Mirror symmetry: A brief history.

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This is reconciled with real-world observations by positing that the universe is of the form

$$\mathbb{R}^{1,3} \times X$$

where $\mathbb{R}^{1,3}$ is usual Minkowski space-time and X is a (very small!) six-dimensional compact manifold.

Properties of X should be reflected in properties of the observed world.

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Supersymmetry suggests (to first approximation) that X should be *Ricci-flat*, i.e., be a *Calabi-Yau manifold*.

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Example

Let

$$\mathbb{CP}^4 = (\mathbb{C}^5 \setminus \{(0,0,0,0,0)\})/\mathbb{C}^*$$

be four-dimensional complex projective space, with coordinates

$$x_0, \ldots, x_4.$$

Let X be the three-dimensional complex manifold defined by the equation

$$x_0^5 + \cdots + x_4^5 = 0.$$

This is a Calabi-Yau manifold, by Yau's proof of the Calabi Conjecture.

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Example

Let X be the quintic, given by

$$x_0^5 + \cdots x_4^5 = 0$$

and consider the group action of

$$G = \{(a_0, \ldots, a_4) | a_i \in \mathbb{Z}/5\mathbb{Z} \text{ and } \sum_i a_i = 0\}$$

on X given by

$$(x_0, \ldots, x_4) \mapsto (\xi^{a_0} x_0, \ldots, \xi^{a_4} x_4), \quad \xi = e^{2\pi i/5}.$$

Then X/G is very singular, but there is a resolution $\check{X} \to X/G$.

$\check{X}$ is the mirror of the quintic discovered by Greene and Plesser.

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Enumerative geometry (19th century).

This is the study of questions of the flavor: "How many geometric gadgets of a given sort are contained in some other gadget, or intersect some collection of gadgets."

For example, given two points in $\mathbb{CP}^2$, there is precisely one line (a subset defined by a linear equation) passing through two points.

(Cayley-Salmon) A smooth cubic surface in $\mathbb{CP}^3$ always contains precisely 27 lines.

e.g., the Clebsch diagonal surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3.$$

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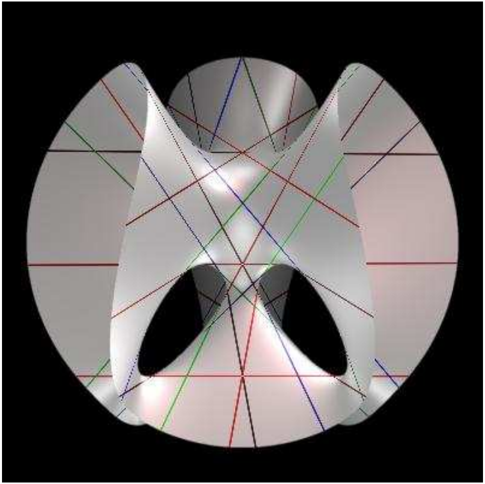

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Let N_1 be the number of lines in X.

Let N_2 be the number of conics in X.

Let N_d be "the number of rational curves of degree d in X". Such a curve is the image of a map $\mathbb{CP}^1 \to \mathbb{CP}^4$ defined by

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$$\int_\alpha \Omega,$$

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This work involved a direct calculation of the numbers N_d. But what is the basic underlying geometry of mirror symmetry?

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Try 1:

V a real finite dim'l vector space	$V^* = \mathrm{Hom}(V, \mathbb{R})$ the dual space.

Try 2:

$V \times V$ with complex structure $J(v_1, v_2) = (-v_2, v_1)$	$V \times V^*$ with symplectic structure $\omega((v_1, w_1), (v_2, w_2)) = \langle w_1, v_2 \rangle - \langle w_2, v_1 \rangle$

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A vector space is not a particularly interesting example.

We can make this more interesting by choosing V to have an integral structure, i.e.,

$$V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$$

where $\Lambda \cong \mathbb{Z}^n$. Set

$$\check{\Lambda} := \{w \in V^* \,|\, \langle w, \Lambda \rangle \subseteq \mathbb{Z}\} \subseteq V^*$$

$X(V) := V \times V/\Lambda$ with complex structure J as before.	$\check{X}(V) := V \times V^*/\check{\Lambda}$ with symplectic structure as before.

While this seems like a very simplistic point of view, in fact this toy example already exhibits rich features of mirror symmetry, which we will explore.

A more general point of view replaces V with a more general manifold with an *affine structure*, and this leads to an extensive program (G.-Siebert) for understanding mirror symmetry in general. We will not go down this route today.

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To continue our exploration, we need to travel to the tropics...

Suppose $L \subseteq V$ is a rationally defined affine linear subspace.

$X(L) := L \times L/(L \cap \Lambda) \subseteq X(V)$ holomorphic submanifold.	$\check{X}(L) := L \times L^{\perp}/(L^{\perp} \cap \check{\Lambda}) \subseteq \check{X}(V)$ Lagrangian submanifold.

