$. ♠

Remark 2.53. The above proposition gives an immediate tropical interpretation for the zero locus of the puncturing ideal, ignoring the scheme structure. Indeed, suppose that the data in the above proposition arises from a punctured curve $f : C^\circ \to X$ with C defined over $W = \mathrm{Spec}(Q \to \kappa)$, with punctures $p_1, \ldots, p_r$. Tropicalizing gives a family of tropical curves (2.15). Fixing $x \in \Sigma(W) = Q^\vee_{\mathbb{R}}$ yields a tropical curve $\Sigma(f) : \Gamma \to \Sigma(X)$. Let η be the generic point of the irreducible component of C containing the punctured point p_i, and v_η the vertex of Γ corresponding to η. Then ev_i can be viewed as the evaluation map $\mathrm{ev}_i : \Sigma(W) \to \Sigma(X)$ of $\Sigma(f)$ at the vertex v_η. The condition in the above proposition on σ then says that for $x \in \mathrm{Int}(\sigma)$, the affine length of the leg of Γ corresponding to each p_i is non-zero.

3. The stack of punctured maps

3.1. Algebraicity.

3.1.1. *The set-up and the statement.* We fix a morphism locally of finite presentation and separated logarithmic schemes $X \to B$ as the target with $\mathcal{M}_X$ Zariski.

Denote by $\mathscr{M}_{g,n}(X/B)$ the category of stable punctured maps to $X \to B$ with genus g, m-marked punctured curves fibered over the category of fine and saturated logarithmic schemes. By Proposition 2.25 this is the pullback of the corresponding category of *basic* stable punctured maps fibered over the category of schemes.

Let $\mathscr{M}_{g,n}(\underline{X}/\underline{B})$ be the algebraic stack over $\underline{B}$ parameterizing usual stable maps to the family of underlying schemes $\underline{X} \to \underline{B}$. We view $\mathscr{M}_{g,n}(\underline{X}/\underline{B})$ as the logarithmic stack equipped with the canonical log structure of its universal curves.

The morphism $X \to \underline{X}$ induces a morphism of fibered categories

$$\mathscr{M}_{g,n}(X/B) \to \mathscr{M}_{g,n}(\underline{X}/\underline{B}). \tag{3.1}$$

We will prove the following theorem:

Theorem 3.1. *The morphism (3.1) is representable by logarithmic algebraic spaces locally of finite presentation. In particular, $\mathscr{M}_{g,n}(X/B)$ is a logarithmic Deligne-Mumford stack locally of finite presentation.*

Lemma 3.2 and Proposition 3.3 below will imply that the morphism is represented by logarithmic algebraic stacks, locally of finite presentation. The representability property is a consequence of Proposition 2.26.

3.1.2. *Reduction to the case of universal target.* Denote by $\mathcal{A}_X$ the relative Artin fan associated to $X \to B$, see Corollary 3.3.5 of [ACMW17]. Write $\mathcal{X} := \mathcal{A}_X \times_{\mathcal{A}_B} B$. Then the morphism $X \to B$ uniquely factors through a strict morphism $X \to \mathcal{X}$. We may replace X by $\mathcal{X}$, and form the fibered category of pre-stable punctured maps $\mathfrak{M}_{g,n}(\mathcal{X}/B)$, and the stack of usual pre-stable maps $\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$. Again, we view $\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$ as a logarithmic stack with the canonical log structure of its universal curve. Similarly, the morphism $\mathcal{X} \to \underline{\mathcal{X}}$ induces a morphism

$$\mathfrak{M}_{g,n}(\mathcal{X}/B) \to \mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B}), \tag{3.2}$$

and the strict morphism $X \to \mathcal{X}$ induces a morphism

$$\mathscr{M}_{g,n}(X/B) \to \mathfrak{M}_{g,n}(\mathcal{X}/B). \tag{3.3}$$

Furthermore, the underlying morphism $\underline{X} \to \underline{\mathcal{X}}$ induces a morphism of stacks

$$\mathscr{M}_{g,n}(\underline{X}/\underline{B}) \to \mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B}). \tag{3.4}$$

Lemma 3.2. *There is a canonical isomorphism of fibered categories*

$$\mathscr{M}_{g,n}(X/B) \to \mathscr{M}_{g,n}(\underline{X}/\underline{B}) \times_{\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})} \mathfrak{M}_{g,n}(\mathcal{X}/B),$$

where the fiber product is in the fine and saturated category.

Proof. The morphism in the statement is given by (3.1) and (3.3). To see the isomorphism, observe that giving a stable punctured map $f : C^\circ \to X/B$ over W is equivalent to giving an underlying stable map $\underline{f} : \underline{C} \to \underline{X}/\underline{B}$ over $\underline{W}$ and a morphism of logarithmic structures $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ compatible with the arrows from $\mathcal{M}_B$. The latter is equivalent to a pre-stable punctured map $C^\circ \to \mathcal{X}/B$ whose underlying map is given by the composition $\underline{C} \to \underline{X} \to \underline{\mathcal{X}}$. ♠

Lemma 3.2 and Proposition 2.26 reduces Theorem 3.1 to the following:

Proposition 3.3. *The morphism $\mathfrak{M}_{g,n}(\mathcal{X}/B) \to \mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$ as in (3.2) is a morphism between logarithmic algebraic stacks locally of finite presentation.*

3.1.3. *Moduli of punctured curves with a fixed log curve.* Let $\pi : C \to W$ be a genus g, n-marked logarithmic curve over W. Define W^p to be the fibered category over $\underline{W}$-schemes defined as follows.

For any strict morphism $T \to W$, the objects in $W^p(T)$ are punctured curves $C_T^\circ \to C_T \to T$ with punctures given by the markings of C_T. Here $C_T = C \times_W T \to T$ is the pull-back of the logarithmic curve $C \to W$. Pull-backs in W^p are defined as pull-backs of punctured curves along strict morphisms over W.

In other words, W^p parameterizes punctured curves with the logarithmic curves given by $C \to W$. We prove that

Proposition 3.4. *The tautological morphism $W^p \to W$ obtained by removing the punctured curve is, locally on W^p, a locally closed embedding.*

Proof. For any object $C_T^\circ \to C_T \to T$ in $W^p(T)$, we will construct a strict and locally closed immersion $V \to W$ with the punctured curve $C_V^\circ \to C_V \to V$ such that $T \to W$ factors through V, and $C_T^\circ \to C_T \to T$ is the pull-back of $C_V^\circ \to C_V \to V$. Furthermore, we will show that such an object $C_V^\circ \to C_V \to V$ is universal with respect to the above property.

Since the statement is local on both W and T, shrinking both W and T, we may assume there is a chart $h : Q = \overline{\mathcal{M}}_{W,w} \to \mathcal{M}_W$ which pulls back to a chart $h_T : Q = \overline{\mathcal{M}}_{T,t} \to \mathcal{M}_T$ for a point $w \in W$ and a fixed point $t \in T$ over w.

For each puncture p and a generic point η in $\underline{C}_t$ with $p \in \mathrm{cl}(\eta)$, consider the generization map $\chi_{\eta,p} : \overline{\mathcal{M}}_{C_t^\circ,p} \to \overline{\mathcal{M}}_\eta = Q$. Let $\overline{\mathcal{I}}_t \subset Q$ be the ideal generated by the image $\chi_{\eta,p}(\overline{\mathcal{M}}_{C_{T,t}^\circ} \smallsetminus \overline{\mathcal{M}}_{C_t})$ for each puncture $p \in \mathbf{p}_t$. Since both $\overline{\mathcal{M}}_{C_{T,t}^\circ}$ and $\overline{\mathcal{M}}_{C_t}$ are fine monoids, the ideal $\overline{\mathcal{I}}_t$ is finitely generated. Using the chart h, the ideal $\overline{\mathcal{I}}_t \subset Q$ generizes to a coherent sheaf of ideals $\overline{\mathcal{I}}_W$ and $\overline{\mathcal{I}}_T$.

Let $\mathcal{I}_W := \mathcal{M}_W \times_{\overline{\mathcal{M}}_W} \overline{\mathcal{I}}_W$ and $\mathcal{I}_T := \mathcal{M}_T \times_{\overline{\mathcal{M}}_T} \overline{\mathcal{I}}_T$ be the corresponding coherent log-ideals on W and T respectively. It follows from the construction that $\mathcal{I}_T$ is the pull-back of the log-ideal $\mathcal{I}_W$ via $T \to W$. Denote by $V \to W$ the strict closed immersion defined by the ideal $\alpha_{\mathcal{M}_W}(\mathcal{I}_W)$. Further shrinking T if necessary, by (2) of Definition 2.1 the image $\alpha_{\mathcal{M}_T}(\mathcal{I}_T)$ is the zero ideal. Thus $T \to W$ factors through $V \subset W$. Denote by i the morphism $T \to V$.

We next construct the punctured curves $C_V^\circ \to C_V \to V$. To construct the sheaf of monoids $\overline{\mathcal{M}}_{C_V^\circ}$, first notice that the inclusion $\overline{\mathcal{M}}_{C_V} \subset \overline{\mathcal{M}}_{C_V^\circ}$ is an isomorphism away from the points of $\mathbf{p}$. For each puncture $p_w \in \mathbf{p}_w$ over w, we define $\overline{\mathcal{M}}_{C_V^\circ,p_w} := \overline{\mathcal{M}}_{C_T^\circ,p_t}$ using the fiber over the fixed point t. Further shrinking T, we may assume there is a chart $\overline{\mathcal{M}}_{C_T^\circ,p_t} \to \mathcal{M}_{C^\circ}|_{p_T}$ along the puncture $p_T \in \mathbf{p}$. Shrinking W hence V accordingly, we may assume that there is a chart $\overline{\mathcal{M}}_{C_w,p_t} \to \mathcal{M}_{C_V}|_{p_V}$. We may then extend the fiber $\overline{\mathcal{M}}_{C_V^\circ,p_w}$ along the punctured marking p_V via generization. This defines the subsheaf of fine monoids $\overline{\mathcal{M}}_{C_V^\circ} \subset \overline{\mathcal{M}}_{C_V}^{\mathrm{gp}}$.

Consider $\mathcal{M}_{C_V^\circ} := \mathcal{M}_{C_V}^{\mathrm{gp}} \times_{\overline{\mathcal{M}}_{C_V}^{\mathrm{gp}}} \overline{\mathcal{M}}_{C_V^\circ}$. Observe that $\mathcal{M}_{C_V} \subset \mathcal{M}_{C_V^\circ}$. We define the structure morphism $\alpha_{\mathcal{M}_{C_V^\circ}} : \mathcal{M}_{C_V^\circ} \to \mathcal{O}_{C_V}$ as follows. First, we require

$\alpha_{\mathcal{M}_{C_V^\circ}}|_{\mathcal{M}_{C_V}} = \alpha_{\mathcal{M}_{C_V}}$. Second, for a local section δ of $\mathcal{M}_{C_V^\circ}$ not contained in $\mathcal{M}_{C_V}$, we define $\alpha_{\mathcal{M}_{C_V^\circ}}(\delta) = 0$. This defines a monoid homomorphism. Indeed, if we use the decomposition $\mathcal{M}_{C_V^\circ} \subseteq \mathcal{M} \oplus_{\mathcal{O}^\times} \mathcal{P}^{\mathrm{gp}}$, writing $\delta = \delta' \cdot \delta''$ with δ' the pull-back of a section of $\mathcal{M}_V$, it is sufficient to check that $\alpha_V(\delta') = 0$. However, this follows from the definition of the defining ideal of V. This defines a logarithmic structure $\mathcal{M}_{C_V^\circ}$ over $\underline{C}_V$. The inclusion of logarithmic structures $\mathcal{M}_{C_V} \subset \mathcal{M}_{C_V^\circ}$ is a puncturing, hence defines the punctured curve $C_V^\circ \to C_V \to V$.

We check that $C_T^\circ \to C_T \to T$ is the pull-back of $C_V^\circ \to C_V \to V$ via $i : T \to V$. Since $C_T \to T$ is given by the pull-back of $C_V \to V$, it suffices to show that $i^*\mathcal{M}_{C_V^\circ} = \mathcal{M}_{C_T^\circ}$ as sub-sheaves of monoids in $\mathcal{M}_{C_T}^{\mathrm{gp}}$. Away from the punctures, the equality clearly holds. Along each puncture $p \in \mathbf{p}_T$, we have the equality $i^*\overline{\mathcal{M}}_{C_V^\circ, p_w} = \overline{\mathcal{M}}_{C_T^\circ, p_t}$ at p_t which extends along the marking p by generization. This proves the desired equality.

Finally, consider another closed immersion $V' \to W$ and a family of punctured curves $C_{V'}^\circ \to C_{V'} \to V'$ such that $C_{V'} = C \times_W V'$, the morphism $T \to W$ factors through V', and $C_T^\circ \to C_T \to T$ is the pull-back of $C_{V'}^\circ \to C_{V'} \to V'$. Then $\alpha_{\mathcal{M}_{V'}}(\mathcal{I}_W|_{V'})$ is the zero ideal on V' as it contains the punctured curve over $t \in T$. Hence the inclusion $V' \to W$ factors through V. The same construction above shows that $C_{V'}^\circ \to C_{V'} \to V'$ is the pull-back of $C_V^\circ \to C_V \to V$. This proves the desired universal property. ♠

3.1.4. *Proof of Proposition 3.3.* By [AW18, ACMW17], the morphism $\mathcal{X} \to B$ is locally of finite presentation, quasi-separated, and having affine stabilizers. By [HR14, Theorem 1.2], the stack $\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$ is an algebraic stack locally of finite presentation. Recall that $\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$ is viewed as a logarithmic algebraic stack equipped with the canonical log structure of its universal curve.

Consider a strict morphism $W \to \mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$. We will show that the product in the fine and saturated category

$$\mathfrak{W} := \mathfrak{M}_{g,n}(\mathcal{X}/B) \times_{\mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})} W$$

is represented by a logarithmic algebraic stack locally of finite presentation.

Consider Olsson's log stack $V := \mathrm{Log}_{W \times_{\underline{B}} B}$ as in [Ols03].[3] Pulling-back the universal families via the composition of logarithmic stacks $V \to W \to \mathfrak{M}_{g,n}(\underline{\mathcal{X}}/\underline{B})$, we obtain a family of underlying pre-stable maps $\underline{f} : \underline{C} \to \underline{\mathcal{X}}/\underline{B}$ over $\underline{V}$, and a logarithmic curve $C \to V$ over $\underline{C} \to \underline{V}$. Denote by $U := V^p$ the logarithmic stack over V introduced in Section 3.1.3. By Proposition 3.4, the stack U is a logarithmic algebraic stack. Pulling-back the universal families, we have an underlying pre-stable map $\underline{f}_U : \underline{C}_U \to \underline{\mathcal{X}}/\underline{B}$ over $\underline{U}$, a punctured curve $C_U^\circ \to C_U \to U$, and a morphism $U \to B$.

[3]Departing from Olsson's notation, we write Log_W for the stack parameterizing only fine and saturated logarithmic structures.

Denote by $\mathcal{M} := \underline{f}_U^*\mathcal{M}_{\mathcal{X}} \oplus_{\mathcal{M}_B} \mathcal{M}_U$ in the category of coherent logarithmic structures. Consider the fibered category

$$H := \mathrm{Hom}_{\mathrm{Sch}/\underline{U}}(\mathcal{M}, \mathcal{M}_{C_U^\circ})$$

which associates, to each strict morphism $T \to U$, the category of morphisms of logarithmic structures $\mathcal{M}|_{C_T^\circ} \to \mathcal{M}_{C_T^\circ}$ where $C_T^\circ := C_U^\circ \times_U T$. By [Wis16, Proposition 2.1], the projection $H \to U$ is representable by algebraic spaces locally of finite presentation. Hence H is a logarithmic algebraic stack locally of finite presentation. The universal morphism $f_H^\flat : \mathcal{M}|_{C_H^\circ} \to \mathcal{M}_{C_H^\circ}$ and the pull-back $\underline{f}_H : \underline{C}_H \to \underline{\mathcal{X}}/\underline{B}$ of $\underline{f} : \underline{C} \to \underline{\mathcal{X}}/\underline{B}$ defines a punctured map $f_H : C_H^\circ \to \mathcal{X}/B$ over H.

The universal punctured maps define a tautological morphism $\mathfrak{W} \to H$. By the construction of H and the universal property of basic objects in Proposition 2.28, this morphism identifies $\mathfrak{W}$ with the sub-stack of H parameterizing pre-stable basic punctured maps. By Proposition 2.15 and Proposition 2.24, $\mathfrak{W}$ is identified with an open sub-stack of H. Therefore, $\mathfrak{W}$ is a logarithmic algebraic stack locally of finite presentation.

This completes the proof of Proposition 3.3.

3.2. **Boundedness.**

3.2.1. *The classes of punctured maps.* In what follows, we will need to make a choice of a notion of *degree data* for curves in X; we will write the group of degree data as $H_2(X)$. This could be 1-cycles on X modulo algebraic or numerical equivalence, or it could be $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$. If we work over $\mathbb{C}$, we can use ordinary singular homology $H_2(X, \mathbb{Z})$. In general, any family of stable maps $\underline{f} : \underline{C}/\underline{W} \to \underline{X}$ should induce a well-defined class $\underline{f}_*[C_{\bar{w}}] \in H_2(X)$ for $\bar{w} \in \underline{W}$ a geometric point. If $\underline{W}$ is connected, this class should be independent of the choice of $\bar{w}$.

Definition 3.5. A *class* of stable punctured maps to X/B with Artin fan $\mathcal{A}_X$ Zariski consists of data $\beta = \big(g, \vec{\mathbf{u}}_{\mathbf{p}} = (\mathbf{u}_{p_j})_{p_j \in \mathbf{p}}, A\big)$ where:

(1) g is the genus of the source curve.
(2) $\mathbf{u}_{p_j}$ is the connected component of contact orders of $\mathcal{A}_X$ along the j-th punctured point p_j.
(3) $A \in H_2(X)$ is a curve class.

Similarly we call $\beta' = \big(g, \vec{\mathbf{u}}_{\mathbf{p}}\big)$ a *class* of punctured maps to $\mathcal{X}/B$.

For simplicity, we may write $\vec{\mathbf{u}} = \vec{\mathbf{u}}_{\mathbf{p}}$. We also introduce the notation $\underline{\beta} = (g, k, A)$ for the discrete data of underlying stable maps to $\underline{X}/\underline{B}$, where k is the number of punctured points.

Since contact orders in $\vec{\mathbf{u}}$ are given by connected components of contact orders, we obtain a decomposition

Lemma 3.6. *There are decompositions by disjoint unions of open and closed substacks*

$$\mathscr{M}(X/B) = \coprod_{\beta} \mathscr{M}(X/B,\beta) \quad \textit{and} \quad \mathfrak{M}(\mathcal{X}/B) = \coprod_{\beta'} \mathfrak{M}(\mathcal{X}/B,\beta')$$

where $\mathscr{M}(X/B,\beta)$ and $\mathfrak{M}(\mathcal{X}/B,\beta')$ parameterize punctured maps with the given classes β and β' respectively.

