

# NOTES ON GLUING PUNCTURED LOGARITHMIC MAPS

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ABSTRACT. This is an incomplete draft, from September 22nd, 2022, of a paper covering some well-behaved cases of gluing using the formalism of [ACGS1, ACGS2]. A last section remains to be written.

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## 1. INTRODUCTION

One of the key motivations for developing logarithmic Gromov-Witten [AC, Ch, GS4] was to generalize the gluing formulae of Li–Ruan [LR01] and Jun Li [Li02] to more general degenerations. The original gluing formulae consider flat families  $\pi : X \rightarrow B$ , where  $B$  is a non-singular curve with a special point  $b_0 \in B$ ,  $\pi$  is a normal crossing morphism with  $\pi|_{\pi^{-1}(B \setminus \{0\})}$  smooth and  $\pi^{-1}(b_0) =: X_0$  a union of two irreducible divisors  $Y_1, Y_2$  meeting transversally. The above-mentioned gluing formulae then relate the Gromov-Witten invariants of the general fibre of  $\pi$  to relative Gromov-Witten invariants of the pairs  $(Y_1, D), (Y_2, D)$ , where  $D = Y_1 \cap Y_2$ . In principal, one would like to allow the fibre  $X_0$  to have many irreducible components and deeper strata, where more than two irreducible components meet.

Logarithmic Gromov-Witten theory defines the notion of a logarithmic stable map to such targets  $X \rightarrow B$ ; more generally, this morphism just needs to be toroidal rather than normal crossings, i.e., log smooth. However, completely satisfactory generalizations of the gluing formulae have remained elusive. The work of Abramovich, Chen,

Gross and Siebert [ACGS1, ACGS2] sets up a framework for thinking about gluing formulae. This requires, in particular, developing *punctured* stable maps, further generalizing logarithmic stable maps. The essential reason for this is, if given a log smooth curve, the restriction of the log structure to an irreducible component need not be log smooth. In particular, this requires allowing somewhat more general domains. As a side benefit, this generalization introduces the notion of negative contact order. In turn, this gives a richer set of invariants which have proved invaluable for mirror symmetry constructions, see e.g., [GS7, GS8].

The basic setup for a gluing problem for log or punctured maps with target  $X \rightarrow B$  is given by the data of a *decorated tropical type*, reviewed in §2. This is data  $\tau = (G, \sigma, \mathbf{u}, \mathbf{A})$ . Here,  $G$  is a dual intersection graph for a domain curve, with vertices  $V(G)$  corresponding to (unions of) irreducible components, edges  $E(G)$  corresponding to nodes and legs  $L(G)$  corresponding to marked or punctured points. The map  $\sigma : V(G) \cup E(G) \cup L(G) \rightarrow \Sigma(X)$  records which stratum of  $X$  (strata of  $X$  being indexed by cones in the tropicalization  $\Sigma(X)$  of  $X$ ) the corresponding curve feature maps into. The data  $\mathbf{u}$  records the contact orders associated to edges and legs, and the data  $\mathbf{A}$  associates a curve class in  $X$  to each vertex of  $G$ . Together, this data determines a moduli space  $\mathcal{M}(X/B, \tau)$  of punctured log maps marked by  $\tau$ , as defined [ACGS2]. This moduli space is a proper Deligne-Mumford stack over  $B$ , assuming  $X$  is projective over  $B$ , and carries a virtual fundamental class  $[\mathcal{M}(X/B, \tau)]^{\text{virt}}$ .

The question of gluing is then as follows. Suppose given a decorated tropical type  $\tau$  as above, and a subset of edges  $\mathbf{E} \subseteq E(G)$ . By splitting  $G$  at the edges of  $\mathbf{E}$ , we obtain connected graphs  $G_1, \dots, G_r$ . Here, each edge  $E \in \mathbf{E}$  with endpoints  $v_1, v_2$  is replaced by two legs with endpoints  $v_1, v_2$  respectively. By restricting  $\sigma, \mathbf{u}$  and  $\mathbf{A}$  to  $G_i$ , we obtain types  $\tau_1, \dots, \tau_r$ . Further, there is a canonical splitting map

$$\mathcal{M}(X/B, \tau) \rightarrow \prod_{i=1}^r \mathcal{M}(X/B, \tau_i).$$

This splitting map is defined by normalizing the domain curves at the nodes corresponding to edges of  $\mathbf{E}$ . The key question is then: can we relate  $[\mathcal{M}(X/B, \tau)]^{\text{virt}}$  in terms of  $\prod_{i=1}^r [\mathcal{M}(X/B, \tau_i)]^{\text{virt}}$ ?

There have been a number of approaches to this question. First, Kim, Lho and Ruddat [KLR] proved the Li–Ruan and Jun Li degeneration formula in the context of logarithmic Gromov-Witten theory. Yixian Wu, in [Wu], gave a very general gluing formula, under the hypothesis that all gluing strata are toric. This has already proven to be very useful in [GS8], but doesn't give a proof of the Li–Ruan/Jun Li formula (unless the divisor  $D$  is in fact a point).

A very different approach has been pursued by Dhruv Ranganathan, using expanded degenerations, in [Ra]. There, the theoretical difficulties of gluing are removed by allowing target expansions in such a way that the gluing always happens along codimension one strata. However, this can result in an explosion in combinatorial complexity of the problem. But it is not clear whether such an explosion can be avoided in any general approach to the gluing problem.

The basic problem is that the description of the glued moduli space  $\mathcal{M}(X/B, \boldsymbol{\tau})$  in terms of the moduli spaces  $\mathcal{M}(X/B, \boldsymbol{\tau}_i)$  involves a fibre product in the category of fs log schemes; this is roughly encapsulated in one of the main gluing theorems of [ACGS2], quoted here in Theorem 2.3. One of the basic difficulties in log geometry is that the underlying scheme of an fs fibre product can be quite far from the underlying scheme of the ordinary fibre product.

Here, we consider a case of gluing in which, at the virtual level, the ordinary fibre product and fs fibre product are not wildly divergent. We study this in §3, where we prove a four-point lemma helpful for our situation. There, we give criteria for non-emptiness of an fs fibre product of log points, as well as a computation for the number of connected components of this fs fibre product.

Happily, these criteria for non-emptiness have a simple tropical interpretation. In §4, we consider gluing a single curve. In other words, we consider a type  $\boldsymbol{\tau}$  as described above, a set of splitting edges  $\mathbf{E}$ , and the types  $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_r$  obtained from splitting at the edges of  $\mathbf{E}$ . We consider log points  $W_i$  and punctured log maps  $f_i : C_i^\circ/W_i \rightarrow X$  of type  $\boldsymbol{\tau}_i$ ,  $1 \leq i \leq r$ . We assume that if  $E \in \mathbf{E}$  with vertices  $v_1, v_2, v_j \in V(G_{i_j})$ ,  $j = 1, 2$ ,  $p_{v_j, E} \in C_{i_j}^\circ$  the punctured point corresponding to the leg of  $G_{i_j}$  indexed by the flag  $v_j \in E$ , we have  $f_{i_1}(p_{v_1, E}) = f_{i_2}(p_{v_2, E})$ . Thus the maps  $f_i$  may be glued schematically. The question is then: how many logarithmic gluings of the  $f_i$ 's are there?