These are not topologically very interesting. For example, if $\dim L = 1$, we obtain holomorphic curves which are cylinders.

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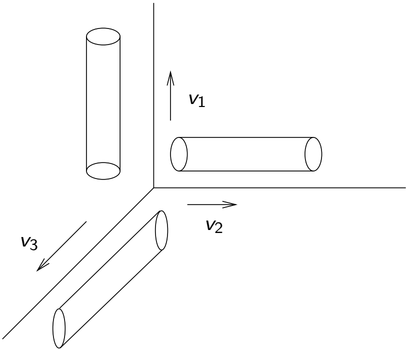
Let's try to get a more interesting "approximate" holomorphic curve by gluing together cylinders, taking three rays meeting at $b \in V$:



We can try to glue the three cylinders by gluing in a surface contained in the fibre $f^{-1}(b)$.

Noting that $H_1(f^{-1}(b),\mathbb{Z}) = \Lambda_b$, the tangent vectors v_1, v_2 and v_3 represent the boundaries of the three cylinders in $H_1(f^{-1}(b),\mathbb{Z})$. Thus the three circles bound a surface if

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Definition

A *parameterized tropical curve* in V is a graph Γ (possibly with non-compact edges with zero or one adjacent vertices) along with

- a weight function w associating a non-negative integer to each edge;
- a proper continuous map $h : \Gamma \to V$

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1. If E is an edge of Γ and $w(E) = 0$, then $h|_E$ is constant; otherwise $h|_E$ is a proper embedding of E into V as a line segment, ray or line of rational slope.
2. *The balancing condition.* For every vertex of Γ with adjacent edges $E_1, \ldots, E_n$, let $v_1, \ldots, v_n \in \Lambda$ be primitive tangent vectors to $h(E_1), \ldots, h(E_n)$ pointing away from $h(V)$. Then

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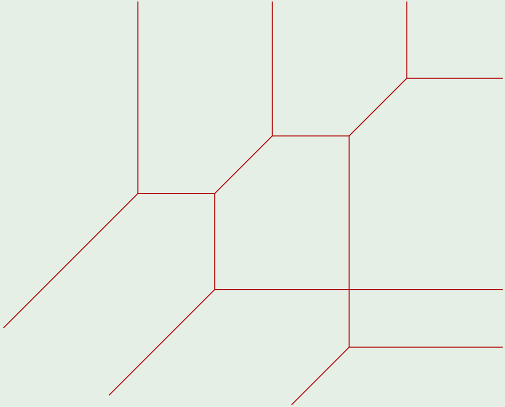
Example

Take $V = \mathbb{R}^2$. Here is a tropical curve:



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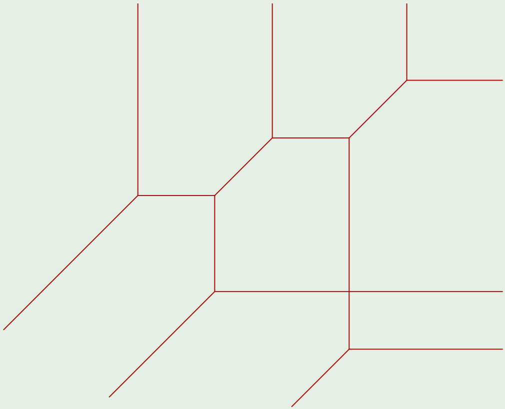

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This curve can be viewed as an approximation to a curve of degree 3 in $\mathbb{CP}^2$.

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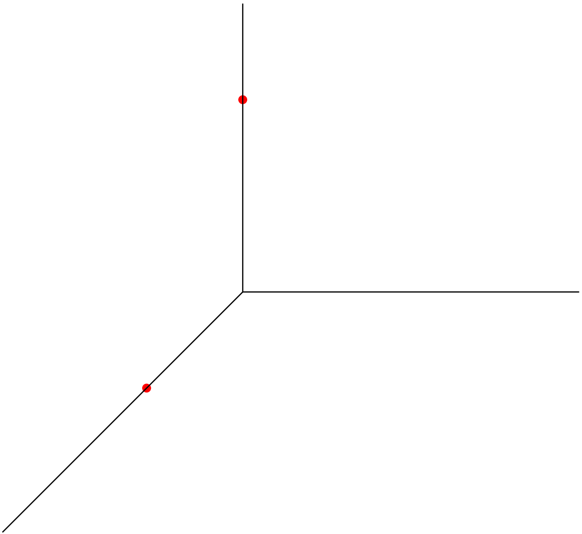

Mirror symmetry for $\mathbb{CP}^2$.

In 1994, Givental gave a mirror for $\mathbb{CP}^2$. This description was enhanced by Barannikov in 1999 to allow mirror calculations to answer the question: "How many rational curves of degree d pass through $3d-1$ points in the complex plane?"