We state our result on boundedness:

Theorem 3.7. *Suppose the underlying family $X \to B$ is projective, the Artin fan $\mathcal{A}_X$ is Zariski, and the sheaf $\overline{\mathcal{M}}_X$ is generated by its global sections. Then the projection $\mathscr{M}(X/B,\beta) \to B$ is of finite type.*

Proof. We split the proof into several steps. This theorem will follow from Propositions 3.10 and 3.11. ♠

Remark 3.8. We remark in the case where all points are marked rather than punctured, [ACMW17] proved this result without any hypotheses on $\overline{\mathcal{M}}_X$. Here, this hypothesis is used in two ways. The first way is similar to its use in [GS13], Theorem 3.8 to bound the numbers of types of tropical curves. Here, we do this in Proposition 3.11. The second use is to apply Proposition 2.46: if a connected contact order has an infinite number of irreducible components, then it may well be that a moduli space of stable punctured maps with such a contact order at a punctured point is not of finite type.

Even if this second issue did not potentially cause problems, we would still be unable to prove the stronger finiteness result of [ACMW17] because we have not shown an analogue of the invariance of punctured invariants under log étale modifications shown in the ordinary marked case in [AW18]. Indeed, the story seems to be rather more subtle in the punctured case, and we leave this to future work.

3.2.2. *Boundedness of $\mathscr{M}(X/B,\beta)$.*

Definition 3.9. A class β is called *combinatorially finite* if the set of types (see Definition 2.20) of stable punctured maps of class β is finite.

Proposition 3.10. *Suppose β is combinatorially finite. Then the forgetful map $\mathscr{M}(X/B,\beta) \to \mathscr{M}(\underline{X}/\underline{B},\underline{\beta})$ is of finite type.*

Proof. The strategy of the proof is similar to those in [GS13, Section 3.2] and [Che14, Section 5.4] by showing that each stratum with constant combinatorial structure is bounded. The proof is largely the same, with extra care needed only in the proof of [GS13, Prop. 3.17]. Let $\mathfrak{f} = (\underline{C}/\underline{W}, \mathbf{p}, \underline{f})$ be a combinatorially constant (in the sense of [GS13, Def. 3.15]) ordinary stable map over an integral, quasi-compact scheme $\underline{W}$. Then $\underline{W} \times_{\mathscr{M}(\underline{X}/\underline{B},\underline{\beta})} \mathscr{M}(X/B,\beta)$ classifies punctured enhancements of the ordinary stable maps parameterized by $\underline{W}$. As the combinatorial type of a log curve is locally constant, we have a decomposition

$$\underline{W} \times_{\mathscr{M}(\underline{X}/\underline{B},\underline{\beta})} \mathscr{M}(X/B,\beta) = \coprod_{\mathbf{u}} \mathscr{M}(X,\mathfrak{f},\mathbf{u})$$

into disjoint open substacks according to the type $\mathbf{u}$. If β is combinatorially finite, this is a finite union, and hence it is sufficient to show quasi-compactness of each $\mathscr{M}(X,\mathfrak{f},\mathbf{u})$. As in the proof of [GS13, Prop. 3.17], it is sufficient to construct a quasi-compact stack Z with a morphism $Z \to \mathscr{M}(X,\mathfrak{f},\mathbf{u})$ which is surjective on geometric points.

To do so, set $Q_1 := \mathbb{N}^k$, where k is the number of nodes of any fibre of $\underline{C} \to \underline{W}$. By Section 2.3.1 and the fact we have fixed the type $\mathbf{u}$, the basic monoid Q is constant on $\mathscr{M}(X,\mathfrak{f},\mathbf{u})$. Then there is a canonical morphism $Q_1 \to Q$ (see Section 2.3.1), which induces a morphism of Artin cones $\mathcal{A}_{Q^\vee} \to \mathcal{A}_{Q_1^\vee}$. We equip $\underline{W}$ with the canonical log structure coming from the family of nodal pre-stable curves $\underline{C} \to \underline{W}$, and consider $Z_1 = \mathcal{A}_{Q^\vee} \times_{\mathcal{A}_{Q_1^\vee}} W$. Pulling back the universal family from W, we obtain a family of log curves $C_1 \to Z_1$ and a usual stable map $\underline{f} : \underline{C}_1 \to \underline{X}/\underline{B}$. Observe that there is a global chart $Q \to \overline{\mathcal{M}}_{Z_1}$. By Theorem 3.1 the morphism (3.1) is locally of finite type, and therefore we can replace Z_1 with its reduction.

The type $\mathbf{u}$ prescribes, for each puncture $p \in \mathbf{p}$, an element u_p which determines over each geometric point of Z_1 a commutative diagram (2.8). Further observe that the commutative diagram (2.8) varies globally constantly along Z_1. We thus obtain a monoid ideal $\overline{\mathcal{K}} \subset Q$ as in (2.11) by taking into account all punctures in $\mathbf{p}$. Denote by $\mathcal{K} = \overline{\mathcal{K}} \times_{\overline{\mathcal{M}}_{Z_1}} \mathcal{M}_{Z_1}$ where the arrow on the left is given by the composition $\overline{\mathcal{K}} \to Q \to \overline{\mathcal{M}}_{Z_1}$ with the last arrow the global chart. Noting that to obtain a family of punctured stable maps of type $\mathbf{u}$ over Z_1 then requires that $\alpha_{Z_1}(\mathcal{K}) = 0$ by Theorem 2.32. Thus in particular if $0 \in \overline{\mathcal{K}}$, then there are no punctured maps of type $\mathbf{u}$ and we can ignore such a $\mathbf{u}$; otherwise, as Z_1 is reduced and $\overline{\mathcal{M}}_{Z_1}$ is locally constant with stalk Q, necessarily $\alpha_{Z_1}(\mathcal{K}) = 0$.

We now construct a punctured family of curves $C_1^\circ \to Z_1$. First, the ghost sheaf $\overline{\mathcal{M}}_{C_1^\circ}$ is identical to $\overline{\mathcal{M}}_{C_1}$ away from punctures. Along each puncture $p \in \mathbf{p}$, we take $\overline{\mathcal{M}}_{C_1^\circ,p} \subset \overline{\mathcal{M}}^{\mathrm{gp}}_{C_1,p}$ to be the smallest fine submonoid generated by $\overline{\mathcal{M}}_{C_1,p}$ and the image of φ_p as in (2.8). As all the characteristic sheaves and morphisms between them are globally constant along Z_1, this yields a well-defined sheaf of monoids $\overline{\mathcal{M}}_{C_1^\circ}$, hence $\mathcal{M}_{C_1^\circ} := \overline{\mathcal{M}}_{C_1^\circ} \times_{\overline{\mathcal{M}}^{\mathrm{gp}}_{C_1}} \mathcal{M}^{\mathrm{gp}}_{C_1}$ over $\underline{C}_1$.

We define the structural morphism $\alpha_{\mathcal{M}_{C_1^\circ}} : \mathcal{M}_{C_1^\circ} \to \mathcal{O}_{C_1}$ as follows. First, we require $\alpha_{\mathcal{M}_{C_1^\circ}}|_{\mathcal{M}_{C_1}} = \alpha_{\mathcal{M}_{C_1}}$. For a local section δ of $\mathcal{M}_{C_1^\circ}$ not contained in $\mathcal{M}_{C_1}$, we defined $\alpha_{\mathcal{M}_{C_1^\circ}}(\delta) = 0$, as away from punctures it generizes to a section in $\mathcal{K}$, hence is the zero section in $\mathcal{O}_{C_1}$. This defines a logarithmic structure $\mathcal{M}_{C_1^\circ}$, hence the desired punctured curve $C_1^\circ \to Z_1$.

The remainder of the proof is now identical to that of [GS13, Prop. 3.17]. ♠

3.2.3. *Finiteness of the combinatorial data.* In order to complete the proof that $\mathscr{M}(X/B,\beta)$ is finite type, it remains to bound the combinatorial data.

Proposition 3.11. *Suppose $\overline{\mathcal{M}}_X$ is generated by its global sections. Then any class β is combinatorially finite.*

Proof. It is sufficient to show that for any combinatorially constant family of ordinary stable maps $(\underline{C}/\underline{W},\mathbf{p},\underline{f})$ in the sense of [GS13, Def. 3.15], there are only finitely many combinatorial types of liftings of such a family to a punctured log curve of type β. This is essentially identical to the proof of [GS13, Thm. 3.8], However, there are two small points of difference. First, in following the argument of [GS13, Thm. 3.8], we must fix the u_p's, as the positivity argument to obtain boundedness for any choice of u_p's does not apply, as the u_p need not be positive. Thus we must use the fact, shown in Proposition 2.46, that a connected contact order only has a finite number of irreducible components to obtain a finite number of possible choices for u_p for each $p\in\mathbf{p}$ for the given family of ordinary stable maps. ♠

3.3. **Valuative criterion.** We now show stable reduction for basic stable punctured maps, which allows us to conclude properness of the moduli spaces of such maps. Recall that for a given class $\beta=(g,\mathbf{u},A)$ of stable punctured maps to $X\to B$, we have the class $\underline{\beta}=(g,k,A)$ for usual stable maps to $\underline{X}\to\underline{B}$ by removing contact orders. We will show that

Theorem 3.12. *The tautological morphism removing all logarithmic structures*

$$\mathscr{M}(X/B,\beta)\to\mathscr{M}(\underline{X}/\underline{B},\underline{\beta})$$

satisfies the weak valuative criterion for properness.

Proof. In what follows, we assume given R a discrete valuation ring over $\underline{B}$ with maximal ideal $\mathfrak{m}$, residue field $\kappa=R/\mathfrak{m}$, and fraction field K. Suppose we have a commutative square of solid arrows of the underlying stacks:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathscr{M}(X/B,\beta) \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathscr{M}(\underline{X}/\underline{B},\underline{\beta}). \end{tikzcd}$$

We want to show that possibly after replacing K with a finite extension $\tilde{K}$ and R by an appropriate discrete valuation ring in $\tilde{K}$, there is a dashed arrow marking the above diagram commutative, and is unique up to a unique isomorphism.

The top arrow of the above diagram yields a stable punctured map

$$(\pi_K: C_K\to\operatorname{Spec}(Q_K\to K),\mathbf{p}_K,f_K)$$

over the logarithmic point $\operatorname{Spec}(Q_K\to K)$. The bottom arrow of the above diagram yields a usual stable map $(\underline{C}/\operatorname{Spec} R,\mathbf{p},\underline{f})$ with its generic fiber given by the underlying stable map of f_K. To construct the dashed arrow, it suffices to extend the stable punctured map f_K across the closed point $0 \in \operatorname{Spec} R$ with the given underlying stable map $\underline{f}$. The task is to then extend the logarithmic structures and morphisms thereof. The proof is almost identical to that of [GS13], Theorem 4.1. Since that proof is quite long, we only note the salient differences.

Section 4.1 of [GS13] accomplishes this extension at the level of ghost sheaves; in particular, [GS13], Proposition 4.3, which states that the type of the central fibre is uniquely determined by the type of the generic fibre, carries through with u_p for a puncture p determined as for marked points. Indeed, if p is a punctured point on $\underline{C}_0$ in the closure of the punctured point p_K on $\underline{C}_K$, then we must have u_p being the composition

$$P_p \longrightarrow P_{p_K} \xrightarrow{u_{p_K}} \mathbb{Z}, \tag{3.5}$$

where the first map is the generization map $(\underline{f}^*\mathcal{M}_X)_p \to (\underline{f}^*\mathcal{M}_X)_{p_K}$. In particular, the contact orders u_p and u_{p_K} are contained in the same connected component specified in β.

By Definition 2.14 and Section 2.2, the type of the central fibre then determines the extension $\overline{\mathcal{M}}_{C^\circ}$ of $\overline{\mathcal{M}}_{C^\circ_K}$ and a map $\bar{f}^\flat : \underline{f}^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_{C^\circ}$ extending the corresponding map on the generic fibre. Here $\overline{\mathcal{M}}_{C^\circ}$ is defined at punctures via Corollary 2.6.

Next, [GS13], §4.2 shows that the logarithmic structure on the base $\operatorname{Spec} R$ is uniquely defined. In this argument, marked points play no role, and the argument remains unchanged in the punctured case. In particular, this produces a unique choice of logarithmic structure $\mathcal{M}_R$ on $\operatorname{Spec} R$, which in addition comes with a morphism of logarithmic structures $\mathcal{M}_R^0 \to \mathcal{M}_R$ where $\mathcal{M}_R^0$ is the basic logarithmic structure (pulled back from the moduli space of pre-stable curves $\mathbf{M}$ with its basic logarithmic structure, see [GS13], Appendix A) associated to the family $\underline{C} \to \operatorname{Spec} R$. In particular, one obtains a logarithmic structure $(\underline{C}, \mathcal{M}'_C) = (\operatorname{Spec} R, \mathcal{M}_R) \times_{(\operatorname{Spec} R, \mathcal{M}_R^0)} (\underline{C}, \mathcal{M}_C^0)$, where $\mathcal{M}_C^0$ is the logarithmic structure pulled back from the basic logarithmic structure of the universal curve over $\mathscr{M}(\underline{X}/\underline{B}, \underline{\beta})$. The logarithmic structure $\mathcal{M}'_C$ then has logarithmic marked points along the punctures p, but there is a sub-logarithmic structure $\mathcal{M}_C \subset \mathcal{M}'_C$ which only differs in that the punctures are no longer marked.

By Corollary 2.6, there is a natural inclusion $\overline{\mathcal{M}}_{C^\circ} \subset (\overline{\mathcal{M}}'_C)^{\mathrm{gp}}$. We form $\mathcal{M}_{C^\circ} := \overline{\mathcal{M}}_{C^\circ} \times_{(\overline{\mathcal{M}}'_C)^{\mathrm{gp}}} (\mathcal{M}'_C)^{\mathrm{gp}}$. We then define a structure homomorphsim $\alpha_{C^\circ} : \mathcal{M}_{C^\circ} \to \mathcal{O}_C$ by $\alpha_{C^\circ}|_{\mathcal{M}'_C} = \alpha_{C'}$ and $\alpha_{C^\circ}(\mathcal{M}_{C^\circ} \setminus \mathcal{M}'_C) = 0$. To show that this is a homomorphism, it is enough to show that if $s \in \mathcal{M}_{C^\circ,p} \setminus \mathcal{M}'_{C,p}$, writing $s = (s_1, s_2)$ as a stalk of $\mathcal{M}_C \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$, then $\alpha_C(s_1) = 0$. But necessarily $(\bar{s}_1, \bar{s}_2) = \bar{f}^\flat(m)$ for some $m \in P_p$ with $u_p(m) < 0$. Write for points $x, x' \in \underline{C}$ with x in the closure of x' the generization map $\chi_{x',x} : P_x \to P_{x'}$. Then $u_{p_K}(\chi_{p_K,p}(m)) = u_p(m)$ by (3.5). Thus $u_{p_K}(\chi_{p_K,p}(m)) < 0$ and necessarily $\alpha_{C_K}(s_1|_{C_K}) = 0$. But since C is reduced and C_K is dense in C, this implies $\alpha_C(s_1) = 0$, as desired. Thus we have a punctured log scheme C°.

We can now extend $f_K^\flat : f_K^*\mathcal{M}_X \to \mathcal{M}_{C_K^\circ}$ to $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_{C^\circ}$ as in §4.3 of [GS13]. ♠

4. The perfect obstruction theory

Throughout this section, we fix a log smooth morphism $X \to B$ with $\mathcal{M}_X$ Zariski and $n \in \mathbb{N}$. As in §3.1.2, $X \to B$ factors over its relative Artin fan $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$. Denote by $\mathscr{M}_n(X/B)$ and by $\mathfrak{M}_n(\mathcal{X}/B)$ the stacks of n-punctured maps to $X \to B$ and to $\mathcal{X} \to B$, respectively. In §§4.1 and 4.2, we construct two perfect relative obstruction theories, in the sense of [BF97, Def. 4.4], one for $\mathscr{M}_n(X/B) \to \mathfrak{M}_n(\mathcal{X}/B)$ and one for a related morphism $\mathscr{M}_n(X/B) \to \mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$, where the latter space incorporates data of maps to X at a set of special points on the domain curve, see (4.12). Working over $\mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$ is crucial for understanding gluing at a virtual level in §5.2.4.

Finally in §4.3, we explore the local structure of $\mathfrak{M}_n(\mathcal{X}/B)$ and in turn the local structure of $\mathfrak{M}_n^{\mathrm{ev}}(\mathcal{X}/B)$, the latter being smooth over $\mathfrak{M}_n(\mathcal{X}/B)$. This is done by studying the forgetful morphism $\mathfrak{M}_n(\mathcal{X}/B) \to \mathbf{M} \times B$, where $\mathbf{M}$ is the Artin stack of prestable basic log curves. In the case that there are no punctures, that is, all points are marked, then in fact this morphism is log smooth, as was shown in [AW18]. Now, however, the morphism is only idealized log smooth, with the idealized structure given by the puncturing log ideal. This tells us that smooth locally $\mathfrak{M}_n(\mathcal{X}/B)$ looks like a closed subscheme of a toric variety defined by the puncturing ideal. We give examples showing that $\mathfrak{M}_n(\mathcal{X}/B)$ need not be pure dimensional. Thus the relative obstruction theorydoes not in general define a virtual fundamental class on $\mathscr{M}_n(X/B)$, but rather a virtual pullback map

$$A_*(\mathfrak{M}_n(\mathcal{X}/B)) \to A_*(\mathscr{M}_n(X/B))$$

via [Man12].

4.1. Obstruction theories for logarithmic maps from pairs.

All cases of interest fit into the following general setup. Let S be a log stack over B and assume we are given a proper and representable morphism of fine log stacks

$$Y \longrightarrow S,$$

with underlying map of ordinary stacks $\underline{Y} \to \underline{S}$ flat and relatively Gorenstein. The fibres of this morphism serve as domains for a space of logarithmic maps. In the application, Y is either the universal curve over $S = \mathfrak{M}_n(\mathcal{X}/B)$ or a union of sections in this universal curve with induced log structure. To avoid adjusting for shifts of dimension in the formulas, we denote by ω_π the relative dualizing complex of a relatively Gorenstein morphism π, that is, the complex with the invertible relative dualizing sheaf shifted to the left by the relative dimension. As a target, we take a composition of log smooth morphisms of fine log stacks

$$V \longrightarrow W \longrightarrow B.$$

We assume further given an S-morphism $Y \to W$ defining a commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & W \arrow[d] \\ S \arrow[r] & B \end{tikzcd}$$

Let M be an open algebraic substack of the stack over S with objects over an affine S-scheme T commutative diagrams

(4.1)
$$\begin{tikzcd} Y_T \arrow[rr] \arrow[d] \arrow[dr] & & V \arrow[d] \\ T \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\ & S \arrow[r] & B \end{tikzcd}$$

where the square formed by Y_T, T, S and Y is cartesian. Thus we are interested in lifting the map $Y \to W$ to V fibrewise relative S.[4] We endow M with the log structure making the morphism $M \to S$ strict. The pull-back of Y to M defines the universal domain $\pi : Y_M \to M$. We have the following 2-commutative diagram of stacks

(4.2)
$$\begin{tikzcd} Y_M \arrow[rr, "f"] \arrow[d, "\pi"'] \arrow[dr] & & V \arrow[d] \\ M \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\ & S \arrow[r] & B \end{tikzcd}$$

Functoriality of log cotangent complexes [Ols05, 1.1(iv)] yields the morphism

$$f^*\Omega_{V/W} = Lf^*\mathbb{L}_{V/W} \longrightarrow \mathbb{L}_{Y_M/Y} = \pi^*\mathbb{L}_{M/S}. \tag{4.3}$$

The equality on the left holds by [Ols05, 1.1 (iii)] since $V \to W$ is log smooth, while the equality on the right follows since $\mathbb{L}_{M/S} = \mathbb{L}_{\underline{M}/\underline{S}}$ and $\mathbb{L}_{Y_M/Y} = \mathbb{L}_{\underline{Y}_M/\underline{Y}}$ by strictness of $M \to S$ [Ols05, 1.1(ii)] and then using compatibility of the ordinary cotangent complexes with flat pull-back by $\underline{\pi}^*$.