The existence of a logarithmic gluing is a difficult question, but the number of such gluings, assuming there is at least one, is an easy question given the four-point lemmas of §3. In particular, in Definition 4.2, we define a map of lattices, the *tropical gluing map*  $\Psi$ , which depends only on the tropical data of  $\boldsymbol{\tau}, \mathbf{E}$ , and define the *tropical multiplicity*  $\mu(\boldsymbol{\tau}, \mathbf{E})$  as the order of the torsion part of  $\text{coker } \Psi$ . This lattice map gives the obstruction to gluing tropical maps of types  $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_q$  to obtain a tropical map of type  $\boldsymbol{\tau}$ . In addition, let  $f : C^\circ/W \rightarrow X$  be the universal gluing of the punctured maps  $f_i$ . Then the first result, Theorem 4.4, is:

**Theorem 1.1.**  *$W$  has  $\mu(\boldsymbol{\tau}, \mathbf{E})$  connected components.*

We remark that this is not complete information about  $W$ , as it may have some non-reduced structure. However, when gluing questions are set up properly, this becomes unimportant, as is seen in [Wu] or §5.

We say the gluing situation is *tropically transverse* if  $\text{coker } \Psi$  is in fact finite. In this case, we have Theorem 4.8, again from the four-point lemmas of §3:

**Theorem 1.2.** *If the gluing situation is tropically transverse, then  $W$  is non-empty.*

These results can be viewed as a generalization of the more hands-on constructions of log stable maps beginning with work of Nishinou–Siebert [NS], Argüz [Ar], and [ACGS1, §4.2]. In fact, versions of tropical gluing maps already appeared in [NS].

In §5, we now apply these observations to gluing moduli spaces, with an aim to describe  $[\mathcal{M}(X/B, \boldsymbol{\tau})]^{\text{virt}}$  in terms of  $\prod_{i=1}^r [\mathcal{M}(X/B, \boldsymbol{\tau}_i)]^{\text{virt}}$ . We define an intermediate moduli space  $\mathcal{M}^{\text{sch}}(X, \boldsymbol{\tau})$ . A point in this moduli space is represented by a point in  $\prod_{i=1}^r \mathcal{M}(X/B, \boldsymbol{\tau}_i)$ , corresponding to a collection of punctured maps  $f_i : C_i^\circ/W_i \rightarrow X$ , such that the  $f_i$  glue schematically. This moduli stack can be defined via a Cartesian diagram in the category of ordinary stacks, see Theorem 5.1 for details. The results of

§4 then apply to give us some information about the natural map  $\phi' : \mathcal{M}(X/B, \boldsymbol{\tau}) \rightarrow \mathcal{M}^{\text{sch}}(X/B, \boldsymbol{\tau})$ . Unfortunately, in general it is difficult to extract useful results from this, but tropical transversality of the gluing situation implies virtual surjectivity of this map, with degree given by the tropical multiplicity, so that

$$\phi'_*[\mathcal{M}(X/B, \boldsymbol{\tau})]^{\text{virt}} = \mu(\boldsymbol{\tau}, \mathbf{E})[\mathcal{M}^{\text{sch}}(X/B, \boldsymbol{\tau})]^{\text{virt}}.$$

On the other hand, if the gluing strata are sufficiently nice (e.g., smooth) and certain other conditions hold, then  $[\mathcal{M}^{\text{sch}}(X/B, \boldsymbol{\tau})]^{\text{virt}}$  can be calculated as an ordinary Gysin pull-back of  $\prod_{i=1}^r [\mathcal{M}(X/B, \boldsymbol{\tau}_i)]^{\text{virt}}$ . See Remark 5.2 for details.

In §5.2, we specialize to the degeneration situation considered in [ACGS1]. Here,  $B$  is a curve or spectrum of a DVR over  $\mathbb{k}$ , with divisorial log structure coming from a closed point  $b_0 \in B$ . Let  $X_0$  be the fibre over  $b_0$ . In [ACGS1], we showed that virtual irreducible components of  $\mathcal{M}(X_0/b_0)$  were indexed by rigid tropical curves. We recast the earlier discussion in the gluing situation provided by a rigid tropical curve.

While I view these results as pretty weak, they in fact appear to be strong enough to be useful in many circumstances. In particular, in §6, we give a very short proof of the Li-Ruan/Jun Li degeneration formula. This is not new even in the logarithmic setup: [KLR] first obtained this result. However, it is pleasant to see that the more general setup proves this special case without pain. The reason this works easily is that the gluing situation is always tropically transverse in this case.

Along the way, in §6, we first prove a more generally useful comparison result between punctured and log invariants for irreducible components of degenerations. Explicitly, given  $X \rightarrow B$  as in the degeneration situation, often one needs to look at moduli spaces of punctured maps into strata of  $X_0$ . In general, this may involve additional information, but if the stratum is an irreducible component  $Y \subseteq X_0$ , life becomes simpler. In particular,  $Y$  carries two possible log structures, one induced from  $X$ , and one the divisorial log structure coming from the union of substrata of  $Y$ . We write this latter log structure as  $\bar{Y}$ . We then obtain in Theorem 6.1 an isomorphism of underlying stacks  $\mathcal{M}(Y/b_0, \boldsymbol{\tau}) \cong \mathfrak{M}(\bar{Y}, \bar{\boldsymbol{\tau}})$ , where the type  $\bar{\boldsymbol{\tau}}$  is derived from the type  $\boldsymbol{\tau}$ . Happily, the latter type does not involve punctures, and hence gives a more familiar moduli space. We note these stacks are not isomorphic as log stacks, but they do carry the same virtual fundamental class.

As a final application, and as an example of what tricks may be applied when we are not in a tropically transverse situation, we give an inductive description of the canonical wall structure of [GS8] for a degeneration of K3 surfaces. This will be of importance in [GHKS].

*Acknowledgements:* Some of the material here, in an earlier form, was originally written with the intention of appearing in [ACGS2]. So it has had a lot of influence from my co-authors of that project, Dan Abramovich, Qile Chen and Bernd Siebert. In addition, the last section was greatly influenced by discussions with my coauthors on [GHKS], i.e., Paul Hacking, Sean Keel, and Bernd Siebert. This paper has also benefited from discussions with Evgeny Goncharov, Sam Johnston, Dhruv Ranganathan, and Yu Wang. It was supported by the ERC Advance Grant MSAG.

## 2. PRELIMINARIES

**2.1. Tropical maps and moduli of punctured curves.** We will work with a relative target space  $X \rightarrow B$ , a proper log smooth morphism. We further assume that the log structure on  $X$  is Zariski, and that  $X$  satisfies assumptions required to guarantee finite type moduli spaces of punctured curves. At the moment, [ACGS2] requires that  $\overline{\mathcal{M}}_X^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  be generated by global sections, so what follows will be written with this assumption. However, see [J22] for finiteness results without this condition. Typically  $B$  itself is taken to be an affine scheme and  $B$  is either log smooth over  $\text{Spec } \mathbb{k}$  or a log point  $\text{Spec}(Q_B \rightarrow B)$ .

We briefly review notation from [ACGS1, ACGS2] for tropical maps to  $\Sigma(X)$  and punctured log maps to  $X$  as developed in [ACGS1, §2.5] and [ACGS2, §2.2].