The mirror is a *Landau-Ginzburg model*, the variety $(\mathbb{C}^*)^2$ along with a function

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To extract enumerative predictions, one needs to consider a family of potentials which are perturbations of the above potential, e.g.,

$$W_{\mathbf{t}} := t_0 + (1 + t_1)W + t_2 W^2,$$

and calculate oscillatory integrals of the form

$$\int_\Gamma e^{W_{\mathbf{t}}/\hbar} f(x, y, \mathbf{t}, \hbar) \frac{dx \wedge dy}{xy},$$

where Γ runs over suitably chosen (possibly unbounded) 2-cycles in $(\mathbb{C}^*)^2$, f is a carefully chosen function which puts the above integrals in some "normalized" form, and the result needs to be expanded in a power series of some specially chosen coordinates on $\mathbf{t}$-space.

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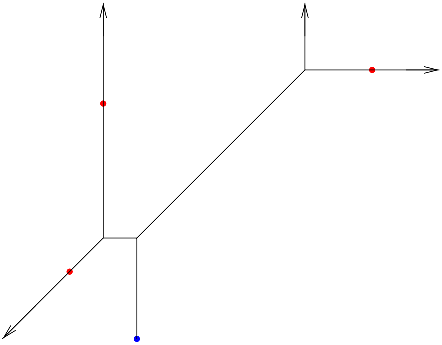

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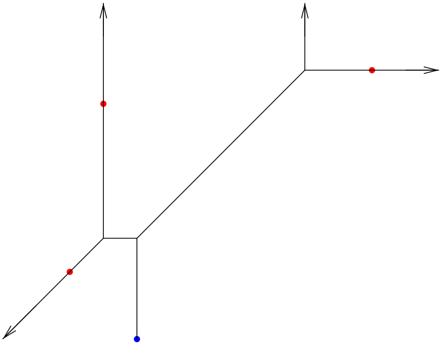

Choose points $P_1, \dots, P_k, Q \in \mathbb{R}^2$ general, and consider all rigid tropical disks passing through some subset of $P_1, \dots, P_k$ and terminating at Q.

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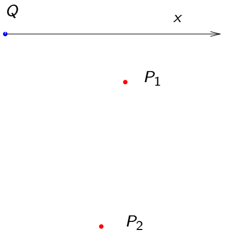
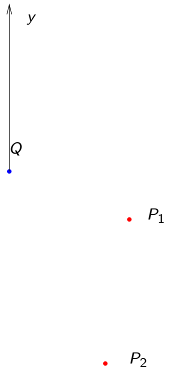
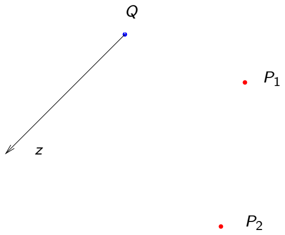


$$W_2 = x$$
$$xyz = \kappa$$



$$W_2 = x + y$$
$$xyz = \kappa$$



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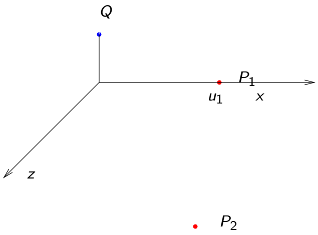

$$W_2 = x + y + z + u_1 xz$$
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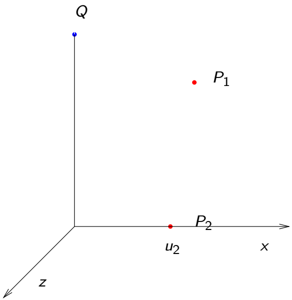

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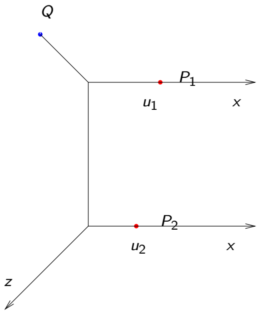

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Take $\Gamma \subset (\mathbb{C}^*)^2$ to be the compact torus

$$\Gamma = \{|x| = |y| = 1\}.$$

Calculate the integral

$$\frac{1}{(2\pi i)^2} \int_\Gamma e^{W_\kappa/\hbar} \frac{dx \wedge dy}{xy}$$

via a Taylor series expansion of the exponential and residues. Via residues, the only terms which contribute are constant on $(\mathbb{C}^*)^2$, i.e., with the same power of x, y and z, using $xyz = \kappa$. The power series expansion selects a set of tropical disks which then must glue at Q to give a tropical curve, the balancing condition being enforced by the integration.

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e.g., $k = 2$:

$$\frac{1}{(2\pi i)^2}\int_\Gamma \Big(1 + \hbar^{-1}(x + y + z + (u_1 + u_2)xz + u_1 u_2 x^2 z)$$
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So we see precisely the contribution from one line (the coefficient of $\kappa\hbar^{-2}(u_1 + u_2)$).