Since $\underline{Y} \longrightarrow \underline{S}$ is relatively Gorenstein by assumption, so is $\underline{Y}_M \longrightarrow \underline{M}$ and we have a natural isomorphism of exact functors $\pi^! = \pi^* \otimes \omega_\pi$. Thus (4.3)

[4]In the application, M is the stack of punctured maps of interest, S is a stack of punctured maps to the relative Artin fan $\mathcal{X}$ of $X \to B$ and $V \to W \to B$ is the composition $X \to \mathcal{X} \to B$. Thus our deformation theory fixes both the domain of the punctured map to X and the map to the relative Artin fan $\mathcal{X}$. In this case, $V \to W$ is strict and we could indeed work with ordinary cotangent complexes throughout, but for possible other applications we do not make this assumption.

is equivalent to a morphism $f^*\Omega_{V/W} \otimes \omega_\pi \to \pi^!\mathbb{L}_{M/S}$, which by adjunction is equivalent to a morphism

$$\Phi : \mathbb{E} \longrightarrow \mathbb{L}_{M/S} \tag{4.4}$$

with $\mathbb{E} = R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi)$. The most transparent proof that Φ is a perfect obstruction theory for M over S is based on the fact that the construction of Φ is functorial. For lack of reference we provide a proof for this well-known property in the following lemma. If $T \to M$ is any map, denote by

$$\Phi_T : \mathbb{E}_T \to \mathbb{L}_{T/S}$$

the morphism in (4.4) constructed from (4.1) instead of (4.2).

Lemma 4.1. *The construction of Φ in (4.4) is functorial in the following sense: Let $\underline{T} \to \underline{M}$ be a morphism of stacks. Denoting $T \to M$ the associated strict morphism of log stacks, we obtain the commutative diagram*

$$\begin{tikzcd}
Y_T \arrow[r, "\tilde{h}"'] \arrow[d, "\pi_T"'] \arrow[rrr, bend left, "f_T"] & Y_M \arrow[rr, "f"] \arrow[d, "\pi"'] \arrow[dr] & & V \arrow[d] \\
T \arrow[r, "h"] & M \arrow[dr] & Y \arrow[r] \arrow[d] & W \arrow[d] \\
& & S \arrow[r] & B
\end{tikzcd}$$

with the two squares of domains cartesian. Then we have a commutative square

$$\begin{tikzcd}
Lh^*\mathbb{E} \arrow[r, "Lh^*\Phi"] \arrow[d, "\beta"] & Lh^*\mathbb{L}_{M/S} \arrow[d] \\
\mathbb{E}_T \arrow[r, "\Phi_T"] & \mathbb{L}_{T/S},
\end{tikzcd}$$

with left-hand vertical arrow a natural isomorphism and the right-hand vertical arrow defined by functoriality of cotangent complexes.

Proof. Naturality of the base change map [Sta17, Rem. 07A7] applied to $f^*\Omega_{V/W} \otimes \omega_\pi \to \mathbb{L}_{Y_M/Y} \otimes \omega_\pi$ together with $f \circ \tilde{h} = f_T$ and $\tilde{h}^*\omega_\pi = \omega_{\pi_T}$, leads to the commutative square

$$\begin{tikzcd}
Lh^*R\pi_*(f^*\Omega_{V/W} \otimes \omega_\pi) \arrow[r] \arrow[d, "\beta"'] & Lh^*R\pi_*(\mathbb{L}_{Y_M/Y} \otimes \omega_\pi) \arrow[d] \\
R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) \arrow[r] & R\pi_{T*}(L\tilde{h}^*\mathbb{L}_{Y_M/Y} \otimes \omega_{\pi_T}).
\end{tikzcd}$$

Now $\mathbb{L}_{Y_M/Y} \simeq \pi^*\mathbb{L}_{M/S}$, as remarked after (4.3), and hence the adjunction counit $R\pi_*\pi^! \to 1$ applied in the construction of Φ in (4.4) is given by the projection formula followed by the trace isomorphism,

$$R\pi_*(\pi^*\mathbb{L}_{M/S} \otimes \omega_\pi) \xrightarrow{\simeq} \mathbb{L}_{M/S} \otimes R\pi_*(\omega_\pi) \xrightarrow{\mathrm{Tr}_{\omega_\pi}} \mathbb{L}_{M/S}.$$

Thus the upper horizontal sequence composed with Lh^* of this adjunction counit isomorphism yields $Lh^*\Phi$.

Similarly, extending the lower horizontal arrow by the map induced by functoriality of cotangent complexes,

$$L\tilde{h}^*\mathbb{L}_{Y_M/Y} \longrightarrow \mathbb{L}_{Y_T/Y} = \pi_T^*\mathbb{L}_{T/S}.$$

composed with the adjunction counit isomorphism $R\pi_{T*}(\pi_T^*\mathbb{L}_{T/S} \otimes \omega_{\pi_T}) \simeq \mathbb{L}_{T/S}$ for π_T retrieves the definition of Φ_T. By compatibility of both the projection formula [Sta17, Lem. 0B6B] and the trace morphism [Sta17, Lem. 0E6C] with base change, the induced map $Lh^*\mathbb{L}_{M/S} \to \mathbb{L}_{T/S}$ agrees with the map defined by functoriality of cotangent complexes. This establishes the claimed commutative diagram.

The claim on β follows from the general base change statement [Sta17, Lem. 0A1K] applied to $\pi : Y_M \to M$, with $f^*\Omega_{V/W}$ for the object in $D_{\mathrm{QCoh}}(\mathcal{O}_{Y_M})$ and with ω_π as complex of π-flat quasi-coherent sheaves. ♠

Proposition 4.2. *The morphism $\Phi : \mathbb{E} \to \mathbb{L}_{M/S}$ constructed in (4.4) is an obstruction theory for $M \to S$ in the sense of* [BF97, Def. 4.4].

Proof. We check the obstruction-theoretic criterion [BF97, Thm. 4.5.3], applied in the setting relative to S, similarly to ordinary logarithmic maps carried out in [GS13, Prop, 5.1].

Assume given a morphism $h : T \to M$, a square zero extension $T \to \overline{T}$ with ideal sheaf $\mathcal{J}$ and a morphism $\overline{T} \to S$, with log structures turning all three morphisms strict. This situation leads to the following commutative diagram:

$$\begin{tikzcd}
 & Y_T \arrow[rr, "\tilde{h}"] \arrow[rrrr, bend left, "f_T"] \arrow[dl] \arrow[dd, "\pi_T"] & & Y_M \arrow[rr, "f"] \arrow[dl] \arrow[dd, "\pi"] & & V \arrow[dl] \\
Y_{\overline{T}} \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] & & W \arrow[dd] & \\
 & T \arrow[rr, "h"] \arrow[dl] & & M \arrow[dl] & & \\
\overline{T} \arrow[rr] & & S \arrow[rr] & & B. &
\end{tikzcd}$$

All sides of the cube on the left are cartesian, but not in general the bottom and top faces.

The obstruction class $\omega(h) \in \mathrm{Ext}^1(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$ for extending h to a map $\overline{T} \to M$ is the composition

$$Lh^*\mathbb{L}_{M/S} \longrightarrow \mathbb{L}_{T/\overline{T}} \longrightarrow \tau_{\geq -1}\mathbb{L}_{\underline{T}/\underline{\overline{T}}} = \mathcal{J}[1]$$

with the first arrow defined by functoriality of the cotangent complex, see [Ill71, Prop. 2.2.4] with $X_0 = T$, $X = \overline{T}$, $Y_0 = Y = M$ and $Z_0 = Z = S$. Because all morphisms are strict we can use the log cotangent complex in this definition [Ols05, 1.1(ii)].

Now $\Phi^*\omega(h)$ is the composition of this morphism with $Lh^*\Phi : Lh^*\mathbb{E} \to Lh^*\mathbb{L}_{M/S}$. By functoriality of our obstruction theory (Lemma 4.1), this composition also has the factorization

$$\mathbb{E}_T = R\pi_{T*}(f_T^*\Omega_{V/W} \otimes \omega_{\pi_T}) \xrightarrow{\Phi_T} \mathbb{L}_{T/S} \longrightarrow \tau_{\geq -1}\mathbb{L}_{T/\overline{T}} = \mathcal{J}[1],$$

which by adjunction is equivalent to the composition

$$f_T^*\Omega_{V/W} \otimes \omega_{\pi_T} \longrightarrow \mathbb{L}_{Y_T/Y} \otimes \omega_{\pi_T} \longrightarrow \tau_{\geq -1}\pi_T^!\mathbb{L}_{T/\overline{T}} = \pi_T^*\mathcal{J}[1] \otimes \omega_{\pi_T},$$

Up to tensoring with ω_{π_T} this is the obstruction class for extending $f_T : Y_T \to V$ to $Y_{\overline{T}}$, as a morphism over W. By our assumption on the objects of M, this extension exists if and only if $T \to M$ extends to $\overline{T}$. This shows the part of the criterion concerning the obstruction.

A similar argument shows that once $\omega(h) = 0$ the space of extensions form a torsor under $\mathrm{Ext}^0(Lh^*\mathbb{L}_{\underline{M}/\underline{S}}, \mathcal{J})$, showing the second part of the criterion. ♠

After this recapitulation of obstruction theories for logarithmic maps with proper and relatively Gorenstein domains, we are now in position to bring in point conditions. Abstractly we consider a composition of proper, representable morphisms of fine log stacks

$$Z \xrightarrow{\iota} Y \longrightarrow S, \tag{4.5}$$

with maps of schemes underlying $Z \to S$ and $Y \to S$ flat and relatively Gorenstein as before. Note that while ι may not be flat and hence cannot be considered relatively Gorenstein following the usual convention, one can still define a relative dualizing sheaf

$$\omega_\iota = \omega_{Z/S} \otimes \iota^*\omega_{Y/S}^*. \tag{4.6}$$

fulfilling relative duality, hence defining a right-adjoint functor $\iota^!$ to $R\iota_*$. This works as in the case of smooth morphisms discussed e.g. in [Huy06, §3.4].

We now have another algebraic stack N over S with objects given by diagrams as in (4.1) with Y replaced by Z. We assume that composition with $\iota : Z \to Y$ defines a morphism of stacks

$$\varepsilon : M \longrightarrow N. \tag{4.7}$$

As in (4.4) we now obtain two obstruction theories, one for $M \to S$, the other for $N \to S$,

$$\Phi : \mathbb{E} \longrightarrow \mathbb{L}_{M/S}, \qquad \Psi : \mathbb{F} \longrightarrow \mathbb{L}_{N/S}. \tag{4.8}$$

In our application, $Y \to S$ is some universal curve and $Z \to Y$ a strict closed embedding with morphism to S scheme-theoretically étale. In this case, Ψ is a trivial obstruction theory for a number of points in V/W and in particular, étale locally $\mathbb{F}$ can be taken as the direct sum of the pull-back of $\Omega_{V/W}$ by scheme-theoretic maps from $\underline{N}$ to $\underline{V}$.

Proposition 4.3. *The two obstruction theories Φ and Ψ in (4.8) fit into a commutative square*

$$\begin{tikzcd} L\varepsilon^*\mathbb{F} \arrow[r, "L\varepsilon^*\Psi"] \arrow[d] & L\varepsilon^*\mathbb{L}_{N/S} \arrow[d] \\ \mathbb{E} \arrow[r, "\Phi"] & \mathbb{L}_{M/S}, \end{tikzcd}$$

with the right-hand vertical morphism given by functoriality of the cotangent complex.

Proof. Consider the following commutative diagram with the left four squares cartesian.

$$\begin{tikzcd} Z \arrow[d, "\iota"'] & Z_N \arrow[l] \arrow[d] \arrow[dd, bend right, "p"'] \arrow[rr, bend left, "g"] & Z_M \arrow[l, "\tilde\varepsilon"] \arrow[r, "h"] \arrow[d, "\iota_M"] \arrow[dd, bend right, "p_M"'] \arrow[dr] & V \arrow[d] \\ Y \arrow[d] & Y_N \arrow[l] \arrow[d] & Y_M \arrow[l] \arrow[r] \arrow[d, "\pi"] \arrow[ur, "f"'] & W \arrow[d] \\ S & N \arrow[l] & M \arrow[l, "\varepsilon"] \arrow[r] & B \end{tikzcd}$$

The left column is the given morphism (4.5) of domains, the lower horizontal row contains the restriction morphism ε from (4.7) and the morphism to the base S, while $f : Y_M \to V$ and $g : Z_N \to V$ are the respective universal morphisms defined on the universal domains $Y_M \to M$ and $Z_N \to N$.

The obstruction theory Ψ in (4.8) was defined by applying $Rp_*(. \otimes \omega_p)$ to $g^*\Omega_{V/W} \to \mathbb{L}_{Z_N/Z} = p^*\mathbb{L}_{N/S}$. By functoriality of obstruction theories (Lemma 4.1), the pull-back $L\varepsilon^*\Psi$ is similarly obtained by applying $Rp_{M*}(\,\cdot\, \otimes \omega_{p_M})$ to

$$h^*\Omega_{V/W} \longrightarrow L\tilde\varepsilon^*\mathbb{L}_{Z_N/Z} = L\tilde\varepsilon^* p^*\mathbb{L}_{N/S} = p_M^* L\varepsilon^*\mathbb{L}_{N/S}, \tag{4.9}$$

followed by the adjunction counit $Rp_{M*}p_M^! \to 1$ using $p_M^! = p_M^* \otimes \omega_{p_M}$. Now consider the composition of the morphism in (4.9) with p_M^* of the functoriality morphism $L\varepsilon^*\mathbb{L}_{N/S} \to \mathbb{L}_{M/S}$ and take the tensor product with ω_{p_M} to obtain

$$h^*\Omega_{V/W} \otimes \omega_{p_M} \longrightarrow p_M^* L\varepsilon^*\mathbb{L}_{N/S} \otimes \omega_{p_M} \longrightarrow p_M^!\mathbb{L}_{M/S}. \tag{4.10}$$

Adjunction turns this sequence into the composition of the upper horizontal and right vertical arrows of the commutative square in the assertion:

$$L\varepsilon^*\mathbb{F} = Rp_{M*}(h^*\Omega_{V/W} \otimes \omega_{p_M}) \xrightarrow{L\varepsilon^*\Psi} L\varepsilon^*\mathbb{L}_{N/S} \longrightarrow \mathbb{L}_{M/S}. \tag{4.11}$$

On the other hand, observing $h = f \circ \iota_M$, $\omega_{p_M} = \iota_M^*\omega_\pi \otimes \omega_{\iota_M}$ and $\iota_M^! = L\iota_M^* \otimes \omega_{\iota_M}$, we can rewrite domain and image of the morphism in (4.10) as

$$h^*\Omega_{V/W} \otimes \omega_{p_M} = \iota_M^* f^*\Omega_{V/W} \otimes \iota_M^*\omega_\pi \otimes \omega_{\iota_M} = \iota_M^!(f^*\Omega_{V/W} \otimes \omega_\pi)$$

and

$$p_M^!\mathbb{L}_{M/S} = p_M^*\mathbb{L}_{M/S}\otimes\omega_{p_M} = \iota_M^!(\pi^*\mathbb{L}_{M/S}\otimes\omega_\pi),$$

respectively. The adjunction counit $R\iota_{M*}\iota_M^! \to \mathrm{id}$ applied to (4.10) thus produces the commutative diagram

$$\begin{tikzcd}
R\iota_{M*}(h^*\Omega_{V/W}\otimes\omega_{p_M}) \arrow[r] \arrow[d, equal] & R\iota_{M*}(p_M^!\mathbb{L}_{M/S}) \arrow[d, equal] \\
R\iota_{M*}\iota_M^!(f^*\Omega_{V/W}\otimes\omega_\pi) \arrow[d] & R\iota_{M*}\iota_M^!(\pi^*\mathbb{L}_{M/S}\otimes\omega_\pi) \arrow[d] \\
f^*\Omega_{V/W}\otimes\omega_\pi \arrow[r] & \pi^*\mathbb{L}_{M/S}\otimes\omega_\pi.
\end{tikzcd}$$

The claimed morphism of obstruction theories now follows by (4.11) from the result of applying $R\pi_*$ to the outer square of this diagram, observing $Rp_{M*} = R\pi_*R\iota_{M*}$. ♠

4.2. **Obstruction theories with point conditions.** We are now in position to define obstruction theories for moduli spaces of stable logarithmic maps with prescribed point conditions. Recall the log smooth morphism $X\to B$ and its factorization over the relative Artin fan $\mathcal{X}\to B$ from the beginning of this section. We want to work relative a stack S of stable punctured maps to $\mathcal{X}/B$. Adopting the notation used otherwise in the paper we now write $\mathfrak{M}$ instead of S for the algebraic stack of domains together with the tuple of points to impose point conditions at. For example, $\mathfrak{M}$ could be $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\beta})$ as introduced in Lemma 3.6 or a similar moduli space of nodal curves with nodes labelled in addition to punctured points. Then $Y\to S=\mathfrak{M}$ is the universal curve, $Z\to Y$ a union of sections, one for each point condition, which we assume to be ordered, and we have a universal diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathfrak{M} \arrow[r] & B.
\end{tikzcd}$$

As our target we now take the composition

$$X \longrightarrow \mathcal{X} \longrightarrow B.$$

Note that $\mathcal{X}\to B$ is log étale and $X\longrightarrow\mathcal{X}$ is strict and log smooth. Hence $\underline{X}\to\underline{\mathcal{X}}$ is smooth as a morphism of schemes and we have a sequence of canonical isomorphisms

$$\mathbb{L}_{X/B} = \Omega_{X/B} = \Omega_{X/\mathcal{X}} = \Omega_{\underline{X}/\underline{\mathcal{X}}} = \mathbb{L}_{\underline{X}/\underline{\mathcal{X}}}.$$

For easier reference later on we also write $\mathscr{M}$ instead of M for the algebraic stack of punctured maps to be considered.

For the moduli space N of point conditions we take the space of factorizations of the composition $Z\to Y\to\mathcal{X}$ via $X\to\mathcal{X}$. Thinking of these factorizations as providing evaluation maps $\mathfrak{M} \to X$ at the marked points given by the sections Z of $Y \to S$, we denote the stack of such factorizations by $\mathfrak{M}^{\mathrm{ev}}$. This stack is algebraic by the fibre product description

$$\mathfrak{M}^{\mathrm{ev}} = \mathfrak{M} \times_{\underline{\mathcal{X}} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{\mathcal{X}}} (\underline{X} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{X}). \tag{4.12}$$

Here the map $\mathfrak{M} \to \underline{\mathcal{X}} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{\mathcal{X}}$ is defined by composing the sections $\mathfrak{M} \to \underline{\mathfrak{M}} \to \underline{Z}$ with the composition $\underline{Z} \to \underline{Y} \to \underline{\mathcal{X}}$ in the given order of the marked points.