In what follows, **Cones** denotes the category of rational polyhedral cones with integral structure, i.e., objects are rational polyhedral cones  $\omega \subseteq N_\omega \otimes_{\mathbb{Z}} \mathbb{R}$  for  $N_\omega$  the lattice of integral tangent vectors to  $\omega$ . Morphisms are maps of cones induced by maps of the corresponding lattices. We write  $\omega_{\mathbb{Z}} = \omega \cap N_\omega$  for the set of integral points of  $\omega$ .

A *generalized cone complex* is a topological space with a presentation as the colimit of an arbitrary diagram in the category **Cones** with all morphisms being face morphisms. If  $\Sigma$  is such a generalized cone complex, we write  $\sigma \in \Sigma$  if  $\sigma$  is a cone in the presentation and  $|\Sigma|$  for the underlying topological space. A morphism of generalized cone complexes is a continuous map  $f : |\Sigma| \rightarrow |\Sigma'|$  such that for each  $\sigma \in \Sigma$ , the induced map  $\sigma \rightarrow |\Sigma'|$  factors through a cone map  $\sigma \rightarrow \sigma' \in \Sigma'$ .

There is a functor from fine saturated log schemes to generalized cone complexes, written as  $X \mapsto \Sigma(X)$ . There is a one-to-one correspondence between elements in the presentation  $\Sigma(X)$  and logarithmic strata of  $X$ . If  $\mathcal{M}_X$  denotes the log structure on  $X$  with ghost sheaf  $\overline{\mathcal{M}}_X$ , and  $\bar{\eta}$  is a geometric generic point of a log stratum, then the corresponding cone is  $\text{Hom}(\overline{\mathcal{M}}_{X, \bar{\eta}}, \mathbb{R}_{\geq 0})$ . If  $\sigma \in \Sigma(X)$ , we write  $X_\sigma \subseteq X$  for the corresponding (closed) stratum.

We consider graphs  $G$ , with sets of vertices  $V(G)$ , edges  $E(G)$  and legs  $L(G)$ . In what follows, we will frequently confuse  $G$  with its topological realisation  $|G|$ . Legs will correspond to marked or punctured points of punctured curves, and are rays in the marked case and compact line segments in the punctured case. We view a compact leg as having only one vertex. An abstract tropical curve over  $\omega \in \mathbf{Cones}$  is data  $(G, \mathbf{g}, \ell)$  where  $\mathbf{g} : V(G) \rightarrow \mathbb{N}$  is a genus function and  $\ell : E(G) \rightarrow \text{Hom}(\omega_{\mathbb{Z}}, \mathbb{N}) \setminus \{0\}$  determines edge lengths.

Associated to the data  $(G, \ell)$  is a generalized cone complex  $\Gamma(G, \ell)$  along with a morphism of cone complexes  $\Gamma(G, \ell) \rightarrow \omega$  with fibre over  $s \in \text{Int}(\omega)$  being a tropical curve, i.e., a metric graph, with underlying graph  $G$  and affine edge length of  $E \in E(G)$  being  $\ell(E)(s) \in \mathbb{R}_{\geq 0}$ . Associated to each vertex  $v \in V(G)$  of  $G$  is a copy  $\omega_v$  of  $\omega$  in  $\Gamma(G, \ell)$ . Associated to each edge or leg  $E \in E(G) \cup L(G)$  is a cone  $\omega_E \in \Gamma(G, \ell)$  with  $\omega_E \subseteq \omega \times \mathbb{R}_{\geq 0}$  and the map to  $\omega$  given by projection onto the first coordinate. This projection fibres  $\omega_E$  in compact intervals or rays over  $\omega$  (rays only in the case of a leg representing a marked point).

A *family of tropical maps* to  $\Sigma(X)$  over  $\omega \in \mathbf{Cones}$  is a morphism of cone complexes

$$h : \Gamma(G, \ell) \rightarrow \Sigma(X).$$

If  $s \in \text{Int}(\omega)$ , we may view  $G$  as the fibre of  $\Gamma(G, \ell) \rightarrow \omega$  over  $s$  as a metric graph, and write

$$h_s : G \rightarrow \Sigma(X)$$

for the corresponding tropical map with domain  $G$ . The *type* of such a family consists of the data  $\tau := (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$  where

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \rightarrow \Sigma(X)$$

associates to  $x \in V(G) \cup E(G) \cup L(G)$  the minimal cone of  $\Sigma(X)$  containing  $h(\omega_x)$ . Further,  $\mathbf{u}$  associates to each (oriented) edge or leg  $E \in E(G) \cup L(G)$  the corresponding *contact order*  $\mathbf{u}(E) \in N_{\boldsymbol{\sigma}(E)}$ , the image of the tangent vector  $(0, 1) \in N_{\omega_E} = N_\omega \oplus \mathbb{Z}$  under the map  $h$ .

As we shall only consider tropicalizations of pre-stable punctured curves (see [ACGS2, Def. 2.5], following [ACGS2, Prop. 2.21] we may assume that for  $L \in L(G)$  with adjacent vertex  $v \in V(G)$  giving  $\omega_L, \omega_v \subseteq \Gamma(G, \ell)$ , we have

$$(2.1) \quad h(\omega_L) = (h(\omega_v) + \mathbb{R}_{\geq 0}\mathbf{u}(L)) \cap \boldsymbol{\sigma}(L) \subseteq N_{\boldsymbol{\sigma}(L), \mathbb{R}}.$$

In other words, the images of legs extend as far as possible inside their cones.

A *decorated type* is data  $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$  where  $\mathbf{A} : V(G) \rightarrow H_2(X)$  associates a curve class to each vertex of  $G$ . The *total curve class* of  $\mathbf{A}$  is  $A = \sum_{v \in V(G)} \mathbf{A}(v)$ .

We also have a notion of a contraction morphism of types  $\phi : \boldsymbol{\tau} \rightarrow \boldsymbol{\tau}'$ , see [ACGS1, Def. 2.24]. This is a contraction of edges on the underlying graphs, and the additional data satisfies some relations as follows. If  $x \in V(G) \cup E(G) \cup L(G)$ , then  $\boldsymbol{\sigma}'(\phi(x)) \subseteq \boldsymbol{\sigma}(x)$  (if  $x$  is an edge, it may be contracted to a vertex by  $\phi$ ). Further, if  $E \in E(G) \cup L(G)$  then  $\mathbf{u}(E) = \mathbf{u}'(\phi(E))$  under the inclusion  $N_{\boldsymbol{\sigma}'(\phi(E))} \subseteq N_{\boldsymbol{\sigma}(E)}$ , provided that  $E$  is not an edge contracted by  $\phi$ .

We say a type  $\boldsymbol{\tau}$  is *realizable* if there exists a family of tropical maps to  $\Sigma(X)$  of type  $\boldsymbol{\tau}$ . We also say  $\boldsymbol{\tau} = (\tau, \mathbf{A})$  is realizable if  $\tau$  is realizable. In this paper, we will only deal with realizable types. As a consequence, we will not need the more general notion of global type discussed in [ACGS2, §3].