With this notation, the composition $M \to N \to S$ considered in the proof of Proposition 4.3 reads

$$\mathscr{M} \xrightarrow{\varepsilon} \mathfrak{M}^{\mathrm{ev}} \longrightarrow \mathfrak{M}.$$

In §4.1 we recalled the construction of obstruction theories for $\mathscr{M}/\mathfrak{M}$ and for $\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}$, which in the situation at hand are perfect of amplitude contained in $[-1, 0]$, and showed their compatibility (Proposition 4.3). As in [Man12, Constr. 3.13], this situation provides perfect obstruction theories for $\mathscr{M}/\mathfrak{M}^{\mathrm{ev}}$ by completing the compatibility diagram in Proposition 4.3 to a morphism of distinguished triangles:

$$\begin{tikzcd}
L\varepsilon^*\mathbb{F} \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d] & \mathbb{G} \arrow[r] \arrow[d, dashed] & L\varepsilon^*\mathbb{F}[1] \arrow[d] \\
L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}} \arrow[r] & \mathbb{L}_{\mathscr{M}/\mathfrak{M}} \arrow[r] & \mathbb{L}_{\mathscr{M}/\mathfrak{M}^{\mathrm{ev}}} \arrow[r] & L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}}[1]
\end{tikzcd} \tag{4.13}$$

Note that while the isomorphism class of $\mathbb{G}$ is unique, the dashed arrow is not, so this recipe potentially provides several different obstruction theories for $\mathscr{M}/\mathfrak{M}^{\mathrm{ev}}$. Uniqueness holds, however, for the induced obstruction theories in the sense of Wise [Wis11], and hence we can ignore this subtlety in the following.

For being explicit and for later use we now work out $\mathbb{G}$. For simplicity of notation write $C \to \mathscr{M}$ for the pull-back $Y_{\mathscr{M}}$ of the universal curve $Y \to \mathfrak{M}$ to $\mathscr{M}$, while in disagreement with the general discussion write $\iota : Z \to C$ for the closed subscheme of special points rather than Z_C. Since $Z \to Y$ is the inclusion of a union of sections of the family of nodal curves $Y \to \mathfrak{M}$, we can write $Z = Z' \amalg Z''$ with Z'' contained in the critical locus of $C \to \mathscr{M}$ and Z' disjoint from it. Recall also from the setup that each connected component of Z maps isomorphically to a connected component of $\mathscr{M}$. Denote by $\kappa : \tilde{C} \to C$ the partial normalization of $\tilde{C}$ that normalizes C along Z'', but otherwise leaves C untouched. Write $\tilde{\pi} = \pi \circ \kappa : \tilde{C} \to \mathscr{M}$, $\tilde{f} = f \circ \kappa : \tilde{C} \to X$ and $\tilde{Z} = \kappa^{-1}(Z)$.

Lemma 4.4. *For the tangent-obstruction bundle in (4.13) it holds*

$$\mathbb{G} \simeq R\pi_*\big(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde{\pi}}(\tilde{Z}))\big) \simeq R\tilde{\pi}_*\big(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\tilde{Z})\big) \simeq (R\tilde{\pi}_*\tilde{f}^*\Theta_{X/B}(-\tilde{Z}))^\vee.$$

Moreover, $\mathbb{G}$ is perfect of amplitude $[-1, 0]$.

Proof. The second isomorphism follows by the projection formula, the third isomorphism by relative duality.

For the first isomorphism we start with the following exact sequence of complexes, all concentrated in degree -1:

$$0 \longrightarrow \omega_\pi \longrightarrow \kappa_*\big(\omega_{\tilde\pi}(\tilde Z)\big) \longrightarrow \iota_*\mathcal{O}_Z[1] \longrightarrow 0. \tag{4.14}$$

On the complement of the nodal locus Z'', this sequence is defined by

$$0 \longrightarrow \omega_\pi \longrightarrow \omega_\pi(Z') \longrightarrow \omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') \longrightarrow 0$$

by means of the canonical isomorphism

$$\omega_\pi \otimes_{\mathcal{O}_C} \iota_*\mathcal{O}_{Z'}(Z') = \iota_*(\iota^*\omega_\pi \otimes_{\mathcal{O}_{Z'}} \omega_\iota) \simeq \iota_*\mathcal{O}_{Z'}[1]$$

coming from the definition of ω_ι in (4.6), with the first equality arising from the projection formula and the fact that $\iota^*\omega^*_{Y/S} = \mathcal{O}_{Z'}(Z')$. Explicitly, the homomorphism $\omega_\pi(Z') \to \mathcal{O}_{Z'}[1]$ takes the residue along Z'. Near the nodal locus, (4.14) is defined by

$$0 \longrightarrow \omega_\pi \xrightarrow{\kappa^*} \kappa_*\big(\omega_{\tilde\pi}(\tilde Z)\big) \longrightarrow \iota_*\mathcal{O}_{Z''}[1] \longrightarrow 0.$$

To obtain this sequence, recall that étale locally $\omega_\pi = \Omega_{C/\mathfrak{M}}[1]$ with $\Omega_{C/\mathfrak{M}}$ the sheaf of relative logarithmic differentials for $C/\mathfrak{M}$, while $\omega_{\tilde\pi} = \Omega_{\underline{\tilde C}/\underline{\mathfrak{M}}}[1]$ with $\Omega_{\underline{\tilde C}/\underline{\mathfrak{M}}}$ the sheaf of relative ordinary differentials for $\underline{\tilde C}/\underline{\mathfrak{M}}$. In fibrewise coordinates z, w for the two branches of C along Z'' on an étale neighbourhood, $\Omega_{C/\mathfrak{M}}$ is locally generated by $z^{-1}dz = -w^{-1}dw$, hence pulls back to ordinary differentials with simple poles along $\kappa^{-1}(Z'') \subseteq \tilde Z$. The map to $\mathcal{O}_Z$ takes the difference of the residues of such rational differential forms on $\tilde C$ along the two preimages of the nodal locus.[5] This establishes sequence (4.14).

Now apply $R\pi_*$ to (4.14) tensored with $f^*\Omega_{X/B}$, and observe $\omega_{p_{\mathscr{M}}} \simeq \mathcal{O}_Z$ since $Z \to \mathscr{M}$ is étale, to obtain the claimed distinguished triangle

$$\begin{array}{ccccc}
\mathbb{E} & & \mathbb{G} & & L\varepsilon^*\mathbb{F}[1] \\
\Vert & & \Vert & & \Vert \\
R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) & \longrightarrow & R\pi_*\big(f^*\Omega_{X/B} \otimes \kappa_*(\omega_{\tilde\pi}(\tilde Z))\big) & \longrightarrow & p_{\mathscr{M}*}(h^*\Omega_{X/B})[1]
\end{array}$$

Taking cohomologies, this diagram also shows the statement about the amplitude of $\mathbb{G}$. ♠

[5]Note that this map depends on an order of the two branches along each connected component of Z''. This local ambiguity of sign is irrelevant for our purposes.

4.3. **Idealized smoothness for $\mathfrak{M}(\mathcal{X}/B) \to \mathbf{M} \times B$.** Denote by $\mathbf{M}$ the moduli stack of pre-stable curves over the ground field $\Bbbk$ with any number of marked points or genus, along with its basic log structure. See [GS13, Appendix A] for details. There is a natural forgetful morphism

$$\mathfrak{M}(\mathcal{X}/B) \to \mathbf{M} \times B \tag{4.15}$$

which remembers only the domain curve as a family of marked curves over B.

Theorem 4.5. *If $\mathbf{M} \times B$ is given the idealized structure with ideal sheaf the empty set, then the forgetful morphism (4.15) is idealized log étale.*[6]

Proof. We use short-hand $\mathfrak{M} := \mathfrak{M}(\mathcal{X}/B)$. According to the definition of idealized log étale, it is sufficient to consider a diagram of solid arrows

$$\begin{tikzcd} T_0 \arrow[r, "g_0"] \arrow[d] & \mathfrak{M} \arrow[d] \\ T \arrow[r, "g"'] \arrow[ur, dashed] & \mathbf{M} \times B \end{tikzcd} \tag{4.16}$$

where $T_0 \hookrightarrow T$ is an idealized strict closed embedding defined by a square-zero ideal J over T. We wish to show that there is a unique dashed arrow making the above diagram commutative.

Let $\mathcal{K}_{T_0} \subset \mathcal{M}_{T_0}$, $\mathcal{K}_T \subset \mathcal{M}_T$ be the ideals of the idealized log structure on T_0 and T. By strictness, $\mathcal{K}_T$ pulls-back to the ideal $\mathcal{K}_{T_0}$. Let $\mathcal{K}$ be the puncturing log ideal on $\mathfrak{M}$. Necessarily $g_0^\flat(\mathcal{K}) \subset \mathcal{K}_{T_0}$.

Denote by $f_{T_0} : C^\circ_{T_0} \to \mathcal{X}$ the punctured map over T_0 corresponding to the morphism g_0. Let $C_{T_0} \to T_0$ be the family of log curves underlying $C^\circ_{T_0} \to T_0$. The morphism $T \to \mathbf{M} \times B$ also induces a family $C_T \to T$ of log curves such that $C_T \times_T T_0 = C_{T_0}$.

We then lift C_T to a punctured curve over T lifting $C^\circ_{T_0}$ as follows. Consider the Cartesian diagram

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M}_{C^\circ_{T_0}} \arrow[d] \\ \mathcal{M}^{\mathrm{gp}}_{C_T} \arrow[r] & \mathcal{M}^{\mathrm{gp}}_{C_{T_0}} \end{tikzcd} \tag{4.17}$$

where the two vertical arrows are inclusion of sheaves of monoids. We next show that $\mathcal{M}'$ is a puncturing along markings in $\mathbf{p}$.

Indeed, as $T_0 \to T$ is a square-zero extension, the induced morphism $\overline{\mathcal{M}}^{\mathrm{gp}}_{C_T} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{C_{T_0}}$ is an isomorphism. Observe that $\mathcal{O}^*_{C_T} \subset \mathcal{M}'$, and write $\overline{\mathcal{M}}' := \mathcal{M}'/\mathcal{O}^*_{C_T}$. Then the induced morphism $\overline{\mathcal{M}}' \to \overline{\mathcal{M}}_{C^\circ_{T_0}}$ is also an isomorphism. Now the inclusion $\mathcal{M}_{C_{T_0}} \hookrightarrow \mathcal{M}_{C^\circ_{T_0}}$ implies an inclusion $\mathcal{M}_{C_T} \hookrightarrow \mathcal{M}'$, and the isomorphism $(\mathcal{M}_{C^\circ_{T_0}})|_{C_{T_0} \setminus \mathbf{p}} = \mathcal{M}_{C_{T_0}}|_{C_{T_0} \setminus \mathbf{p}}$ implies $\mathcal{M}'|_{C_T \setminus \mathbf{p}} = \mathcal{M}_{C_T}|_{C_T \setminus \mathbf{p}}$.

[6]For the definition of idealized log étale, see [Ogu18, IV §3.1].

We then check that the structure morphism $\mathcal{M}_{C_T} \to \mathcal{O}_T$ extends to $\alpha_{\mathcal{M}'}: \mathcal{M}' \to \mathcal{O}_T$. Let $\alpha_{\mathcal{M}'}|_{\mathcal{M}_{C_T}} = \alpha_{\mathcal{M}_{C_T}}$. Let e be a local section of $\mathcal{M}'$ around a puncture $p \in \mathbf{p}$ which is not contained in $\mathcal{M}_{C_T}$. Then its image $\bar{e}$ in $\overline{\mathcal{M}}' \cong \overline{\mathcal{M}}_{C^\circ_{T_0}}$ is not contained in $\overline{\mathcal{M}}_{C_{T_0}}$. By Corollary 2.6, the section $\bar{e}$ is of the form $\bar{e} = \bar{a} + \bar{f}^\flat_{T_0}(\bar{\delta})$ for some local section $\bar{a} \in \overline{\mathcal{M}}_{C_{T_0}}$ and $\bar{\delta} \in f^*_{T_0}\overline{\mathcal{M}}_{\mathcal{X}}$ such that $u_p(\bar{\delta}) < 0$. By (2.11), $\bar{f}^\flat_{T_0}(\bar{\delta}) = \bar{b} + u_p(\bar{\delta})\bar{x}_p$ for some $\bar{b} \in g^\bullet\overline{\mathcal{K}} \subset \overline{\mathcal{K}}_{T_0}$ and $\bar{x}_p \in \overline{\mathcal{M}}_{C_{T_0}}$ corresponding to the local coordinate of the puncture p. As the morphism $\overline{\mathcal{K}}_T \to \overline{\mathcal{K}}_{T_0}$ is an isomorphism, we see that any lift $b \in \mathcal{M}_T$ of $\bar{b}$ is contained in $\mathcal{K}_T$ whose image in $\mathcal{O}_T$ is zero. We thus define $\alpha_{\mathcal{M}'}(e) = 0$. This makes $C^\circ := (\underline{C}_T, \mathcal{M}')$ a punctured curve over T extending $C^\circ_{T_0} \to T_0$.

Consider the commutative diagram of solid arrows

$$\begin{tikzcd}
C^\circ_{T_0} \arrow[r, "f_{T_0}"] \arrow[d] & \mathcal{X} \arrow[d] \\
C^\circ_T \arrow[r] \arrow[ur, dashed, "f_T"'] & B
\end{tikzcd}$$

To construct the unique dashed arrow in (4.17), it remains to construct a unique dashed arrow f_T lifting f_{T_0}. Since $\mathcal{X} \to B$ is log étale, by the infinitesimal lifting property of log étale morphisms, such f_T exists and is unique. This completes the proof. ♠

Remark 4.6. Of course $\mathbf{M} \times B$ is log smooth over B. Thus Theorem 4.5 implies that $\mathfrak{M}(\mathcal{X}/B)$ is idealized log smooth over B. This implies by [Ogu18], IV Variant 3.3.5, that locally in the smooth topology, the morphism $\mathfrak{M}(\mathcal{X}/B) \to B$ is modelled on a morphism of the form $\operatorname{Spec} \Bbbk[Q]/K \to \operatorname{Spec} \Bbbk[R]$. Here B locally has a chart given by a morphism to $\operatorname{Spec} \Bbbk[R]$. In particular, if $B = \operatorname{Spec} \Bbbk$, we may take $R = 0$, in which case a smooth neighbourhood of a point $\bar{x}$ of $\mathfrak{M}(\mathcal{X}/B)$ is smooth over $\operatorname{Spec} \Bbbk[Q]/K$, where Q is the basic monoid at $\bar{x}$ and $K \subseteq Q$ is the puncturing ideal. Thus the puncturing ideal gives a key local description for $\mathfrak{M}(\mathcal{X}/B)$.

Example 4.7. Take $B = \operatorname{Spec} \Bbbk$, and consider X a smooth surface with log structure coming from a smooth rational curve $D \subseteq X$ with $D^2 = 2$. Consider a type of punctured curve of genus 0, underlying curve class $[D]$, and four punctures, $p_1, \ldots, p_4$, with contact orders $-1, -1, 2$ and 2 respectively. Consider a punctured curve $f: C^\circ \to X$ where $C = C_1 \cup C_2 \cup C_3$ has three irreducible components and two nodes $q_1 = C_1 \cap C_2$, $q_2 = C_1 \cap C_3$. We have $p_1, p_3 \in C_2$, $p_2, p_4 \in C_3$. Finally, $\underline{f}$ identifies C_1 with D and contracts C_2 and C_3. It is not difficult to check such a curve exists with $u_{q_1} = u_{q_2} = 1$.

The corresponding tropical curve Γ has three vertices, v_1, v_2, v_3, edges E_{q_1}, E_{q_2}, and legs $E_{p_1}, \ldots, E_{p_4}$. The moduli space of tropical curves of this combinatorial type is $\mathbb{R}^3_{\geq 0}$, with coordinates ρ, ℓ_1, ℓ_2, where ρ gives the distance of the image of v_1 from the origin of $\Sigma(X) = \mathbb{R}_{\geq 0}$, and ℓ_1, ℓ_2 give the length of the edges E_{q_1}, E_{q_2}.

In particular, the basic monoid for this punctured log curve is $Q = \mathbb{N}^3$, generated by ρ, ℓ_1, ℓ_2.

In this case we may easily calculate the puncturing ideal. We have contributions from each of the two punctures. Using the definition (2.11), we note that at the puncture p_i, $i = 1$ or 2, the map $\varphi_{\bar\eta} \circ \chi_{\eta,p_i} : P_{p_i} = \mathbb{N} \to Q$ is dual to $\mathrm{ev}_i : Q^\vee_{\mathbb{R}} \to (P_{p_i})^\vee = \mathbb{R}_{\geq 0}$ evaluating the tropical curve parameterized by a point at $Q^\vee_{\mathbb{R}}$ at v_i, see Proposition 2.52. Thus for $m \in Q^\vee_{\mathbb{R}}$, $\mathrm{ev}_i(m) = \rho(m) + \ell_i(m)$. Dually $\varphi_{\bar\eta} \circ \chi_{\eta,p_i} : P \to Q$ is given by $1 \mapsto \rho + \ell_i$. As $u_{p_i}(1) = -1$, $i = 1, 2$, we see the puncturing ideal K is generated by $\rho + \ell_1, \rho + \ell_2$. Writing $\Bbbk[Q] = \Bbbk[x, y, z]$, with the three variables corresponding to ρ, ℓ_1, ℓ_2 respectively, we see $\operatorname{Spec} \Bbbk[Q]/K = \operatorname{Spec} \Bbbk[x, y, z]/(xy, xz)$, which has two irreducible components of differing dimension.

To understand why it is natural to have two irreducible components, let us assume that D can be deformed inside X to a curve transversal to D. We then have two ways to deform the map f. By smoothing one or both of the nodes, we obtain a (partial) smoothing of the domain, with at least one pair p_1, p_3 or p_2, p_4 now being distinct points on the component of the domain mapping surjectively to D. Since this component now contains a negative contact order point, its image cannot be deformed away from D by Remark 2.19.

On the other hand, if one keeps the domain of f fixed, one may deform the image of C_1 away from D, so that it meets D transversally in two points (provided the geometry of X allows this). The remaining components C_2 and C_3 are then contracted to the points of intersection of $f(C_1)$ with D. It is then no longer possible to smooth the nodes.

The point of the puncturing ideal is that it captures these intrinsic singularities of the moduli space: the example given above may well be unobstructed.

5. Splitting and gluing

5.1. **Splitting punctured log maps.** The origin of the notion of puncturing arises from the fact that a stable log map, split at a node, can no longer be viewed as a stable log map. Thus punctured maps are the correct category in which to work.

Definition 5.1. Suppose given a family of punctured curves $\pi : C^\circ \to W$. A *nodal section* $q : \underline{W} \to \underline{C}$ of π is a section with image a node, so that étale locally near the image of q, $\underline{C}$ takes the form $\mathbf{Spec}\, \mathcal{O}_W[x, y]/(xy)$. The *normalization* of $\underline{C}$ at q is a morphism $\nu : \widetilde{\underline{C}} \to \underline{C}$ which is an isomorphism away from $q(\underline{W})$ and is given étale locally at the image of q by

$$\mathbf{Spec}\, \mathcal{O}_W[x] \amalg \mathbf{Spec}\, \mathcal{O}_W[y] \to \mathbf{Spec}\, \mathcal{O}_W[x, y]/(xy).$$

Note that $\nu^{-1}(q(\underline{W})) \to q(\underline{W})$ is an étale double cover, and we say that the node q is *of splitting type* if $\nu^{-1}(q(\underline{W})) \cong q(\underline{W}) \amalg q(\underline{W})$.

Proposition 5.2. *Let $(C \to W, \mathbf{p})$ be a family of punctured curves equipped with nodal sections $q_1, \ldots, q_n$ of splitting type. Let $\nu : \underline{\widetilde{C}} \to \underline{C}$ be the normalization of $\underline{C}$ at the nodes $q_1, \ldots, q_n$. Setting $\mathcal{M}_{\widetilde{C}^\circ} = \nu^* \mathcal{M}_{C^\circ}$, we obtain a family of (possibly disconnected) punctured curves*

$$(\widetilde{C}^\circ \to W, \mathbf{p}, p_{11}, p_{12}, \ldots, p_{n1}, p_{n2}),$$

with $\nu \circ p_{ij} = q_i$ for $1 \leq i \leq n$.

Proof. For each nodal section q_i, we have distinct sections $p_{i1}, p_{i2} : \underline{W} \to \underline{\widetilde{C}}$ with $\nu \circ p_{ij} = q_i$. To show that $\widetilde{C}^\circ \to W$ is a punctured curve with the given punctures, we first note that this is obvious away from the images of the p_{ij} in $\widetilde{C}$.

At an image of p_{ij}, note that étale locally along the node q_i, $\mathcal{M}_{C^\circ}$ is generated by $\pi^* \mathcal{M}_W$, s_x and s_y, where s_x, s_y are local sections of $\mathcal{M}_{C^\circ}$ near the node induced by the coordinates x, y. These are subject to the relation $s_x s_y = s_\rho$ for some section s_ρ of $\pi^* \mathcal{M}_W$, and hence $\pi^* \mathcal{M}_W$, s_y locally generate $\mathcal{M}_{C^\circ}^{\mathrm{gp}}$ as a group, with $s_x = s_\rho s_y^{-1}$. Pulling back to $\underline{\widetilde{C}}$, along the branch $x = 0$, (with $y = 0$ giving the image of the section p_{ij}), we have $(\nu^* \mathcal{M}_{C^\circ})^{\mathrm{gp}}$ locally generated by $\pi^* \mathcal{M}_W$ and $\nu^* s_y$. Further, $\nu^* s_y$ is also a section of $\mathcal{P}$, the divisorial log structure given by p_{ij}, and the image of $\nu^* s_y$ in $\overline{\mathcal{P}}$ generates $\overline{\mathcal{P}}$ as a monoid. Thus locally near p_{ij},

$$\pi^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\underline{\widetilde{C}}}} \mathcal{P} \subset \nu^* \mathcal{M}_{C^\circ} \subset \pi^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\underline{\widetilde{C}}}} \mathcal{P}^{\mathrm{gp}}.$$

Further, any element of $\nu^* \mathcal{M}_{C^\circ}$ not contained in $\pi^* \mathcal{M}_W \oplus_{\mathcal{O}^\times_{\underline{\widetilde{C}}}} \mathcal{P}$ can be written in the form $s_x^a s_y^b s_W$ with $a > 0$, $b \geq 0$ and s_W a section of $\pi^* \mathcal{M}_W$. Since $\alpha(s_x) = 0$ when $x = 0$, we see that α applied to any such element is zero. Thus $\widetilde{C}$ is a punctured curve at p_{ij}. ♠

Proposition 5.3. *Let $(C \to W, \mathbf{p}, f : C^\circ \to X)$ be a pre-stable punctured map equipped with nodal sections $q_1, \ldots, q_n$ of splitting type. Let $\nu : \underline{\widetilde{C}} \to \underline{C}$ be the normalization of $\underline{C}$ at the nodes $q_1, \ldots, q_n$. Then there is an induced pre-stable punctured log map (with possibly disconnected domain)*

$$(\widetilde{C}^\circ \to W, \mathbf{p}, p_{11}, p_{12}, \ldots, p_{n1}, p_{n2}, \tilde{f}),$$

and $\nu \circ p_{ij} = q_i$ for $1 \leq i \leq n$. Further, there is a canonical isomorphism $\mathcal{M}^{\mathrm{gp}}_{\widetilde{C}^\circ} \cong (\nu^* \mathcal{M}_{C^\circ})^{\mathrm{gp}}$. If f is stable, so is $\tilde{f}$.

Proof. We first apply Proposition 5.2 to split the domain; we obtain a strict morphism $\nu : \widetilde{C}^\circ \to C^\circ$ which we may compose with f to obtain $f : \widetilde{C}^\circ \to X$. This may not be pre-stable in the sense of Definition 2.14, but we can replace $\mathcal{M}_{\widetilde{C}^\circ}$ with a smaller log structure with the same group using Proposition 2.4 and obtain a pre-stable morphism. ♠

5.2. **Gluing punctured log maps.**

5.2.1. *The general setup.* We now want to reverse the procedure described in §5.1. To do so, we fix a target space X. For the purposes of this discussion, X will be a Zariski log scheme. However, all statements go through *mutatis mutandis* when X is replaced by a Zariski Artin fan $\mathcal{X} = \mathcal{A}_X$, or, in the case of a family of targets $X \to B$, by $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$, where $\mathcal{A}_X$ is the relative Artin fan of X over B.

For this discussion, we fix a *combinatorial type of a gluing situation*:

Definition 5.4. A *combinatorial type of a gluing situation* is a tuple $(G, \mathbf{g}, u, \mathbf{A})$, where

(1) G is a connected graph, with a set of vertices $V(G)$, a set of edges $E(G)$, and a set of legs $L(G)$.
(2) $\mathbf{g} : V(G) \to \mathbb{N}$ assigns a genus to each vertex of G;
(3) $\mathbf{A} : V(G) \to H_2(X)$ assigns an underlying curve class to each vertex.
(4) u assigns to each flag $v \in E \in E(G) \cup L(G)$ a connected component of contact orders $u_{v,E}$. We require that if $E \in E(G)$ with vertices v, v', then $u_{v,E}$ and $u_{v',E}$ are opposite contact orders in the sense of Definition 2.47.

Note that given a combinatorial type of gluing situation, associated to each vertex v we have a class of stable punctured curve $\boldsymbol{\beta}(v)$. This includes the underlying curve class $\mathbf{A}(v)$, the genus $\mathbf{g}(v)$, and the collection of contact orders $u_{v,E}$ for each flag $v \in E \in E(G) \cup L(G)$.

In such a situation, we can define a *glued class* of stable punctured curve $\boldsymbol{\beta}^{\mathrm{gl}}$ whose underlying curve class is $A = \sum_{v \in V(G)} \mathbf{A}(v)$, underlying genus is $g = b_1(G) + \sum_{v \in G} \mathbf{g}(v)$, and with one puncture for each $E \in L(G)$, with contact order $u_{v,E}$.

Definition 5.5. Given a combinatorial type of a gluing situation, we define a *gluing situation* to be the additional data of fs log schemes or fs log algebraic stacks W_v for $v \in V(G)$ equipped with morphisms $W_v \to \mathscr{M}(X/B, \boldsymbol{\beta}(v))$ (or $\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}(v))$ or other variants as appropriate). In particular, we are given a family of punctured maps

$$(\pi_v : C_v \to W_v, \mathbf{p}_v = \{p_{v,E} \,|\, v \in E \in E(G) \cup L(G)\}, f_v : C_v^\circ \to X)$$

over B.

In what follows, the parameter spaces W_v can be taken to be fs log schemes or fs log algebraic stacks, but to simplify the language here, we will stick to the category of log schemes. The statements are true *mutatis mutandis* in the category of fs log stacks.

Definition 5.6. Given a gluing situation as above, we define the *stack of gluings* to be the category fibred over Log_B, whose objects consist of log schemes T defined over B along with the following data:

(1) A morphism $T \to \mathscr{M}(X/B, \boldsymbol{\beta}^{\mathrm{gl}})$ inducing a punctured map
$$(\pi_T : C_T \to T, \mathbf{p} = \{p_{v,E} \,|\, v \in E \in L(G)\}, f_T : C_T^\circ \to X)$$
over B. We are also given nodal sections of splitting type $q_E : \underline{T} \to \underline{C}_T$ indexed by $E \in E(G)$.

(2) For each vertex $v \in E(G)$ a morphism $\psi_{v,T} : T \to W_v$, yielding a pull-back punctured map $f_{v,T} : C_{v,T}^\circ \to X$.

(3) Let $\tilde{f}_T : \widetilde{C}_T^\circ \to X$ denote the splitting of f_T along the nodes q_E, in the sense of Proposition 5.3. Then we are also given an isomorphism of punctured maps
$$(\tilde{f}_T : \widetilde{C}_T^\circ \to X) \cong (\coprod f_{v,T} : \coprod_{v \in V(G)} C_{v,T}^\circ \to X)$$
compatible with an isomorphism
$$\underline{C}_T \cong \left(\coprod_{v \in V(G)} \underline{C}_{v,T} \right) / \langle p_{v_1,E} = p_{v_2,E} \rangle.$$
where we range over all edges E with endpoints v_1, v_2. Under this isomorphism, the nodal section q_E has image $p_{v_1,E} = p_{v_2,E}$ for v_1, v_2 the endpoints of E.

Morphisms in the category of gluings are given by strict morphisms $T_1 \to T_2$ over B with isomorphisms of the data over T_1 with the pull-back of the data over T_2.

We will show that the stack of gluings is represented by a log algebraic stack W.

In the special case that the given morphism $W_v \to \mathscr{M}(X/B, \boldsymbol{\beta}(v))$ is the identity, we write the stack of gluings as $\mathscr{M}^{\mathrm{gl}}(X/B, G, \boldsymbol{\beta})$.

Proposition 5.7. *The log stack $\mathscr{M}^{\mathrm{gl}}(X/B, G, \boldsymbol{\beta})$ is algebraic, and the canonical morphism $\mathscr{M}^{\mathrm{gl}}(X/B, G, \boldsymbol{\beta}) \to \mathscr{M}(X/B, \boldsymbol{\beta}^{\mathrm{gl}})$ is finite, representable and strict.*

Proof. Denoting as usual by $\mathbf{M}_{g,n}$ the stack of pre-stable logarithmic curves of genus g and n marked points, with its basic log structure, we write
$$\underline{\mathbf{M}}^{\mathrm{gl}} = \prod_{v \in V(G)} \underline{\mathbf{M}}_{\mathbf{g}(v), n(v)},$$
where $n(v)$ is the valency of the vertex v. There is an obvious gluing map
$$\underline{\mathrm{gl}} : \underline{\mathbf{M}}^{\mathrm{gl}} \to \underline{\mathbf{M}}_{g,n}$$
where g is the genus of $\boldsymbol{\beta}^{\mathrm{gl}}$ and $n = \#L(G)$. This gluing map identifies the marked point $p_{v,E}$ with $p_{v',E}$, whenever v, v' are the two vertices of an edge $E \in E(G)$. We then give $\mathbf{M}^{\mathrm{gl}}$ the pull-back log structure, yielding a strict morphism gl. Note that gl is finite and representable. Further, the pull-back of the universal curve $\mathbf{C}^{\mathrm{gl}} \to \mathbf{M}^{\mathrm{gl}}$ comes with nodal sections $q_E : \underline{\mathbf{M}}^{\mathrm{gl}} \to \underline{\mathbf{C}}^{\mathrm{gl}}$ of splitting type, with the image of q_E the glued node produced by gluing $p_{v,E}$ and $p_{v',E}$.

There is a forgetful morphism $\mathscr{M}(X/B,\boldsymbol{\beta}^{\mathrm{gl}})\to\mathbf{M}_{g,n}$, and consider a morphism

$$T\to\mathscr{M}(X/B,\boldsymbol{\beta}^{\mathrm{gl}})\times_{\mathbf{M}_{g,n}}\mathbf{M}^{\mathrm{gl}}.$$

Giving such a morphism is equivalent to giving the following data:

- A punctured map $f_T: C_T^\circ\to X$ as in Definition 5.6, (1).
- For each $v\in V(G)$, a family of curves $C_{v,T}\to T$ which is the pull-back of the universal curve over $\mathbf{M}_{\mathbf{g}(v),n(v)}$ via the composed morphism

$$T\to\mathscr{M}(X/B,\boldsymbol{\beta}^{\mathrm{gl}})\times_{\mathbf{M}_{g,n}}\mathbf{M}^{\mathrm{gl}}\to\mathbf{M}^{\mathrm{gl}}\to\mathbf{M}_{\mathbf{g}(v),n(v)},$$

  where the last morphism is projection to $\underline{\mathbf{M}}_{\mathbf{g}(v),n(v)}$ at the level of underlying stacks. It is defined at the logarithmic level by noting that after partially normalising the universal curve $\mathbf{C}^{\mathrm{gl}}$ at the nodes q_E and restricting to a connected component indexed by v, one obtains a family of curves of genus $\mathbf{g}(v)$ with $n(v)$ marked points over $\mathbf{M}^{\mathrm{gl}}$, and hence a tautological morphism $\mathbf{M}^{\mathrm{gl}}\to\mathbf{M}_{\mathbf{g}(v),n(v)}$.
- Let $\widetilde{C}_T$ denote the partial normalization of C_T at the nodes q_E, viewing the points of $\widetilde{C}_T$ mapping to normalized nodes as marked points. Then we are given an isomorphism $\widetilde{C}_T\cong\coprod_{v\in V(G)}C_{v,T}$ compatible with an isomorphism

$$\underline{C}_T\cong\left(\coprod_{v\in V(G)}\underline{C}_{v,T}\right)/\langle p_{v_1,E}=p_{v_2,E}\rangle.$$

  where we range over all edges E with endpoints v_1,v_2.

By splitting the morphism f_T along the nodes of splitting type q_E, $E\in E(G)$, one obtains punctured maps $f_{v,T}: C_{v,T}^\circ\to X$. Now the class of this punctured map need not be $\boldsymbol{\beta}(v)$. However, as the class of a punctured map is locally constant in families, there is an open and closed substack $\mathscr{M}^{\mathrm{gl}}(X/B,G,\boldsymbol{\beta})$ of $\mathscr{M}(X/B,\boldsymbol{\beta}^{\mathrm{gl}})\times_{\mathbf{M}_{g,n}}\mathbf{M}^{\mathrm{gl}}$ such that if the morphism from T factors through this open and closed substack, $f_{v,T}$ is of class $\boldsymbol{\beta}(v)$. In this case the punctured map induces a morphism $\psi_{v,T}:T\to\mathscr{M}(X/B,\boldsymbol{\beta}(v))$ for each v, and hence we obtain the data (1)-(3) of Definition 5.6. Thus it is clear that the log stack $\mathscr{M}^{\mathrm{gl}}(X/B,G,\boldsymbol{\beta})$ represents the stack of gluings. Further, since $\mathbf{M}^{\mathrm{gl}}\to\mathbf{M}_{g,n}$ is strict, finite, and representable, the morphism $\mathscr{M}^{\mathrm{gl}}(X/B,G,\boldsymbol{\beta})\to\mathscr{M}(X/B,\boldsymbol{\beta}^{\mathrm{gl}})$ is also strict, finite and representable. ♠

Proposition 5.8. *Suppose given a general gluing situation. Then the stack of gluings is represented by a log algebraic stack. If each family $f_v: C_v^\circ/W_v\to X$ is basic, then so is the universal glued family $f: C^\circ/W\to X$.*

Proof. We consider a Cartesian diagram in the fs log category

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d] & \prod_v W_v \arrow[d] \\ \mathscr{M}^{\mathrm{gl}}(X/B, G, \boldsymbol{\beta}) \arrow[r, "\psi"'] & \prod_v \mathscr{M}(X, \boldsymbol{\beta}(v)) \end{tikzcd}$$

Here the lower morphism $\psi = \prod_v \psi_v$ is given by the universal splitting map. It is then clear that W represents the stack of gluings of the families parameterized by the W_v.

In the basic case, the right-hand vertical arrow is strict, and hence so is the left-hand vertical arrow. Further $\mathscr{M}^{\mathrm{gl}}(X/B, G, \boldsymbol{\beta}) \to \mathscr{M}(X/B, \boldsymbol{\beta}^{\mathrm{gl}})$ is strict. This shows the family of glued punctured maps over W is basic. ♠

There are two significant issues concerning gluing left unresolved in the previous discussion. The first is that to be useful, one needs to understand how virtual fundamental classes behave under gluing. The second is that often one needs an explicit description of glued families, and the previous subsection is only useful at a theoretical level. We deal with these two issues in the following sub-sections.

5.2.2. *Gluing via fibred products.* In a slightly more restrictive situation, we can describe, given a gluing situation, the gluing via a fibre product, in analogy with the case of ordinary stable maps. Indeed, suppose given a gluing situation, and suppose we wish to glue the underlying stable maps $\underline{f}_v : \underline{C}_v \to \underline{X}$. Then there is a standard Cartesian diagram

(5.1)
$$\begin{tikzcd} \underline{V} \arrow[r] \arrow[d] & \prod_v \underline{W}_v \arrow[d, "\mathrm{ev}"] \\ \prod_{E\in E(G)} \underline{X} \arrow[r, "\Delta"'] & \prod_{v\in E\in E(G)} \underline{X} \end{tikzcd}$$

where $\underline{V}$ parameterizes the glued family of ordinary stable maps to $\underline{X}$. Here ev is the evaluation map, with the component indexed by $v \in E \in E(G)$ the composition of the projection $\prod_v \underline{W}_v \to \underline{W}_v$ and the evaluation map $f_v \circ p_{v,E}$: $\underline{W}_v \to \underline{X}$ of the stable map f_v at $p_{v,E}$. The diagonal Δ is the product of the diagonal morphisms $\underline{X} \to \underline{X} \times \underline{X}$, taking the copy of X indexed by $E \in E(G)$ to the product of copies of X indexed by $v_1 \in E$, $v_2 \in E$.

In the case that we are working with a family of targets $\underline{X} \to \underline{B}$, the glued stable map $\underline{f} : \underline{C}/\underline{V} \to \underline{X}$ can be composed with $\underline{X} \to \underline{B}$. This composition is constant on fibres of $\underline{C} \to \underline{V}$ as the fibres are connected, since G is assumed to be connected. Thus we obtain a glued stable map to the family of targets $\underline{X} \to \underline{B}$.

Unfortunately, the story is slightly more complex in the logarithmic category. In particular, we will require:

Assumption 5.9. *Suppose given a (logarithmic) gluing situation. Then for every geometric point $\bar{w} \in |\underline{V}|$ where $\underline{V}$ is defined by (5.1) with images $\bar{w}_v \in \underline{W}_v$, let f_v :*

$(\underline{C}_v)_{\bar w_v} \to \underline{X}$ be the induced stable map. We necessarily have $f_v(p_{v,E}) = f_{v'}(p_{v',E})$ for v, v' the two vertices of an edge $E \in E(G)$. We then assume that the contact orders $u_{v,E} : P_{p_{v,E}} \to \mathbb{Z}$ and $u_{v',E} : P_{p_{v,E}} \to \mathbb{Z}$ satisfy the relation $u_{v',E} = -u_{v,E}$.

Remark 5.10. The assumption that contact orders be opposite is insufficient in general to guarantee that the assumption holds. The problem only arises in fairly unusual situations, such as in the Möbius example of Remark 2.45. By Remark 2.48, the contact order u given in the example is in fact opposite to itself, so if that contact order is specified in a gluing situation, one might have $u_{v',E} = u_{v,E}$ or $u_{v',E} = -u_{v,E}$ depending on the point $\bar w \in |\underline{V}|$.