If a type  $\boldsymbol{\tau}$  is realizable, then there is a universal family of tropical maps of type  $\boldsymbol{\tau}$ , parameterized by an object of  $\mathbf{Cones}$ . Hopefully without confusion, we will generally write this cone as  $\tau$ . Hence we have a cone complex  $\Gamma(G, \ell)$  equipped with a map to  $\tau$  and a map of cone complexes  $h = h_\tau : \Gamma(G, \ell) \rightarrow \Sigma(X)$ . Generally we write  $h$  rather than  $h_\tau$  when unambiguous. Note that for each  $x \in E(G) \cup L(G) \cup V(G)$ , we thus obtain  $\tau_x \in \Gamma(G, \ell)$  the corresponding cone.

We write  $\mathcal{A}_X$  for the Artin fan of  $X$ , see [ACMW], as well as [ACGS1, §2.2] for a summary. With  $X \rightarrow B$  log smooth with  $X$  Zariski, we obtain a morphism of Artin fans  $\mathcal{A}_X \rightarrow \mathcal{A}_B$  and define

$$\mathcal{X} := \mathcal{A}_X \times_{\mathcal{A}_B} B.$$

We refer to [ACGS2, Defs. 2.10, 2.13, 2.14] for the notion of a family  $\pi : C^\circ \rightarrow W$  of punctured curves and pre-stable or stable punctured log maps  $f : C^\circ/W \rightarrow X$  or  $f : C^\circ/W \rightarrow \mathcal{X}$  defined over  $B$ .

Given a punctured log map with domain  $C^\circ \rightarrow W$  and  $W = \text{Spec}(Q \rightarrow \kappa)$  for  $\kappa$  an algebraically closed field and target  $X$  or  $\mathcal{X}$ , we obtain by functoriality of tropicalizations a family of tropical maps

$$(2.2) \quad \begin{array}{ccc} \Sigma(C) = \Gamma(G, \ell) & \longrightarrow & \Sigma(X) \\ \downarrow & & \downarrow \\ \Sigma(W) = \omega = Q_{\mathbb{R}}^{\vee} & \longrightarrow & \Sigma(B) \end{array}$$

parameterized by  $W$ . The *type* of the punctured map is then the type  $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$  of this family of tropical maps. We recall that the punctured map  $f : C^\circ/W \rightarrow X$  is *basic* if (2.2) is the universal family of tropical maps of type  $\tau$ .

Given a realizable type  $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ , [ACGS2, Def. 3.7] defines the notion of a *marking* or *weak marking* of a punctured map by  $\tau$ .<sup>1</sup> Roughly, a *weak marking* of a punctured map  $f : C^\circ/W \rightarrow X$  involves the following information. (1) A marking of the underlying domain curve  $\underline{C}$  by  $G$ . In other words, we have a pre-stable curve  $\underline{C}_v$  for each  $v \in V(G)$  of genus  $\mathbf{g}(v)$ , a marked point  $p_L \in \underline{C}_v$  for each leg  $L \in L(G)$  adjacent to  $v$ , and marked points  $q_{E,1}, q_{E,2}$  in  $\underline{C}_{v_1}, \underline{C}_{v_2}$  for each edge  $E$  connecting  $v_1$  to  $v_2$ . Further,  $\underline{C}$  is obtained as a marked curve by identifying pairs  $q_{E,1}, q_{E,2}$  for all  $E \in E(G)$ . (2) For each subcurve or punctured or nodal section  $Z$  of  $\underline{C}$ , indexed by an element  $x \in V(G) \cup L(G) \cup E(G)$ , the morphism  $\underline{f}|_Z$  factors through the closed stratum  $X_{\boldsymbol{\sigma}(x)}$  of  $X$ . (3) For any geometric point  $\bar{w} \rightarrow W$  giving a curve of type  $\tau_{\bar{w}} = (G_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$ , the contraction morphism  $G_{\bar{w}} \rightarrow G$  induced by the marking of the domain yields a contraction morphism of types  $\tau_{\bar{w}} \rightarrow \tau$ .

A *marking* of a punctured map  $f : C^\circ/W \rightarrow X$  is a weak marking satisfying an additional requirement that a certain natural monoid ideal on  $\mathcal{M}_W$  defines an idealized log structure on  $W$ . See [ACGS2, Def. 3.7] for full details.

In either case, if further  $\boldsymbol{\tau} = (\tau, \mathbf{A})$  is a decoration of  $\tau$ , then  $f : C^\circ/W \rightarrow X$  is (weakly)  $\boldsymbol{\tau}$ -marked if in addition to be (weakly)  $\tau$ -marked, for each  $v \in V(G)$ , the curve class associated the the stable map  $\underline{f}$  restricted to the subcurve indexed by  $v$  is  $\mathbf{A}(v)$ .

In particular, this gives rise to the following moduli spaces:

- (1)  $\mathcal{M}(X, \boldsymbol{\tau})$  (resp.  $\mathcal{M}'(X, \boldsymbol{\tau})$ ) the moduli space of (weakly)  $\boldsymbol{\tau}$ -marked stable punctured maps.
- (2)  $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$  (resp.  $\mathfrak{M}'(\mathcal{X}, \boldsymbol{\tau})$ ) the moduli space of (weakly)  $\boldsymbol{\tau}$ -marked punctured maps to  $\mathcal{X}$ .
- (3)  $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$  (resp.  $\mathfrak{M}'(\mathcal{X}, \boldsymbol{\tau})$ ) the moduli space of (weakly)  $\boldsymbol{\tau}$ -marked punctured maps to  $\mathcal{X}$ . Note here that while curve classes in  $\mathcal{X}$  are meaningless, the

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<sup>1</sup>The cited reference applies to global types, but by [ACGS2, Lem. 3.5], giving a realizable global type is the same as giving a realizable type. Since all our types are realizable here, we ignore the notion of global type.

decoration  $\mathbf{A}$  on  $\tau$  affects the notion of isomorphism in the categories  $\mathfrak{M}(\mathcal{X}, \tau)$  or  $\mathfrak{M}'(\mathcal{X}, \tau)$ , and there is an étale morphism  $\mathfrak{M}(\mathcal{X}, \tau) \rightarrow \mathfrak{M}(\mathcal{X}, \tau)$ .

In general, we are always working over  $B$ , but when we need to be more precise, we write  $\mathcal{M}(X/B, \tau)$ .

By [ACGS2, Thm. 3.10], all of the above moduli spaces are algebraic stacks and  $\mathcal{M}(X, \tau)$  and  $\mathcal{M}'(X, \tau)$  are Deligne-Mumford. Further, there are natural morphisms

$$(2.3) \quad \begin{aligned} \varepsilon &: \mathcal{M}(X, \tau) \rightarrow \mathfrak{M}(\mathcal{X}, \tau) \\ \varepsilon &: \mathcal{M}'(X, \tau) \rightarrow \mathfrak{M}'(\mathcal{X}, \tau) \end{aligned}$$

given by composing a punctured log map  $C^\circ \rightarrow X$  with the canonical map  $X \rightarrow \mathcal{X}$ . [ACGS2, §4] then gives a perfect relative obstruction theory for  $\varepsilon$ .