However, if $\overline{\mathcal{M}}_X$ is generated by global sections, as is currently required (Theorem 3.7) for moduli spaces to be of finite type, then the above assumption always hold. Indeed, as in the proof of Proposition 2.46, a connected component of contact orders determines a composed morphism $v : \Gamma(\underline{X}, \overline{\mathcal{M}}_X) \to \Gamma(\mathcal{Z}, \overline{\mathcal{M}}_{\mathcal{Z}}) \xrightarrow{\mathbf{u}} \mathbb{Z}$, and if $\mathbf{u}$, $\mathbf{u}'$ are opposite contact orders, then $v = -v'$. In particular, in the situation of Assumption (5.9), there are then factorizations $\Gamma(\underline{X}, \overline{\mathcal{M}}_X) \to P_{p_{v,E}} \xrightarrow{u_{v,E}} \mathbb{Z}$ and $\Gamma(\underline{X}, \overline{\mathcal{M}}_X) \to P_{p_{v,E}} \xrightarrow{u_{v',E}} \mathbb{Z}$ of v and v' respectively, where the first map takes germs of global sections. Thus $u_{v,E} = -u_{v',E}$.

We now partially describe the construction of the glued family W. The chief difficulty is that there do not exist evaluation maps at the log level $\mathrm{ev}_{v,E} : W_v \to X$ evaluating f_v at $p_{v,E}$. To rectify this, let W_v^E denote the log scheme with underlying scheme $\underline{W}_v$ and log structure $p_{v,E}^* \mathcal{M}_{C_v^\circ}$. Let $E_1, \ldots, E_n$ be the edges adjacent to v. Then define $\widetilde{W}_v$ to be the saturation (see [Ogu18, III Prop. 2.1.5]) of the fibre product in the category of fine log schemes

$$\widetilde{W}_v^{\mathrm{fine}} := W_v^{E_1} \times_{W_v} \cdots \times_{W_v} W_v^{E_n}. \tag{5.2}$$

Note that there are natural composed morphisms

$$\mathrm{ev}_{v,E} : \widetilde{W}_v \longrightarrow \widetilde{W}_v^{\mathrm{fine}} \longrightarrow W_v^E \xrightarrow{f_v \circ p_{v,E}} X, \tag{5.3}$$

which can be viewed as an evaluation map at the puncture $p_{v,E}$. This induces, by ranging over all E containing a given vertex v, a morphism

$$\mathrm{ev}_v : \widetilde{W}_v \to \prod_E X,$$

where the product is over all edges containing v. Finally, taking the product over all v gives a morphism

$$\mathrm{ev} : \prod_{v \in V(G)} \widetilde{W}_v \to \prod_{v \in E \in E(G)} X. \tag{5.4}$$

We now have a diagram Cartesian in the fs log category

$$\begin{tikzcd} \widetilde{W} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \prod_{v\in V(G)} \widetilde{W}_v \arrow[d, "\mathrm{ev}"] \\ \prod_{E\in E(G)} X \arrow[r, "\Delta"'] & \prod_{v\in E\in E(G)} X \end{tikzcd} \tag{5.5}$$

The morphism Δ is the product of the diagonal morphisms $X \to X \times X$, taking the copy of X indexed by $E \in E(G)$ to the product of copies of X indexed by $v_1 \in E$, $v_2 \in E$.

Before stating the main gluing result, we need the following standard fact:

Proposition 5.11. *Let X_1, X_2 and Y be fs log schemes, $p_i : X_i \to Y$ morphisms, and $W = X_1 \times_Y X_2$ the product in the category of fs log schemes with projections $\pi_i : W \to X_i$. If $\bar{w} \in W$, let $Q = \overline{\mathcal{M}}_{W,\bar{w}}$, $Q_i = \overline{\mathcal{M}}_{X_i,\pi_i(\bar{w})}$, $P = \overline{\mathcal{M}}_{Y,p_i\circ\pi_i(\bar{w})}$. Then Q is the saturated image of $Q_1 \oplus Q_2$ in $(Q_1^{\mathrm{gp}} \oplus Q_2^{\mathrm{gp}})/R$, where R is the saturation of the image of $P^{\mathrm{gp}} \to Q_1^{\mathrm{gp}} \oplus Q_2^{\mathrm{gp}}$, $m \mapsto \big(\bar{p}_1^\flat(m), -\bar{p}_2^\flat(m)\big)$.*

Proof. By [ACGS16, Proposition 6.3.5], $Q^\vee = Q_1^\vee \times_{P^\vee} Q_2^\vee$. Since Q is a sharp fine saturated monoid, $Q = Q^{\vee\vee}$, and the latter is precisely the stated monoid. ♠

The main gluing result is then:

Theorem 5.12. *Given a gluing situation as above satisfying Assumption 5.9, there is a log scheme $W = (\underline{\widetilde{W}}, \mathcal{M}_W)$ with $\mathcal{M}_W \subset \mathcal{M}_{\widetilde{W}}$, equipped with morphisms $\psi_v : W \to W_v$ and a universal glued family $(\pi : C \to W, \mathbf{p}, f : C \to X)$.*

Proof. **Step 1. Gluing ordinary stable maps.**

At the level of underlying schemes, $\mathrm{ev}_{v,E}$ is the composition of the projection $\underline{\widetilde{W}}_v \to \underline{W}_v$ and the evaluation map $\underline{f}_v \circ p_{v,E}$. Thus we obtain a canonical morphism $\underline{\widetilde{W}} \to \underline{V}$ where $\underline{V}$ is the ordinary gluing defined in (5.1). Thus we may glue the underlying stable maps to obtain $\underline{f} : \underline{C} \to \underline{X}$, with target relative to $\underline{B}$ if the original punctured maps are defined relative to B.

Step 2. Construction of $\mathcal{M}_W$ and the morphisms ψ_v. Consider the composed morphisms, for $v \in E$,

$$\psi_v^E : \widetilde{W} \to \widetilde{W}_v \to W_v^E.$$

By construction, there are canonical inclusions $\mathcal{M}_{W_v}, \mathcal{P}_{v,E} \subset \mathcal{M}_{W_v^E}$ where $\mathcal{P}_{v,E}$ is the pull-back via $p_{v,E}$ of the divisorial log structure $(\underline{C}_v, p_{v,E}(\underline{W}_v))$. Note that $\mathcal{P}_{v,E}$ is the DF(1) log structure associated to the pull-back of the conormal bundle $p_{v,E}^*\mathcal{N}^\vee_{p_{v,E}(\underline{W}_v)/\underline{C}_v}$.[7] Thus we obtain morphisms of log structures

$$(\psi_v^E)^*\mathcal{M}_{W_v} \to \mathcal{M}_{\widetilde{W}}. \tag{5.6}$$

[7]Here we mean the log structure with ghost sheaf the constant sheaf $\mathbb{N}$ which is the pull-back of the standard log structure on $B\mathbb{G}_m$ via the morphism $\underline{W}_v \to B\mathbb{G}_m$ given by the line bundle $p_{v,E}^*\mathcal{N}^\vee_{p_{v,E}(\underline{W}_v)/\underline{C}_v}$.

and

$$(\psi_v^E)^*\mathcal{P}_{v,E} \to \mathcal{M}_{\widetilde{W}}. \tag{5.7}$$

We obtain for each edge E two log structures on $\underline{\widetilde{W}}$

$$\mathcal{N}_E \subset \widetilde{\mathcal{N}}_E := \big((\psi_{v_1}^E)^*\mathcal{P}_{v_1,E}\big) \oplus_{\mathcal{O}^\times_{\widetilde{W}}} \big((\psi_{v_2}^E)^*\mathcal{P}_{v_2,E}\big)$$

with the induced inclusion of ghost sheaves being $\underline{\mathbb{N}} \subseteq \underline{\mathbb{N}} \oplus \underline{\mathbb{N}}$ the diagonal. In particular $\mathcal{N}_E$ is the DF(1) log structure induced by the line bundle

$$(p_{v_1,E} \circ \psi_{v_1}^E)^*\mathcal{N}^\vee_{p_{v_1,E}(\underline{W}_{v_1})/\underline{C}_{v_1}} \otimes (p_{v_i,E} \circ \psi_{v_2}^E)^*\mathcal{N}^\vee_{p_{v_2,E}(\underline{W}_{v_2})/\underline{C}_{v_2}}.$$

We have morphisms of log structures induced by (5.7)

$$\mathcal{N}_E \to \mathcal{M}_{\widetilde{W}} \tag{5.8}$$

and

$$\widetilde{\mathcal{N}}_E \to \mathcal{M}_{\widetilde{W}}. \tag{5.9}$$

We then define $\mathcal{M}_W \subseteq \mathcal{M}_{\widetilde{W}}$ to be the fine saturated log structure generated by the images of all the morphisms (5.6), (5.7) and (5.8). Because $\mathcal{M}_W$ contains the image of (5.6), the morphism $\pi_v \circ \psi_v^E$ factors through $\widetilde{W} \to W$, giving the desired morphism $\psi_v : W \to W_v$.

Step 3. Analysis of the monoids.

For future use in the proof, we give a more detailed description of the ghost sheaves. Let $\bar{w}$ be a geometric point of $\underline{W}$. Write $\widetilde{Q}$ and Q for the stalks of $\overline{\mathcal{M}}_{\widetilde{W}}$ and $\overline{\mathcal{M}}_W$ at $\bar{w}$. We will describe $\widetilde{Q}^{\mathrm{gp}}$ and Q. Let $(\bar{x}_E)_{E\in E(G)}$ be the image of $\bar{w}$ under the projection morphism pr_1 of (5.5), and $(\bar{w}_v)_{v\in V(G)}$ the image of $\bar{w}$ under the projection morphism pr_2. From Proposition 5.11, if $\bar{w}'_v$ denotes the image of $\bar{w}_v$ in W_v and $Q_v = \overline{\mathcal{M}}_{W_v,\bar{w}'_v}$, then

$$\overline{\mathcal{M}}^{\mathrm{gp}}_{\widetilde{W}_v,\bar{w}_v} = Q_v^{\mathrm{gp}} \oplus \bigoplus_{E:\, v\in E} \mathbb{Z}.$$

Finally, for $v \in E$, write $P_E = P_{v,E}$ for the stalk of $\overline{\mathcal{M}}_X$ at $\bar{x}_E$; note this does not depend on v. Again from Proposition 5.11 and the diagram (5.5), we can write

$$\widetilde{Q}^{\mathrm{gp}} = \left(\bigoplus_{E\in E(G)} P_E^{\mathrm{gp}} \oplus \bigoplus_{v\in V(G)} Q_v^{\mathrm{gp}} \oplus \bigoplus_{v\in E\in E(G)} \mathbb{Z} \right) \Big/ R$$

where R is the saturation of the subgroup generated by elements of the form

$$\big((0,\ldots,0,-p,0,\ldots,0),(0,\ldots,\varphi_v(p),\ldots,0),(0,\ldots,u_{p_{v,E}}(p),\ldots,0)\big)$$

where $p \in P^{\mathrm{gp}}_{v,E}$, and the non-zero entries lie in the terms P_E^{gp}, Q_v^{gp}, and the copy of $\mathbb{Z}$ indexed by $v \in E$ respectively. Here $\varphi_v : P_{v,E} \to Q_v$ is induced by

$f_v^\flat : \overline{\mathcal{M}}_{X,\bar{x}_E} \to \overline{\mathcal{M}}_{C_v^\circ, p_{v,E}(\bar{w}'_v)} \subset Q_v \oplus \mathbb{Z}$ followed by the first projection. Note this is isomorphic to

$$\widetilde{Q}^{\mathrm{gp}} = \left(\bigoplus_{v\in V(G)} Q_v^{\mathrm{gp}} \oplus \bigoplus_{v\in E\in E(G)} \mathbb{Z} \right) \Big/ R'$$

where R' is the saturation of the subgroup generated by elements of the form, for $p \in P_E^{\mathrm{gp}}$, v_1, v_2 the two vertices of E,

$$\big((\ldots,0,\varphi_{v_1}(p),\ldots,-\varphi_{v_2}(p),0,\ldots),(\ldots,0,u_{p_{v_1,E}}(p),\ldots,-u_{p_{v_2,E}}(p),0,\ldots)\big)$$

where the non-zero entries lie in $Q_{v_1}^{\mathrm{gp}}$, $Q_{v_2}^{\mathrm{gp}}$, and the copies of $\mathbb{Z}$ indexed by $v_1 \in E$ and $v_2 \in E$ respectively. By Assumption 5.9, $-u_{p_{v_2,E}} = u_{p_{v_1,E}}$. This implies that R' is contained in Q^{gp}, and we are thus able to describe Q as the saturated image of $\bigoplus_{v\in V(G)} Q_v \oplus \bigoplus_{E\in E(G)} \mathbb{N}$ in

$$Q^{\mathrm{gp}} = \left(\bigoplus_{v\in V(G)} Q_v^{\mathrm{gp}} \oplus \bigoplus_{E\in E(G)} \mathbb{Z} \right) \Big/ R''$$

where R'' is the saturation of the subgroup generated by elements of the form

$$\big((\ldots,0,\varphi_{v_1}(p),\ldots,-\varphi_{v_2}(p),0,\ldots),(\ldots,0,u_{p_{v_1,E}}(p),0,\ldots)\big).$$

Here the copy of $\mathbb{Z}$ indexed by E in the above description is embedded diagonally in $\mathbb{Z}\oplus\mathbb{Z}$ in the description of $\widetilde{Q}^{\mathrm{gp}}$, with the two copies of $\mathbb{Z}$ indexed by $v_1 \in E$ and $v_2 \in E$.

Step 4. Construction of the glued log structure on $\underline{C}$.

Following the notation of the proof of Proposition 5.7, we glue the domains by constructing a morphism $\mu : W \to \mathbf{M}^{\mathrm{gl}}$. To do so, we need to be explicit about the log structure on $\mathbf{M}^{\mathrm{gl}}$. Recalling that $\underline{\mathbf{M}}^{\mathrm{gl}} = \prod_v \underline{\mathbf{M}}_{g(v),n(v)}$, write

$$(\underline{\mathbf{M}}^{\mathrm{gl}}, \mathcal{M}_{\mathrm{prod}}) := \prod_v \mathbf{M}_{g(v),n(v)}.$$

Let $\mathcal{N}'_E$ denote the DF(1) log structure on $\underline{\mathbf{M}}^{\mathrm{gl}}$ induced by the tensor product of conormal bundles $\mathcal{N}^\vee_{p_{v_1,E}(\mathbf{M}_{g(v),n(v)})/\mathbf{C}_{v_1}} \otimes \mathcal{N}^\vee_{p_{v_2,E}(\mathbf{M}_{g(v),n(v)})/\mathbf{C}_{v_2}}$, where $\mathbf{C}_v \to \mathbf{M}_{g(v),n(v)}$ is the universal curve. Then the log structure on $\mathbf{M}^{\mathrm{gl}}$ is the fibred sum (over $\mathcal{O}^\times$) of $\mathcal{M}_{\mathrm{prod}}$ and the $\mathcal{N}'_E$ for $E \in E(G)$.

We can now define a log morphism

$$\mu : W \to \mathbf{M}^{\mathrm{gl}}$$

as follows. First, if $\mu_v : W_v \to \mathbf{M}_{g(v),n(v)}$ is the tautological map induced by C_v/W_v, the composed morphisms $\mu_v \circ \psi_v : W \to \mathbf{M}_{g(v),n(v)}$ yield a morphism

$$\mu' : W \to (\underline{\mathbf{M}}^{\mathrm{gl}}, \mathcal{M}_{\mathrm{prod}}).$$

Thus it is sufficient to define morphisms of log structures $(\mu')^*\mathcal{N}'_E \to \mathcal{M}_W$. But there is a canonical isomorphism $(\mu')^*\mathcal{N}'_E \to \mathcal{N}_E$, which combined with the map (5.8) yields the desired morphism of log structures.

The universal log curve $\mathbf{C}^{\mathrm{gl}} \to \mathbf{M}^{\mathrm{gl}}$ can now be pulled back via μ to obtain a log curve $\pi : C \to W$, with underlying scheme being the glued curve $\underline{C}$.

Step 5. Construction of the morphism $f : C^\circ \to X$.

Recall that from Step 1, we already have $\underline{f} : \underline{C} \to \underline{X}$. We need to lift this to log schemes.

Denote the nodal sections of $\underline{C} \to \underline{W}$ produced by the gluing by q_E, $E \in E(G)$. Let $C^\circ_{v,W} = C^\circ_v \times_{W_v} W$, and let $\nu : \underline{\widetilde{C}} \to \underline{C}$ denote the normalization along the nodal sections q_E, so that we have a canonical isomorphism $\underline{\widetilde{C}} \cong \coprod_v \underline{C}^\circ_{v,W}$. Denote by $\widetilde{C}^\circ$ the log structure on $\underline{\widetilde{C}}$ pulled back via this isomorphism. There is a morphism $\tilde{f} = \coprod_v f_{v,W} : \widetilde{C}^\circ \to X$, where $f_{v,W} : C^\circ_{v,W} \to X$ is induced by f_v.

As $\underline{\widetilde{C}} \setminus \bigcup_{v\in E\in E(G)} p_{v,E}(W) \cong \underline{C} \setminus \bigcup_{E\in E(G)} q_E(W)$, we obtain a puncturing C° of C at its marked points so that $\widetilde{C}^\circ \setminus \bigcup_{v\in E\in E(G)} p_{v,E}(W) \to C^\circ \setminus \bigcup_{E\in E(G)} q_E(W)$ is strict. Thus $\tilde{f}$ induces a morphism $f : C^\circ \setminus \bigcup_E q_E(\underline{W}) \to X$. We only need to extend this morphism across the nodal sections.

We do this by showing the morphism $\tilde{f} : \widetilde{C}^\circ \to X$ induces a morphism $\tilde{f} : (\underline{\widetilde{C}}, \nu^*\mathcal{M}_{C^\circ}) \to X$ and that for each edge E, the restriction of $\tilde{f}$ to the section $p_{v_i,E}(\underline{W})$ is independent of i, for v_1, v_2 the vertices of E.

First note that by Proposition 5.2, $\mathcal{M}^{\mathrm{gp}}_{\widetilde{C}^\circ}$ and $(\nu^*\mathcal{M}_{C^\circ})^{\mathrm{gp}}$ can be canonically identified, since $\mathcal{M}_{\widetilde{C}^\circ}$ and $\nu^*\mathcal{M}_{C^\circ}$ are both puncturings along the same set of punctures. Fix $E \in E(G)$ with vertices $v_1, v_2 \in E$. Set

$$\begin{aligned} W^E &= (\underline{W}, q_E^*\mathcal{M}_{C^\circ}) = (\underline{W}, p^*_{v_i,E}\nu^*\mathcal{M}_{C^\circ}) \\ W^{v,E} &= (\underline{W}, p^*_{v,E}\mathcal{M}_{C^\circ_{v,W}}) \end{aligned}$$

We still have a canonical identification $\beta^\flat_{v,E} : \mathcal{M}^{\mathrm{gp}}_{W^{v,E}} \cong \mathcal{M}^{\mathrm{gp}}_{W^E}$. By the discussion of the previous paragraph, it is enough to show (1) $\beta^\flat_{v,E}$ induces a log morphism $\beta_{v,E} : W^E \to W^{v,E}$; (2) $f_{v_1,W} \circ \beta_{v_1,E} = f_{v_2,W} \circ \beta_{v_2,E}$. Thus we only need to show the following. Consider the diagram of sheaves of monoids on W

$$\begin{tikzcd} (\underline{f} \circ q_E)^*\mathcal{M}_X \arrow[r, "f^\flat_{v_1,W}"] \arrow[d, "f^\flat_{v_2,W}"'] & \mathcal{M}_{W^{v_1,E}} \arrow[d, "\beta^\flat_{v_1,E}"] \\ \mathcal{M}_{W^{v_2,E}} \arrow[r, "\beta^\flat_{v_2,E}"'] & \mathcal{M}^{\mathrm{gp}}_{W^E} \end{tikzcd} \tag{5.10}$$

It is sufficient to show (1) $\beta^\flat_{v_i,E} \circ f^\flat_{v_i,W}$ has image lying in $\mathcal{M}_{W^E}$, and (2) the diagram (5.10) is commutative.

To do so, note that by the construction of Step 2, $\mathcal{N}_E$ is a sub-log structure of both $\mathcal{M}_W$ and $\widetilde{\mathcal{N}}_E$. Then

$$\mathcal{M}_{W^E} = \mathcal{M}_W \oplus_{\mathcal{N}_E} \widetilde{\mathcal{N}}_E. \tag{5.11}$$

Further, the map (5.9) then induces a homomorphism $\mathcal{M}_{W^E} \to \mathcal{M}_{\widetilde{W}}$. If this homomorphism is injective, then to check commutativity of (5.10) we may replace

$\mathcal{M}^{\mathrm{gp}}_{W^E}$ with $\mathcal{M}^{\mathrm{gp}}_{\widetilde{W}}$. The injectivity can be checked at the level of stalks of ghost sheaves, and thus follows from the explicit descriptions of these monoids in Step 3. The commutativity statement (2) then follows from commutativity of the diagram (5.5), by tracing the effects of the morphisms on the copies of $\mathcal{M}_X$ on $\prod_{v\in E\in E(G)} X$ indexed by $v_1 \in E$ and $v_2 \in E$.

To show (1), we can use the commutativity of (2). Let $\bar{w}$ be a geometric point of $\underline{W}$, and let s be a section of $(\underline{f} \circ q_E)^*\mathcal{M}_X$ defined in a neighbourhood of $\bar{w}$. Let $\bar{s}$ be the induced section of $(\underline{f} \circ q_E)^{-1}\overline{\mathcal{M}}_X$. We have

$$f^\flat_{v_i,W} : (\underline{f} \circ q_E)^*\mathcal{M}_X \to \mathcal{M}_{W^{v_i,E}} \subset \mathcal{M}_W \oplus_{\mathcal{O}^\times_W} \psi^*_{v_i}\mathcal{P}^{\mathrm{gp}}_{v_i,E},$$

so a priori we can write $f^\flat_{v_i,W}(s) = s_1 \cdot s_2$, with s_1 a section of $\mathcal{M}_W$ and s_2 a section of $\psi^*_{v_i}\mathcal{P}^{\mathrm{gp}}_{v_i,E}$.

Now $u_{p_{v_1,E}}, u_{p_{v_2,E}} \in \overline{\mathcal{M}}^*_{X,\underline{f}(w)}$ are related by $u_{p_{v_2,E}} = -u_{p_{v_1,E}}$ by Assumption 5.9. Thus $u_{p_{v_i,E}}(\bar{s}) \geq 0$ for some i. Then for this i, in fact

$$s_2 \in \psi^*_{v_i}\mathcal{P}_{v_i,E} \subset \widetilde{\mathcal{N}}_E \subset \mathcal{M}_{W^E},$$

the composition of the latter two inclusions being induced by $\beta^\flat_{v_i,E}$. Thus $\beta^\flat_{v_i,E} \circ f^\flat_{v_i,W}(s) \in \mathcal{M}_{W^E}$. By commutativity of (5.10), the choice of i is irrelevant.

To recap, by construction we have obtained the data $(\pi : C \to W, \mathbf{p}, f : C^\circ \to X)$ with nodal sections q_E, morphisms $\psi_v : W \to W_v$, and an isomorphism of $\coprod_{v\in V(G)} C^\circ_{v,W}$ with the splitting $\widetilde{C}^\circ$ of C° along the q_E. Thus we have constructed a glued family in the sense of Definition 5.6.

Step 6. $f : C^\circ \to X$ is a morphism over B.

As the original underlying morphisms $\underline{f}_v : \underline{C}_v \to X$ were defined over $\underline{B}$ and G is connected, in fact $\underline{f}$ is a family of stable maps defined over $\underline{B}$, i.e., there is a morphism $\underline{W} \to \underline{B}$ compatible with $\underline{f}$. Indeed, this morphism can be taken to be the composition of $\underline{X} \to \underline{B}$ with $\underline{f} \circ q_E$ for any gluing node q_E, and all these maps coincide. To see that f is defined over B at the logarithmic level, it is sufficient to check that the image of $\mathcal{M}_B$ in $\mathcal{M}_{C^\circ}$, under the composition $C^\circ \to X \to B$ is contained in $\pi^*\mathcal{M}_W$. However, this can be checked after pull-back to $\widetilde{C}$, where the claim holds because each stable map $f_{v,W} : C^\circ_{v,W} \to X$ is defined over B.

Step 7. Verification of the universal property.

Assume given any glued family over a log scheme T, i.e., a family $(C_T \to T, \mathbf{p}, f_T : C_T \to X)$, morphisms $\psi_{v,T} : T \to W_v$, and an isomorphism of punctured maps between the splitting $\tilde{f}_T : \widetilde{C}^\circ_T \to X$ of f_T and $\coprod f_{v,T} : \coprod_v C^\circ_{v,T} \to X$.

In analogy with W^E, $W^{v,E}$, set

$$\begin{aligned} T^E &= (\underline{T}, q^*_E\mathcal{M}_{C^\circ_T}),\\ T^{v,E} &= (\underline{T}, p^*_{v,E}\mathcal{M}_{C^\circ_{v,T}}),\\ \widetilde{T} &= T^{E_1} \times_T \cdots \times_T T^{E_n}, \end{aligned}$$

where $E_1, \ldots, E_n$ is an enumeration of the edges of G. Note that the underlying schemes of $\widetilde{T}$ and T are the same as the morphisms $T^{E_i} \to T$ are integral and saturated, see [Ogu18, I Theorem 4.8.14].

Note we have a morphism

$$\beta^T_{v,E} : T^E \to T^{v,E}$$

which is the identity on underlying schemes. This morphism exists at the log level as the induced $f_{v,T} \circ p_{v,E} : T^{v,E} \to X$ is pre-stable, so we may apply Proposition 2.4.

We also have the following morphisms:

$$\begin{aligned}
&\mathrm{pr}_{v,T} : T^{v,E} \to W^E_v,\\
&\mathrm{pr}_E : \widetilde{T} \to T^E,\\
&\psi^E_{v,T} = \mathrm{pr}_{v,T} \circ \beta^T_{v,E} \circ \mathrm{pr}_E : \widetilde{T} \to W^E_v,\\
&\mathrm{ev}_{E,T} = f_T \circ q_E \circ \mathrm{pr}_E : \widetilde{T} \to X,
\end{aligned}$$

where $\mathrm{pr}_{v,T}$ is induced by the projection $C^\circ_{v,T} \to C^\circ_v$ and pr_E is the projection onto T^E.

We also have a commutative diagram

$$\begin{tikzcd}
\widetilde{T} \arrow[r, "\psi^E_{v,T}"] \arrow[d] & W^E_v \arrow[d] \\
T \arrow[r, "\psi_{v,T}"'] & W_v
\end{tikzcd} \tag{5.12}$$

where the vertical maps are the canonical ones. Thus the composition $\widetilde{T} \to W^E_v \to W_v$ is independent of the edge E, and we obtain a morphism $\widetilde{T} \to \widetilde{W}^{\mathrm{fine}}_v$. Since $\widetilde{T}$ is saturated, this morphism factors through the saturation to give

$$\widetilde{T} \to \widetilde{W}_v.$$

This in turn gives a morphism

$$\widetilde{\psi}' : \widetilde{T} \to \prod_v \widetilde{W}_v.$$

On the other hand, we have a morphism

$$\mathrm{ev}_T = \prod_E \mathrm{ev}_{E,T} : \widetilde{T} \to \prod_E X.$$

Using (5.5), we can now define a morphism

$$\widetilde{\psi}_T : \widetilde{T} \to \widetilde{W}$$

by noting that $\mathrm{ev} \circ \widetilde{\psi}' = \Delta \circ \mathrm{ev}_T$. Indeed, this follows from the equality of the morphisms $\mathrm{ev}_{E,T} = f_T \circ q_E \circ \mathrm{pr}_E : \widetilde{T} \to X$ and $f_v \circ p_{v,E} \circ \psi^E_{v,T} : \widetilde{T} \to X$, which holds because $f_{v,T}$ is induced from f_T via the splitting of f along the nodes q_E.

Finally, we show that $\widetilde{\psi}_T$ induces a morphism

$$\psi_T : T \to W.$$

This is done by showing that under $\widetilde{\psi}_T^\flat$, $\mathcal{M}_W \subset \mathcal{M}_{\widetilde{W}}$ is mapped to $\mathcal{M}_T \subset \mathcal{M}_{\widetilde{T}}$, which can be checked on the generating set of $\mathcal{M}_W$. By commutativity of (5.12), the image of (5.6) is mapped into $\mathcal{M}_T$ by $\psi_T^\flat$. Further, analogously to (5.11)

$$\mathcal{M}_{T^E} = \mathcal{M}_T \oplus_{\mathcal{N}_E^T} \widetilde{\mathcal{N}}_E^T,$$

where $\mathcal{N}_E^T$, $\widetilde{\mathcal{N}}_E^T$ are pull-backs of $\mathcal{N}_E$, $\widetilde{\mathcal{N}}_E$ under $\underline{\widetilde{\psi}}_T$, with

$$\widetilde{\mathcal{N}}_E^T = \Big((\psi_{v_1}^E \circ \widetilde{\psi}_T)^* \mathcal{P}_{v_1,E}\Big) \oplus_{\mathcal{O}_{\widetilde{W}}^\times} \Big((\psi_{v_2}^E \circ \widetilde{\psi}_T)^* \mathcal{P}_{v_2,E}\Big)$$

Note that $\widetilde{\psi}_T$ is compatible with $\psi_{v_i,T}^E$ and hence $\widetilde{\psi}_T^\flat$ maps the image of (5.7) to the image of $(\psi_{v_1}^E \circ \widetilde{\psi}_T)^* \mathcal{P}_{v_1,E}$ in $\mathcal{M}_{\widetilde{T}}$. This implies that $\widetilde{\psi}_T^\flat$ maps $\mathcal{N}_E \subset \mathcal{M}_W$ to $\mathcal{N}_E^T \subset \mathcal{M}_T$. Thus $\mathcal{M}_W$ is mapped via $\widetilde{\psi}_T^\flat$ into $\mathcal{M}_T$, as desired, defining the morphism ψ_T.

We then have $\psi_{v,T} = \psi_v \circ \psi_T$, which follows from the equality $\psi_{v,T}^E = \psi_v^E \circ \widetilde{\psi}_T$. By construction, the data of the gluing over T is the pull-back of the universal gluing over W via ψ_T. ♠

5.2.3. *Gluing with evaluation maps.* In this subsection we consider a minor variant of the gluing procedure of the previous subsection. Suppose given a family of targets $X \to B$ with X Zariski. Set $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$ with $\mathcal{A}_X$ the relative Artin fan. Suppose also given a gluing situation of punctured log maps to $\mathcal{X}/B$ satisfying Assumption 5.9. We may glue to obtain a family of maps to $\mathcal{X}/B$ using the fibre product description of the previous section, but as we shall see in the following subsections, often there is a more useful gluing setup.

Let $\mathbf{p}_v$ be a subset of the punctures of the given type $\boldsymbol{\beta}(v)$, and assume that whenever $E \in E(G)$ is an edge with vertex v we have $p_{v,E} \in \mathbf{p}_v$. Let $\mathbf{p}'_v \subseteq \mathbf{p}_v$ be the subset of punctures corresponding to legs with endpoint v. Let $\mathbf{q} = \{q_E \,|\, E \in E(G)\}$ be the collection of nodal sections of a curve resulting from gluing the given data, and let

$$\mathbf{p}' := \bigcup_{v \in V(G)} \mathbf{p}'_v.$$

This is a subset of the set of punctures of $\boldsymbol{\beta}^{\mathrm{gl}}$.

We introduce short-hand: for a set $\mathbf{p}$, we write

$$\prod_{p \in \mathbf{p}} \underline{X} := \underline{X} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{X}, \quad \prod_{p \in \mathbf{p}} \underline{\mathcal{X}} := \underline{\mathcal{X}} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{\mathcal{X}},$$

with a one-to-one correspondence between factors in the product and elements of $\mathbf{p}$. Thus, following (4.12), we set

$$W_v^{\mathrm{ev}(\mathbf{p}_v)} := W_v \times_{\prod_{p \in \mathbf{p}_v} \underline{\mathcal{X}}} \prod_{p \in \mathbf{p}_v} \underline{X} \tag{5.13}$$

Here the morphism $W_v \to \prod_{p\in\mathbf{p}_v} \underline{\mathcal{X}}$ is just the product of the schematic evaluation maps at each of the punctures of the domain curve in $\mathbf{p}_v$.

Note that giving a morphism $\psi_{v,T} : T \to W_v^{\mathrm{ev}(\mathbf{p}_v)}$ is equivalent to giving (1) a morphism $T \to W_v$ yielding via pull-back a family of punctured maps $f_v : C^\circ_{v,T}/T \to \mathcal{X}$; (2) for each puncture $p_{v,E} \in \mathbf{p}_v$ of $C^\circ_{v,T}$, a factorization of $\underline{f}_v \circ p_{v,E} : W_v \to \underline{\mathcal{X}}$ as $W_v \to \underline{X} \to \underline{\mathcal{X}}$.

Similarly, we can define, for W the gluing of the W_v,

$$W^{\mathrm{ev}(\mathbf{q},\mathbf{p}')} = W \times_{\prod \underline{\mathcal{X}}} \prod \underline{X}, \tag{5.14}$$

where the product is over all nodes in $\mathbf{q}$ and punctures in $\mathbf{p}'$; again, the map $W \to \prod \underline{\mathcal{X}}$ is given by the schematic evaluation maps at these points.

The main observation of this subsection is that $W^{\mathrm{ev}(\mathbf{q},\mathbf{p}')}$ can be constructed from the spaces $W^{\mathrm{ev}(\mathbf{p}_v)}$ as a fibre product as in the previous subsection. We define

$$\widetilde{W}_v^{\mathrm{ev}(\mathbf{p}_v)} := \widetilde{W}_v \times_{\prod_{p\in\mathbf{p}_v}\underline{\mathcal{X}}} \prod_{p\in\mathbf{p}_v} \underline{X}$$

and

$$\widetilde{W}^{\mathrm{ev}(\mathbf{q},\mathbf{p}')} := W \times_{\prod \underline{\mathcal{X}}} \prod \underline{X}.$$

Clearly $W^{\mathrm{ev}(\mathbf{q},\mathbf{p}')}$ is just a sub-log structure of $\widetilde{W}^{\mathrm{ev}(\mathbf{q},\mathbf{p}')}$ with the same underlying stack structure by Theorem 5.12.

Theorem 5.13. *There is a Cartesian diagram in the category of fs log stacks*

$$\begin{tikzcd} \widetilde{W}^{\mathrm{ev}(\mathbf{q},\mathbf{p}')} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \prod_{v\in V(G)} \widetilde{W}_v^{\mathrm{ev}(\mathbf{p}_v)} \arrow[d, "\mathrm{ev}"] \\ \prod_{E\in E(G)} X \arrow[r, "\Delta"'] & \prod_{v\in E\in E(G)} X \end{tikzcd} \tag{5.15}$$

Further, $\widetilde{W}_v^{\mathrm{ev}(\mathbf{p}_v)}$ can be constructed from the space $W_v^{\mathrm{ev}(\mathbf{p}_v)}$ via (5.2) and pr_2 is finite and representable.

Proof. The first statement follows immediately from the Cartesian diagram (5.5) and properties of fibre product.

For the second statement, certainly if $v \in E$, then $(W_v^{\mathrm{ev}(\mathbf{p}_v)})^E \cong W_v^E \times_{\prod_{p\in\mathbf{p}_v}\underline{\mathcal{X}}} \prod_{p\in\mathbf{p}_v} \underline{X}$. Thus $(\widetilde{W_v^{\mathrm{ev}(\mathbf{p}_v)}})^{\mathrm{fine}} \cong \widetilde{W}_v^{\mathrm{fine}} \times_{\prod \underline{\mathcal{X}}} \prod \underline{X}$ by standard properties of fibre product. Finally, saturation commutes with strict base-change. This shows the desired isomorphism.

For the final statement that pr_2 is finite and representable, note that $\underline{\mathrm{pr}}_2$ factors as

$$\underline{\widetilde{W}}^{\mathrm{ev}(\mathbf{q},\mathbf{p}')} \to \prod_v \underline{\widetilde{W}}_v^{\mathrm{ev}(\mathbf{p}_v)} \times_{\prod_E \underline{X}} \prod_{v\in E} \underline{X} \to \prod_v \underline{\widetilde{W}}_v^{\mathrm{ev}(\mathbf{p}_v)}$$

where the first morphism is the result of integralization and saturation of a log structure on the ordinary fibre product, hence finite and representable, and the second morphism is a base-change of Δ, hence a closed immersion, and in particular finite and representable. Thus pr_2 is finite and representable. ♠

5.2.4. *Gluing at the virtual level.* We fix a combinatorial type of gluing situation $G, \boldsymbol{\beta}$ with target $X \to B$, which we assumed to be log smooth with X Zariski. Further, to guarantee that moduli spaces are of finite type, we will assume in this subsection that $\overline{\mathcal{M}}_X$ is globally generated.

As in the previous subsection, we set $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$ with $\mathcal{A}_X$ the relative Artin fan as usual. In addition, we continue with $\mathbf{p}_v$ a subset of the punctures of the given type $\boldsymbol{\beta}(v)$ as in the previous subsection, yielding subsets $\mathbf{p}'_v \subseteq \mathbf{p}_v$ and the set of nodal sections $\mathbf{q} = \{q_E \mid E \in E(G)\}$ obtained from gluing.

Definition 5.14. As in (5.13), (5.14), we define

$$\mathfrak{M}^{\mathrm{ev}(\mathbf{p}_v)}(\mathcal{X}, \boldsymbol{\beta}(v)) := \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}(v)) \times_{\prod_{p\in\mathbf{p}_v} \underline{\mathcal{X}}} \prod_{p\in\mathbf{p}_v} \underline{X},$$

$$\mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \boldsymbol{\beta}) := \mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}) \times_{\prod_{x\in\mathbf{q}\cup\mathbf{p}'} \underline{\mathcal{X}}} \prod_{x\in\mathbf{q}\cup\mathbf{p}'} \underline{X}.$$

Note that we have obvious strict factorizations

$$\mathscr{M}(X, \boldsymbol{\beta}(v)) \xrightarrow{\varepsilon_v} \mathfrak{M}^{\mathrm{ev}(\mathbf{p}_v)}(\mathcal{X}, \boldsymbol{\beta}(v)) \longrightarrow \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}(v))$$

and

$$\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \xrightarrow{\varepsilon_{\mathrm{gl}}} \mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, \boldsymbol{\beta}) \longrightarrow \mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, \boldsymbol{\beta})$$

with the compositions being the canonical morphisms given by composition of a stable map to X with the morphism $X \to \mathcal{X}$.