In general, it appears that the  $\tau$ -marked moduli spaces are more important than the weakly  $\tau$ -marked moduli spaces. In particular, [ACGS2, Prop. 3.31] shows that  $\mathfrak{M}(\mathcal{X}, \tau)$  is a closed substack of  $\mathfrak{M}'(\mathcal{X}, \tau)$  defined by a nilpotent ideal. Further, in the cases of greatest interest for this paper ( $B$  log smooth over  $\mathrm{Spec} \mathbb{k}$  or  $B = \mathrm{Spec} \mathbb{k}^\dagger$ , the standard log point),  $\mathfrak{M}(\mathcal{X}, \tau)$  is in fact reduced and pure-dimensional, see [ACGS2, Prop. 3.28]. In fact, while  $\mathfrak{M}(\mathcal{X}, \tau)$  may be quite poorly behaved globally, it has a simple local structure coming from the fact that it is idealized log smooth over  $B$ , see [ACGS2, Thm. 3.24, Rem. 3.25]. The main point for including the weakly marked moduli spaces is that they naturally occur in the gluing formalism.

If there is a contraction morphism between decorated global types  $\phi : \tau \rightarrow \tau'$ , we obtain a forgetful map  $\mathcal{M}(X, \tau) \rightarrow \mathcal{M}(X, \tau')$ . This gives rise to a stratified description of these moduli spaces, see [ACGS2, Rem. 3.29].

If  $\tau = (G, \sigma, \mathbf{u}, \mathbf{A})$  denotes a choice of decorated type, and  $I \subseteq E(G) \cup L(G)$  is a collection of edges and legs, then we write

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau) = \mathfrak{M}^{\mathrm{ev}(I)}(\mathcal{X}, \tau) := \mathfrak{M}(\mathcal{X}, \tau) \times_{\underline{\mathcal{X}}^I} \underline{X}^I.$$

Here  $\underline{\mathcal{X}}^I$  denotes the product of  $|I|$  copies of  $\underline{\mathcal{X}}$  over  $\underline{S}$ , and similarly  $\underline{X}^I$ ; the morphism  $\mathfrak{M}(\mathcal{X}, \tau) \rightarrow \underline{\mathcal{X}}^I$  is given by evaluation at the nodes and punctured points indexed by elements of  $I$ , and  $\underline{X}^I \rightarrow \underline{\mathcal{X}}^I$  is induced by the canonical smooth map  $\underline{X} \rightarrow \underline{\mathcal{X}}$ . The map  $\varepsilon$  then factors as

$$(2.4) \quad \mathcal{M}(X, \tau) \xrightarrow{\varepsilon^{\mathrm{ev}}} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau) \longrightarrow \mathfrak{M}(\mathcal{X}, \tau).$$

The second morphism is smooth, while  $\varepsilon^{\mathrm{ev}}$  also possesses a relative obstruction theory compatible with the morphism  $\varepsilon$  of (2.3), see [ACGS2, §4.2].

**2.2. The gluing formalism.** We may now describe the key gluing formalism of [ACGS2]. We begin with the *standard gluing situation*.

**Notation 2.1** (The standard gluing situation). We fix a target  $X \rightarrow B$ , a proper log smooth morphism. We further assume that the log structure on  $X$  is Zariski, and that  $X$  satisfies assumptions required to guarantee finite type moduli spaces of punctured curves. Further we assume that  $B$  is either log smooth over  $\mathrm{Spec} \mathbb{k}$  or  $B = \mathrm{Spec} \mathbb{k}^\dagger$ , the standard log point. Fix a realisable type  $\tau = (G, \mathbf{g}, \sigma, \mathbf{u})$  of tropical map to  $\Sigma(X)/\Sigma(B)$ . We select a set of splitting edges  $\mathbf{E} \subseteq E(G)$ , and let  $G_1, \dots, G_r$  be the



connected components of the graph obtained by splitting  $G$  at the edges of  $\mathbf{E}$ , i.e., replacing each edge  $E \in \mathbf{E}$  with endpoints  $v_1, v_2$  with two legs with endpoints  $v_1, v_2$  respectively. We write these two legs as flags  $(E, v_1)$  and  $(E, v_2)$ . We then let  $\tau_1, \dots, \tau_r$  be the induce set of decorated types with underlying graphs  $G_1, \dots, G_r$ . Let  $\mathbf{L} \subseteq \bigcup_{i=1}^r L(G_i)$  be the subset of all legs obtained from splitting edges, and  $\mathbf{L}_i = \mathbf{L} \cap L(G_i)$ . For  $v \in V(G)$ , let  $i(v) \in \{1, \dots, r\}$  denote the connected component  $G_i$  containing  $v$ .

For each  $E \in \mathbf{E}$  denote by  $\mathfrak{M}'_E(\mathcal{X}, \tau)$  the image of the nodal section  $s_E : \mathfrak{M}'(\mathcal{X}, \tau) \rightarrow \mathfrak{C}'^\circ(\mathcal{X}, \tau)$  with the restriction of the log structure on the universal domain  $\mathfrak{C}'^\circ(\mathcal{X}, \tau)$ . Denote further by  $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau)$  the fs fiber product

$$(2.5) \quad \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) = \mathfrak{M}'_{E_1}(\mathcal{X}, \tau) \times_{\mathfrak{M}'(\mathcal{X}, \tau)}^{\text{fs}} \cdots \times_{\mathfrak{M}'(\mathcal{X}, \tau)}^{\text{fs}} \mathfrak{M}'_{E_r}(\mathcal{X}, \tau),$$

where  $E_1, \dots, E_r \in E(G)$  are the edges in  $\mathbf{E}$ . With this enlarged log structure, the pull-back  $\widetilde{\mathfrak{C}}'^\circ(\mathcal{X}, \tau) \rightarrow \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau)$  of the universal domain has sections  $\tilde{s}_E$ ,  $E \in \mathbf{E}$ , in the category of log stacks. Moreover,  $\underline{\text{ev}}_{\mathbf{E}}$  lifts to a logarithmic evaluation morphism

$$(2.6) \quad \text{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) \longrightarrow \prod_{E \in \mathbf{E}} \mathcal{X},$$

with  $E$ -component equal to  $\tilde{f} \circ \tilde{s}_E$  for  $\tilde{f} : \widetilde{\mathfrak{C}}'^\circ(\mathcal{X}, \tau) \rightarrow \mathcal{X}$  the universal punctured morphism.

Similarly, for each of the types  $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \mathbf{u}_i)$  obtained by splitting and  $L \in L(G_i)$ , denote by  $\mathfrak{M}'_L(\mathcal{X}, \tau_i)$  the image of the punctured section  $s_L : \mathfrak{M}'(\mathcal{X}, \tau_i) \rightarrow \mathfrak{C}'^\circ(\mathcal{X}, \tau_0)$  defined by  $L$ , again endowed with the pull-back of the log structure on  $\mathfrak{C}'^\circ(\mathcal{X}, \tau_0)$ . With  $L_1, \dots, L_s$  the legs of  $\mathbf{L}_i$ , define the stack

$$\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) = \left( \mathfrak{M}'_{L_1}(\mathcal{X}, \tau_i) \times_{\mathfrak{M}'(\mathcal{X}, \tau_i)}^{\text{f}} \cdots \times_{\mathfrak{M}'(\mathcal{X}, \tau_i)}^{\text{f}} \mathfrak{M}'_{L_s}(\mathcal{X}, \tau_i) \right)^{\text{sat}},$$

where sat denotes saturation. This stack differs from  $\mathfrak{M}'(\mathcal{X}, \tau_i)$  by adding the pull-back of the log structure of each puncture, so that the pull-back  $\widetilde{\mathfrak{C}}'^\circ(\mathcal{X}, \tau_i) \rightarrow \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i)$  of the universal curve now has punctured sections in the category of log stacks. We define the evaluation morphism

$$(2.7) \quad \text{ev}_{\mathbf{L}} : \prod_{i=1}^r \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) \longrightarrow \prod_{E \in \mathbf{E}} \mathcal{X} \times \mathcal{X},$$

by taking as  $E$ -component the evaluation at the corresponding two sections  $s_{E,v}, s_{E,v'}$ , observing the chosen orientation of  $E$ .