Theorem 5.15. *There is a Cartesian diagram*

$$\begin{tikzcd} \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \arrow[r, "\delta"] \arrow[d, "\varepsilon_{\mathrm{gl}}"'] & \prod_{v\in V(G)} \mathscr{M}(X, \boldsymbol{\beta}(v)) \arrow[d, "\varepsilon=\prod_v \varepsilon_v"] \\ \mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \boldsymbol{\beta}) \arrow[r, "\delta'"] & \prod_{v\in V(G)} \mathfrak{M}^{\mathrm{ev}(\mathbf{p}_v)}(\mathcal{X}, \boldsymbol{\beta}(v)) \end{tikzcd} \tag{5.16}$$

with vertical maps strict. The morphisms δ and δ' are finite and representable.

Proof. The diagram is clearly commutative, and hence if

$$W = \mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \boldsymbol{\beta}) \times_{\prod_v \mathfrak{M}^{\mathrm{ev}(\mathbf{p}_v)}(\mathcal{X},\boldsymbol{\beta}(v))} \prod_v \mathscr{M}(X, \boldsymbol{\beta}(v)),$$

we obtain a morphism $\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \to W$. It is thus enough to construct an inverse morphism.

Note that giving a morphism $T \to W$ is the same as giving: (1) A family of punctured maps $f : C^\circ_T \to \mathcal{X}$ of class $\boldsymbol{\beta}^{\mathrm{gl}}$ equipped with nodal sections of splitting type q_E which splits as a collection of punctured maps $f_v : C^\circ_{v,T} \to \mathcal{X}$ of class $\boldsymbol{\beta}(v)$; (2) For each edge $E \in E(G)$, a morphism $\underline{f}_E : \underline{T} \to \underline{X}$ and an isomorphism between the composition of $\underline{f}_E$ with $\underline{X} \to \underline{\mathcal{X}}$ and $\underline{f} \circ q_E : \underline{T} \to \underline{\mathcal{X}}$. For each leg

$E \in L(G)$ corresponding to a puncture $p_E \in \mathbf{p}'$, a morphism $\underline{f}_E : \underline{T} \to \underline{X}$ and an isomorphism between the composition of $\underline{f}_E$ with $\underline{X} \to \underline{\mathcal{X}}$ and $\underline{f} \circ p_E : \underline{T} \to \underline{\mathcal{X}}$. (3) Punctured maps $\tilde{f}_v : C^\circ_{v,T} \to X$ and an isomorphism between the composition of $\tilde{f}_v$ with $X \to \mathcal{X}$ and f_v. This isomorphism is such that it induces an isomorphism between $\underline{\tilde{f}}_v \circ p_{v,E} : \underline{T} \to \underline{X}$ and $\underline{f}_E$ whenever $v \in E$ corresponds to a puncture $p_{v,E} \in \mathbf{p}_v$.

Given such data, we show that it induces a unique morphism $T \to \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta})$ compatible with all maps. To do this, it is sufficient to lift f to a morphism $\tilde{f} : C^\circ_T \to X$ such that (1) $\underline{\tilde{f}} \circ q_E = \underline{f}_E$; (2) the splittings $C^\circ_{v,T} \to X$ of $\tilde{f}$ agree with $\tilde{f}_v$.

To give a lifting $\tilde{f}$ of f, it is enough to give a factorization

$$\underline{C}_T \xrightarrow{\tilde{f}} \underline{X} \longrightarrow \underline{\mathcal{X}}$$

of $\underline{f}$ because $X \to \mathcal{X}$ is strict. However, this is immediate as the morphisms $\underline{\tilde{f}}_v : \underline{C}_{v,T} \to \underline{X}$ can be glued as ordinary stable maps precisely because of the isomorphism of $\underline{\tilde{f}}_v \circ p_{v,E}$ with $\underline{f}_E$ for all flags $v \in E \in E(G)$. Thus we obtain a morphism $T \to \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta})$, and hence a morphism $W \to \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta})$ which is clearly inverse to the canonical morphism $\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \to W$.

The finiteness and representability of δ and δ' follow immediately from Theorem 5.13. ♠

We now analyze the obstruction theories in (5.16). For short-hand, write

$$\begin{aligned}
\mathfrak{M}_v &:= \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}(v)), & \mathfrak{M}^{\mathrm{ev}}_v &:= \mathfrak{M}^{\mathrm{ev}(\mathbf{p}_v)}(\mathcal{X}, \boldsymbol{\beta}(v)),\\
\overline{\mathfrak{M}} &:= \prod_{v \in V(G)} \mathfrak{M}_v, & \overline{\mathfrak{M}}^{\mathrm{ev}} &:= \prod_{v \in V(G)} \mathfrak{M}^{\mathrm{ev}}_v\\
\mathfrak{M}^{\mathrm{gl}} &:= \mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}), & \mathfrak{M}^{\mathrm{gl,ev}} &:= \mathfrak{M}^{\mathrm{gl,ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \boldsymbol{\beta})\\
\mathscr{M}_v &:= \mathscr{M}(X, \boldsymbol{\beta}(v)), & \overline{\mathscr{M}} &:= \prod_{v \in V(G)} \mathscr{M}_v\\
\mathscr{M}^{\mathrm{gl}} &:= \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}).
\end{aligned}$$

Denote by $C_v \to \mathscr{M}_v$ and $C \to \mathscr{M}^{\mathrm{gl}}$ the universal curves over $\mathscr{M}_v$ and $\mathscr{M}^{\mathrm{gl}}$, respectively, by $\overline{C}_v \to \overline{\mathscr{M}}$ the pull-back of C_v under the projection from the product $\overline{\mathscr{M}}$ to $\mathscr{M}_v$ and write $\overline{\pi} : \overline{C} = \coprod_v \overline{C}_v \to \overline{\mathscr{M}}_v$. We also have universal morphisms $f : C \to X$, $\overline{f} : \overline{C} \to X$, and the subschemes of special points to be considered $\iota : Z \to C$, $\overline{\iota} : \overline{Z} \to \overline{C}$ and projections $p = \pi \circ \iota$ and $\overline{p} = \overline{\pi} \circ \overline{\iota}$. Here Z is the union of the images of the punctured sections in $\mathbf{p}'$ and the nodal sections in $\mathbf{q}$, while $\overline{Z}$ is the union of punctured sections in $\bigcup_v \mathbf{p}_v$. With κ the partial normalization along the nodal locus $Z'' \subset Z$ as defined before Lemma 4.4, there is also the universal morphism $\tilde{f} = f \circ \kappa : \tilde{C} \to X$ and the subscheme $\tilde{Z} = \kappa^{-1}(Z) \to \tilde{C}$ of special points on $\tilde{C}$ with projection $\tilde{p} : \tilde{Z} \to \mathscr{M}^{\mathrm{gl}}$. We have the following commutative diagram with two cartesian squares:

(5.17)
$$
\begin{tikzcd}
\tilde{C} \arrow[r] \arrow[d, "\kappa"] \arrow[dd, bend right, "\tilde{\pi}"'] \arrow[rr, bend left, "\tilde{f}"] & \overline{C} = \coprod_v \overline{C}_v \arrow[r, "\overline{f}"] \arrow[dd, "\overline{\pi}"] & X \\
C \arrow[d, "\pi"] \arrow[urr, bend right, "f"'] & & \\
\mathscr{M}^{\mathrm{gl}} \arrow[r, "\delta"] \arrow[d, "\varepsilon_{\mathrm{gl}}"'] & \overline{\mathscr{M}} = \prod_v \mathscr{M}_v \arrow[d, "\varepsilon"] & \\
\mathfrak{M}^{\mathrm{gl,ev}} \arrow[r] & \overline{\mathfrak{M}}^{\mathrm{ev}} = \prod_v \mathfrak{M}_v^{\mathrm{ev}}. &
\end{tikzcd}
$$

The discussion in §4.2 provides an obstruction theory $\overline{\mathbb{G}} \to \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}^{\mathrm{ev}}}$ for $\overline{\mathscr{M}}$ relative to $\overline{\mathfrak{M}}$ with

$$\overline{\mathbb{G}} = R\overline{\pi}_*\big(\overline{f}^*\Omega_{X/B} \otimes \omega_{\overline{\pi}}(\overline{Z})\big). \tag{5.18}$$

Recall that this obstruction theory is obtained by taking the cone of a morphism of perfect obstruction theories provided by Proposition 4.3:

$$
\begin{tikzcd}
L\varepsilon^*\overline{\mathbb{F}} \arrow[r] \arrow[d, "L\varepsilon^*\overline{\Psi}"'] & \overline{\mathbb{E}} \arrow[d, "\overline{\Phi}"] \\
L\varepsilon^*\mathbb{L}_{\overline{\mathfrak{M}}^{\mathrm{ev}}/\overline{\mathfrak{M}}} \arrow[r] & \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}}
\end{tikzcd}
$$

Pulling back to $\mathscr{M}^{\mathrm{gl}}$, we now have four deformation/obstruction situations with corresponding perfect obstruction theories. Given $T \to \mathscr{M}^{\mathrm{gl}}$ a morphism from an affine scheme and $f_T : C_T \to X$, $h_T : Z_T \to X$, $\tilde{f}_T : \tilde{C}_T \to X$, $\tilde{h}_T : \tilde{Z}_T \to X$ the pull-back of the universal morphisms from the universal curve and universal sections and their pull-backs to $\tilde{C}$, respectively, these are as follows. All deformation situations are relative $\mathfrak{M}^{\mathrm{gl}}$.

$(\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}})$ Deforming $f_T : C_T \to X$:

$$\mathbb{E} = R\pi_*(f^*\Omega_{X/B} \otimes \omega_\pi) \longrightarrow \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}}}.$$

$(\mathfrak{M}^{\mathrm{gl,ev}}/\mathfrak{M}^{\mathrm{gl}})$ Deforming $h_T : Z_T \to X$:

$$L\varepsilon^*\mathbb{F} = p_*(h^*\Omega_{X/B}) \longrightarrow L\varepsilon^*\mathbb{L}_{\mathfrak{M}^{\mathrm{gl,ev}}/\mathfrak{M}^{\mathrm{gl}}}.$$

$(\overline{\mathscr{M}}/\overline{\mathfrak{M}})$ Deforming $\tilde{f}_T : \tilde{C}_T \to X$:

$$L\delta^*\overline{\mathbb{E}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}) \longrightarrow L\delta^*\mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}}.$$

$(\overline{\mathfrak{M}}^{\mathrm{ev}}/\overline{\mathfrak{M}})$ Deforming $\tilde{h}_T : \tilde{Z}_T \to X$:

$$L\delta^*L\varepsilon^*\overline{\mathbb{F}} = \tilde{p}_*(\tilde{h}^*\Omega_{X/B}) \longrightarrow L\delta^*L\varepsilon^*\mathbb{L}_{\overline{\mathfrak{M}}^{\mathrm{ev}}/\overline{\mathfrak{M}}}.$$

Lemma 5.16. *There is a morphism of distinguished triangles*

$$\begin{array}{ccccccc} L\delta^* L\varepsilon^*\overline{\mathbb{F}} & \longrightarrow & L\delta^*\overline{\mathbb{E}} & \longrightarrow & \mathbb{G} & \longrightarrow & L\delta^* L\varepsilon^*\overline{\mathbb{F}}[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ L\varepsilon^*\mathbb{F} & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{G} & \longrightarrow & L\varepsilon^*\mathbb{F} \end{array}$$

with $\mathbb{G} = L\delta^*\overline{\mathbb{G}} = R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}(\tilde{Z}))$.

Proof. Recall we write $Z = Z' \cup Z''$, where Z' is the union of the images of the punctured sections in $\mathbf{p}'$ and Z'' is the union of the images of the nodal sections in $\mathbf{q}$. With $\tilde{Z}' = \kappa^{-1}(Z')$ and $\tilde{Z}'' = \kappa^{-1}(Z'')$ we have $\tilde{Z} = \tilde{Z}' \cup \tilde{Z}''$ and hence the following commutative diagram of $\mathcal{O}_{\tilde{C}}$-modules with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\tilde{C}}(-\tilde{Z}'') & \longrightarrow & \mathcal{O}_{\tilde{C}}(\tilde{Z}') & \longrightarrow & \mathcal{O}_{\tilde{Z}}(\tilde{Z}') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\tilde{C}} & \longrightarrow & \mathcal{O}_{\tilde{C}}(\tilde{Z}') & \longrightarrow & \mathcal{O}_{\tilde{Z}'}(\tilde{Z}') & \longrightarrow & 0. \end{array}$$

The statement then follows by tensoring this diagram with $\kappa^*\omega_\pi \otimes \tilde{f}^*\Omega_{X/B}$ and arguing similarly as in the proof of Lemma 4.4. For example, since $\kappa^*\omega_\pi(-\tilde{Z}'') = \omega_{\tilde{\pi}}$, taking $R\tilde{\pi}_*$ of this tensor product leads to

$$R\tilde{\pi}_*(\tilde{f}^*\Omega_{X/B} \otimes \omega_{\tilde{\pi}}) = L\delta^*\overline{\mathbb{E}}.$$

Further details are left to the reader. ♠

Theorem 5.17. *In the above situation and notation, we have*

(1) *There is a commutative diagram*

$$\begin{array}{ccc} \mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) & \longrightarrow & \mathscr{M}(X, \boldsymbol{\beta}^{\mathrm{gl}}) \\ \downarrow & & \downarrow \\ \mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}) & \longrightarrow & \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}^{\mathrm{gl}}) \end{array}$$

which exhibits $\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta})$ as an open and closed substack of the fibre product $\mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}) \times_{\mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}^{\mathrm{gl}})} \mathscr{M}(X, \boldsymbol{\beta}^{\mathrm{gl}})$. The relative obstruction theory for $\mathscr{M}(X, \boldsymbol{\beta}^{\mathrm{gl}}) \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}^{\mathrm{gl}})$ pulls back to give a relative obstruction theory for $\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \to \mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta})$, which is the obstruction theory

$$\mathbb{E} \to \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}(X,G,\beta)/\mathfrak{M}^{\mathrm{gl}}(X,G,\beta)}$$

described above.

(2) *The obstruction theory*

$$\mathbb{G} \to \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}(X,G,\beta)/\mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(X,G,\beta)}$$

for

$$\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta}) \to \mathfrak{M}^{\mathrm{gl},\mathrm{ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \boldsymbol{\beta})$$

coincides with the pull-back of the obstruction theory

$$\overline{\mathbb{G}} \to \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}^{\mathrm{ev}}}$$

described above.

(3) *If $\varepsilon^!_{\mathrm{gl}}$ and $\varepsilon^!$ denote Manolache's virtual pull-back defined using the two given obstruction theories, then for $\alpha \in A_*\big(\mathfrak{M}^{\mathrm{gl,ev}(\mathbf{q},\mathbf{p}')}(\mathcal{X}, G, \beta)\big)$, we have the identity*

$$\varepsilon^! \delta'_*(\alpha) = \delta_* \varepsilon^!_{\mathrm{gl}}(\alpha)$$

Proof. (1) It is clear that the fibre product $\mathfrak{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}) \times_{\mathfrak{M}(\mathcal{X},\boldsymbol{\beta}^{\mathrm{gl}})} \mathscr{M}(X, \boldsymbol{\beta}^{\mathrm{gl}})$ consists of the disjoint union of all $\mathscr{M}^{\mathrm{gl}}(\mathcal{X}, G, \boldsymbol{\beta}')$, where $\boldsymbol{\beta}'$ runs over all collections of data $(\mathbf{A}'(v), \mathbf{g}'(v), u'_{v,E})$ such that $\mathbf{g}'(v) = \mathbf{g}(v)$, $u'_{v,E} = u_{v,E}$ and $\sum_v \mathbf{A}'(v) = \sum_v \mathbf{A}(v)$, where $\boldsymbol{\beta}$ is the collection of data $(\mathbf{A}(v), \mathbf{g}(v), u_{v,E})$. Indeed, this is because punctured maps to $\mathcal{X}$ do not remember curve classes. This shows that $\mathscr{M}^{\mathrm{gl}}(X, G, \boldsymbol{\beta})$ must be a union of connected components of the fibre product, giving the first claim.

The statement about obstruction theories then follows from the functoriality statement Lemma 4.1 and the construction in §4.2 of the relative obstruction theory for $\mathscr{M}(X, \boldsymbol{\beta}^{\mathrm{gl}}) \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\beta}^{\mathrm{gl}})$.

(2) The morphism of triangles in Lemma 5.16 form the back face of the following diagram with the solid arrows given:

$$\begin{tikzcd}
L\delta^* L\varepsilon^* \overline{\mathbb{F}} \arrow[rr] \arrow[dd] \arrow[dr] & & L\delta^* \overline{\mathbb{E}} \arrow[rr] \arrow[dd] \arrow[dr] & & \mathbb{G} \arrow[dd, equal] \arrow[dr, dashed] & \\
& L\delta^* L\varepsilon^* \mathbb{L}_{\overline{\mathfrak{M}}^{\mathrm{ev}}/\overline{\mathfrak{M}}} \arrow[rr] \arrow[dd] & & L\delta^* \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}} \arrow[rr] \arrow[dd] & & L\delta^* \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}^{\mathrm{ev}}} \arrow[dd, "\simeq"] \\
L\varepsilon^* \mathbb{F} \arrow[rr] \arrow[dr] & & \mathbb{E} \arrow[rr] \arrow[dr] & & \mathbb{G} \arrow[dr, dashed] & \\
& L\varepsilon^* \mathbb{L}_{\mathfrak{M}^{\mathrm{gl}}/\mathfrak{M}} \arrow[rr] & & \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}} \arrow[rr] & & \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}}}
\end{tikzcd}$$

The four arrows facing to the front in the cube on the left are the four obstruction theories listed above. The top and bottom faces of this cube are commutative by Proposition 4.3. The front face is the morphism of distinguished triangles of the cotangent complexes for the compositions $\mathscr{M}^{\mathrm{gl}} \to \mathfrak{M}^{\mathrm{gl}} \to \mathfrak{M}$ and $\overline{\mathscr{M}} \to \overline{\mathfrak{M}}^{\mathrm{ev}} \to \overline{\mathfrak{M}}$ and hence is also commutative.

The pull-back by δ of the obstruction theory with point conditions $\overline{\mathbb{G}} \to \mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}}$ now provides the dashed morphism of triangles on the top face of this diagram. Since the lower square in (5.17) is cartesian, $L\delta^*\mathbb{L}_{\overline{\mathscr{M}}/\overline{\mathfrak{M}}^{\mathrm{ev}}} \to \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}}}$ is an isomorphism. Thus we also obtain the dashed arrow $\mathbb{G} \to \mathbb{L}_{\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}}}$ on the lower right, which makes the lower face commutativeand defines a perfect obstruction theory for $\mathscr{M}^{\mathrm{gl}}/\mathfrak{M}^{\mathrm{gl}}$ as claimed.

(3) This follows from the morphism δ' being finite and representable, hence projective, and the push-pull formula of [Man12, Thm.4.1,(iii)]. ♠

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