It is worth noting the following (see [ACGS2, Prop. 5.5]).

**Proposition 2.2.** *The canonical map  $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) \rightarrow \mathfrak{M}'(\mathcal{X}, \tau)$  induces an isomorphism of underlying stacks, while the canonical maps  $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) \rightarrow \mathfrak{M}'(\mathcal{X}, \tau_i)$  induces an isomorphism on reductions.*

There are evaluation space versions of this. We set

$$\widetilde{\mathfrak{M}}'^{\text{ev}}(\mathcal{X}, \tau_i) = \mathfrak{M}'^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) \times_{\mathfrak{M}'(\mathcal{X}, \tau_i)} \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i)$$

and

$$\widetilde{\mathfrak{M}}'^{\text{ev}}(\mathcal{X}, \tau) = \mathfrak{M}'^{\text{ev}(\mathbf{E})}(\mathcal{X}, \tau) \times_{\mathfrak{M}'(\mathcal{X}, \tau)} \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau).$$

There is a natural splitting map

$$(2.8) \quad \delta_{\mathfrak{M}} : \mathfrak{M}'(\mathcal{X}, \tau) \rightarrow \prod_{i=1}^r \mathfrak{M}'(\mathcal{X}, \tau_i)$$

as defined in [ACGS2, Prop. 5.4], taking a  $\tau$ -marked map and splitting it at the nodes marked by the edges in  $\mathbf{E}$ . This lifts to

$$(2.9) \quad \delta_{\mathfrak{M}} : \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{E})}(\mathcal{X}, \tau) \rightarrow \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i),$$

and is shown to be a finite morphism in [ACGS2, Cor. 5.13]. Passing to the ev-spaces is key here: the morphism in (2.8) is rarely finite or even proper. This also gives an upgrading of the evaluation morphisms:

$$(2.10) \quad \begin{aligned} \text{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{E})}(\mathcal{X}, \tau) &\rightarrow \prod_{E \in \mathbf{E}} X, \\ \text{ev}_{\mathbf{L}} : \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) &\rightarrow \prod_{E \in \mathbf{E}} X \times X. \end{aligned}$$

We review the key gluing results of [ACGS2]. We first describe the glued moduli space as an fs fibre product:

**Theorem 2.3.** *Suppose given a gluing situation as in Notation 2.1. Then the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{L})}(\mathcal{X}, \tau) & \xrightarrow{\delta_{\mathfrak{M}}} & \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) \\ \text{ev}_{\mathbf{E}} \downarrow & & \downarrow \text{ev}_{\mathbf{L}} \\ \prod_{E \in \mathbf{E}} X & \xrightarrow{\Delta} & \prod_{E \in \mathbf{E}} X \times X \end{array}$$

with  $\Delta$  the product of diagonal embeddings and the other arrows defined in (2.9) and (2.10), is cartesian in the category of fs log stacks. We remind the reader that all products in this square are taken over  $B$ .

An analogous statement holds for  $\tau$  replaced by a decorated global type  $\boldsymbol{\tau} = (\tau, \mathbf{A})$ , or replacing  $\widetilde{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau)$ ,  $\widetilde{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \tau_i)$  with the analogous moduli spaces of stable maps to  $X/B$ ,  $\widetilde{\mathcal{M}}'(X, \tau)$ ,  $\widetilde{\mathcal{M}}'(X, \tau_i)$ .

We remark that in this theorem, the fact that all fibre products are over  $B$  can make the actual calculation of fibre products more difficult than they need to be. But it is often enough to work over  $\text{Spec } \mathbb{k}$ , as the following proposition (see [ACGS2, Prop. 5.11]) shows:

**Proposition 2.4.** *Let  $B$  be an affine log scheme equipped with a global chart  $P \rightarrow \mathcal{M}_B$  inducing an isomorphism  $P \cong \Gamma(B, \overline{\mathcal{M}}_B)$ . Let  $\tau$  be a type of tropical map for  $X/B$ , with underlying graph connected. Then there are isomorphisms  $\mathfrak{M}(\mathcal{X}, \tau) \cong \mathfrak{M}(\mathcal{X}/\text{Spec } \mathbb{k}, \tau)$  and  $\mathcal{M}(X, \tau) \cong \mathcal{M}(X/\text{Spec } \mathbb{k}, \tau)$ .*

Finally, to make contact with stable punctured maps to  $X$ , we have [ACGS2, Prop. 5.15]:

**Theorem 2.5.** *In the situation of Theorem 2.3 there is a cartesian diagram*

$$(2.11) \quad \begin{array}{ccc} \mathcal{M}(X, \tau) & \xrightarrow{\delta} & \prod_{i=1}^r \mathcal{M}(X, \tau_i) \\ \varepsilon^{\text{ev}} \downarrow & & \downarrow \hat{\varepsilon} = \prod_i \varepsilon_i^{\text{ev}} \\ \mathfrak{M}^{\text{ev}(\mathbf{E})}(\mathcal{X}, \tau) & \xrightarrow{\delta^{\text{ev}}} & \prod_{i=1}^r \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) \end{array}$$

with horizontal arrows the canonical splitting maps (see [ACGS2, Prop. 5.4], and the vertical arrows the canonical strict morphisms of (2.4).

Analogous statements hold for decorated and for weakly marked versions of the moduli stacks, see [ACGS2, Def. 3.8].

The evaluation spaces  $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau)$  etc. play a crucial role here. First, if instead one used the spaces  $\mathfrak{M}(\mathcal{X}, \tau)$ , there would be no way to obtain a Cartesian diagram, as the splitting map on the level of punctured maps to Artin fans has no way to impose a matching condition at the schematic level. Using the evaluation spaces allows  $\delta^{\text{ev}}$  to impose both schematic and logarithmic matching conditions.

Second, the splitting map at the level of the spaces of punctured maps to Artin fans is very poorly behaved, being neither representable nor proper. However,  $\delta^{\text{ev}}$ , as this theorem states, is in fact finite and representable. Hence we may use it to push-forward Chow classes, and in particular, as [ACGS2, Thm. 5.17] points out, the compatibility of obstruction theories then allows a calculation

$$\delta_*([\mathcal{M}(X, \tau)]^{\text{virt}}) = \hat{\varepsilon}^! \delta_*^{\text{ev}}[\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau)]$$

of the virtual fundamental class of  $\mathcal{M}(X, \tau)$ .

Thus the main task is finding a useful expression for  $\delta_*^{\text{ev}}[\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau)]$ . While the Artin stacks of the type  $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau)$  may seem very forbidding, in a certain sense they are very well-behaved: they are idealized log smooth over  $B$ , see [ACGS2, Thm. 3.24]. This means that there are local descriptions of these stacks as unions of strata of toric varieties. Further, these local descriptions can be determined very explicitly from the tropical description of the types  $\tau$  and  $\tau_i$ .

This effective description of these moduli spaces has led Yixian Wu [Wu], in the case that all gluing strata are toric, to give an effective formula for the Chow class  $\delta_*[\mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau)]$  as a weighted sum of strata of  $\prod_{i=1}^r \mathfrak{M}^{\text{ev}}(\mathcal{X}, \tau_i)$ . This formula has already proved to be very useful, see e.g., [GS8].

Here we wish to develop a gluing formalism in a complementary direction, which, although very far from general, is also useful.

### 3. FOUR-POINT LEMMAS: FIBRE PRODUCTS OF LOG POINTS

The key point of this note is to understand the gluing fibre diagram of Theorem 2.3 by studying fibre products of log points. We carry this study out in this section; unfortunately, this is rather dry. An fs fibre product of log points can be quite subtle. Even determining whether such a fibre product is non-empty is difficult, as a “four-point lemma” does not hold widely in log geometry, see [Og, III Prop. 2.2.3] for such a result. Here we will generally be interested in the number of connected components of a fibre product for application to specific gluing situations.

We start with a small lemma:

**Lemma 3.1.** *Let  $W := \text{Spec}(Q \rightarrow \kappa)$  be a log point with  $Q$  a sharp fine monoid. Then  $W^{\text{sat}}$  is a disjoint union of possibly non-reduced points, and the number of connected components of  $W^{\text{sat}}$  is  $|(Q^{\text{gp}})_{\text{tors}}|$ .*

*Proof.* By construction of the saturation, [Og, III Prop. 2.1.5],

$$W^{\text{sat}} = W \times_{\text{Spec } \kappa[Q]} \text{Spec } \kappa[Q^{\text{sat}}],$$

where  $Q^{\text{sat}}$  is the saturation of  $Q$  inside  $Q^{\text{gp}}$ . Note that the morphism  $W \rightarrow \text{Spec } \kappa[Q]$  identifies  $W$  with the closed point of  $\text{Spec } \kappa[Q]$  corresponding to the maximal monomial ideal  $\mathfrak{m} = \langle z^q \mid q \in Q \setminus \{0\} \rangle$ .

The monomial ideal  $I \subseteq \kappa[Q^{\text{sat}}]$  generated by the image of  $\mathfrak{m}$  then satisfies  $\sqrt{I} = \mathfrak{m}' = \langle z^q \mid q \in Q^{\text{sat}} \setminus (Q^{\text{sat}})^\times \rangle$ . Indeed, for any element  $q \in Q^{\text{sat}} \setminus (Q^{\text{sat}})^\times$ , there exists a positive integer  $n$  such that  $nq \in Q$ . Further,  $nq \neq 0$  since otherwise  $q$  is torsion in  $Q^{\text{sat}}$  and hence invertible. Thus  $z^{nq}$  is a generator of  $\mathfrak{m}$ , so  $z^{nq} \in I$ . This shows that  $\mathfrak{m}' \subseteq \sqrt{I}$ . The converse holds as  $I \subseteq \mathfrak{m}'$  and  $\mathfrak{m}'$  is a radical ideal, as  $\kappa[Q^{\text{sat}}]/\mathfrak{m}' = \kappa[(Q^{\text{gp}})_{\text{tors}}]$  is reduced.

Thus we see that  $(W^{\text{sat}})_{\text{red}} = \text{Spec } \kappa[Q^{\text{sat}}]/\mathfrak{m}' = \text{Spec } \kappa[(Q^{\text{gp}})_{\text{tors}}]$ . However, the latter consists of  $|(Q^{\text{gp}})_{\text{tors}}|$  points.  $\spadesuit$

**Lemma 3.2.** *Let  $W_1, W_2, X$  be fs log schemes with morphisms  $W_1, W_2 \rightarrow X$ . Then there is a canonical isomorphism*

$$(W_{1,\text{red}} \times_{X_{\text{red}}}^{\text{fs}} W_{2,\text{red}})_{\text{red}} \cong (W_1 \times_X^{\text{fs}} W_2)_{\text{red}}.$$

*Proof.* Considering the strict closed immersion  $(W_1)_{\text{red}} \rightarrow W_1$ , we obtain via base change a strict closed immersion  $(W_1)_{\text{red}} \times_X^{\text{fs}} W_2 \rightarrow W_1 \times_X^{\text{fs}} W_2$ . Repeating with  $W_2$ , we obtain a strict closed immersion  $(W_1)_{\text{red}} \times_X^{\text{fs}} (W_2)_{\text{red}} \rightarrow W_1 \times_X^{\text{fs}} W_2$ . Since the morphisms  $(W_i)_{\text{red}} \rightarrow X$  factor through  $X_{\text{red}}$ , the former log scheme is isomorphic to  $(W_1)_{\text{red}} \times_{X_{\text{red}}}^{\text{fs}} (W_2)_{\text{red}}$ .

Thus we obtain a strict closed immersion  $(W_{1,\text{red}} \times_{X_{\text{red}}}^{\text{fs}} W_{2,\text{red}})_{\text{red}} \rightarrow (W_1 \times_X^{\text{fs}} W_2)_{\text{red}}$ , which we must now prove is an isomorphism, which we do by showing this map induces a bijection on geometric points. Indeed, the set of geometric points of  $W_1 \times_X^{\text{fs}} W_2$  and  $(W_1 \times_X^{\text{fs}} W_2)_{\text{red}}$  are the same, and a strict geometric point  $\text{Spec}(Q \rightarrow \kappa) \rightarrow W_1 \times_X^{\text{fs}} W_2$  clearly induces a strict closed point  $\text{Spec}(Q \rightarrow \kappa) \rightarrow (W_1)_{\text{red}} \times_{X_{\text{red}}}^{\text{fs}} (W_2)_{\text{red}}$ . This shows the claim.  $\spadesuit$

**Lemma 3.3.** *Let  $W_1, W_2, X$  be finite length connected fs log schemes over  $\text{Spec } \kappa$  with  $\kappa$  an algebraically closed field. Write the ghost sheaf monoids as  $Q_1, Q_2$  and  $P$  respectively. Suppose given morphisms  $f_i : W_i \rightarrow X$  inducing  $\theta_i = \bar{f}_i^{\flat} : P \rightarrow Q_i$ . Set*

$$\theta := (\theta_1^{\text{gp}}, -\theta_2^{\text{gp}}) : P^{\text{gp}} \rightarrow Q_1^{\text{gp}} \oplus Q_2^{\text{gp}}.$$

Then

- (1) *If the fs fibre product  $W_1 \times_X^{\text{fs}} W_2$  is non-empty, it has  $|\text{coker}(\theta)_{\text{tors}}|$  connected components.*
- (2)  $|\text{coker}(\theta)_{\text{tors}}| = |\text{coker}(\theta^t)_{\text{tors}}|$ .

*Proof.* (1) By Lemma 3.2, we may assume  $W_i$  and  $X$  are (reduced) log points. According to the construction of the fs fibre product in [Og, III, §2.1], we proceed in a couple of steps. Let  $W, W^{\text{int}}$  and  $W^{\text{fs}}$  denote the fibre product  $W_1 \times_X W_2$  in the category of log schemes, fine log schemes, and fs log schemes. Then  $W^{\text{int}}$  is the integralization of  $W$  and  $W^{\text{fs}}$  is the saturation of  $W^{\text{int}}$ .

First,  $\underline{W}$  agrees with the fibre product  $\underline{W}_1 \times_X \underline{W}_2 = \text{Spec } \kappa$ . Integralization involves passing to a closed subscheme of  $\text{Spec } \kappa$ . Thus either  $\underline{W}^{\text{int}} = \text{Spec } \kappa$  or is the empty scheme. We rule out the latter as we have assumed that the fs fibre product is non-empty. In this case,  $W^{\text{int}} = \text{Spec}(Q \rightarrow \kappa)$ , where

$$Q = Q_1 \oplus_P^{\text{fine}} Q_2$$

and  $\oplus^{\text{fine}}$  denotes push-out in the category of fine monoids. This is constructed (see [Og, I, Prop. 1.3.4] and its proof) as the fine submonoid of

$$\text{coker } \theta = Q_1^{\text{gp}} \oplus_{P^{\text{gp}}} Q_2^{\text{gp}} = Q^{\text{gp}}$$

generated by the images of  $Q_1$  and  $Q_2$ .

As  $W^{\text{fs}}$  is the saturation of  $W^{\text{int}}$ , (1) now follows from Lemma 3.1.

(2) is easy homological algebra:  $0 \rightarrow \ker \theta \rightarrow P^{\text{gp}} \rightarrow Q_1^{\text{gp}} \oplus Q_2^{\text{gp}} \rightarrow \text{coker}(\theta) \rightarrow 0$  is a free resolution of  $\text{coker}(\theta)$ , and  $\text{Ext}^1(\text{coker}(\theta), \mathbb{Z})$  is isomorphic to the torsion part of  $\text{coker}(\theta)$ . However, this Ext group is calculated as the middle cohomology of the complex  $Q_1^* \oplus Q_2^* \rightarrow P^* \rightarrow \ker(\theta)^*$ , and as the kernel of  $P^* \rightarrow \ker(\theta)^*$  is a saturated sublattice of  $P^*$ , the torsion part of  $\text{coker}(\theta^t)$  agrees with  $\text{Ext}^1(\text{coker}(\theta), \mathbb{Z})$ . ♠

The following is a somewhat technically complicated criterion for non-emptiness for  $W_1 \times_X^{\text{fs}} W_2$ , which is tailored for our gluing needs.

**Lemma 3.4.** *Let  $W_i, X$  be as in Lemma 3.3, and suppose also given log points  $W'_i, X'$  over  $\text{Spec } \kappa$  similarly with maps  $f'_i : W'_i \rightarrow X'$ . Write  $Q'_i, P'$  for the corresponding monoids,  $\theta'_i : P' \rightarrow Q'_i$  for the induced maps. Suppose further given a commutative diagram*

$$(3.1) \quad \begin{array}{ccccc} W_1 & \xrightarrow{f_1} & X & \xleftarrow{f_2} & W_2 \\ g_1 \downarrow & & g \downarrow & & \downarrow g_2 \\ W'_1 & \xrightarrow{f'_1} & X' & \xleftarrow{f'_2} & W'_2 \end{array}$$

Suppose that:

- (1)  $W'_1 \times_{X'}^{\text{fs}} W'_2$  is non-empty.
- (2) The projections  $Q_1^{\vee} \times_{P^{\vee}} Q_2^{\vee} \rightarrow Q_i^{\vee}$  have image intersecting the interior of  $Q_i^{\vee}$ .
- (3) The maps  $\bar{g}^{\flat} : P' \rightarrow P$  and  $\bar{g}'_i : Q'_i \rightarrow Q_i$  are all injective, and the map induced by  $\theta$ ,

$$(3.2) \quad P^{\text{gp}} / \bar{g}^{\flat}((P')^{\text{gp}}) \rightarrow Q_1^{\text{gp}} / \bar{g}'_1((Q'_1)^{\text{gp}}) \oplus Q_2^{\text{gp}} / \bar{g}'_2((Q'_2)^{\text{gp}})$$

is injective.

Then  $W_1 \times_X^{\text{fs}} W_2$  is non-empty.

*Proof.* Given the hypotheses, we will construct a morphism  $\mathrm{Spec} \kappa^\dagger \rightarrow W_1 \times_X^{\mathrm{fs}} W_2$  from the standard log point. Recall that giving a morphism of log points  $g : \mathrm{Spec}(Q \rightarrow \kappa) \rightarrow \mathrm{Spec}(P \rightarrow \kappa)$  is equivalent to giving  $g^b : P \times \kappa^\times \rightarrow Q \times \kappa^\times$  written as

$$g^b(p, t) = (\bar{g}^b(p), \chi_g(p) \cdot t)$$

for  $\bar{g}^b : P \rightarrow Q$  a local homomorphism and  $\chi_g : P \rightarrow \kappa^\times$  an arbitrary homomorphism. Note also that  $\theta : P \rightarrow Q$  being a local homomorphism can be characterized dually by the statement that  $\theta^t(Q^\vee)$  intersects the interior of  $P^\vee$ .

First, let  $Q$  be the stalk of the ghost sheaf of any point of  $W_1 \times_X^{\mathrm{fs}} W_2$  if this log scheme were non-empty, i.e.,  $Q = (Q_1 \oplus_P^{\mathrm{fs}} Q_2)/\mathrm{tors}$ . Let  $Q'$  be the corresponding monoid for the primed data. Then  $Q^\vee = Q_1^\vee \times_{P^\vee} Q_2^\vee$  ([ACGS1, Proposition 6.3.5]), with a similar expression for  $(Q')^\vee$ .

Choose an element  $q \in \mathrm{Int}(Q^\vee)$ . Then necessarily the image  $q_i$  of  $q$  in  $Q_i^\vee$  lies in  $\mathrm{Int}(Q_i^\vee)$  by condition (2). Further,  $q_1$  and  $q_2$  have the same image in  $P^\vee$ . Let  $q'_i = (\bar{g}_i^b)^t(q_i)$ . Necessarily  $q'_i \in \mathrm{Int}((Q'_i)^\vee)$  as  $\bar{g}_i^b : Q'_i \rightarrow Q_i$  is a local homomorphism. Further,  $q'_1$  and  $q'_2$  have the same image in  $(P')^\vee$  because of commutativity of (3.1). This gives an element  $q' := (q'_1, q'_2)$  of  $(Q')^\vee = (Q'_1)^\vee \times_{(P')^\vee} (Q'_2)^\vee$ . In particular  $q'$  lies in the interior of  $(Q')^\vee$ .

Hence we obtain a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{q} & \mathbb{N} \\ \uparrow & \nearrow q' & \\ Q' & & \end{array}$$

with  $q, q'$  local homomorphisms. Note that  $Q'$  is the stalk of the ghost sheaf at any point of  $W'_1 \times_{X'}^{\mathrm{fs}} W'_2$ . By condition (1) this latter log scheme is non-empty, and so there is a morphism  $\mathrm{Spec} \kappa^\dagger \rightarrow W'_1 \times_{X'}^{\mathrm{fs}} W'_2$  which induces the map  $q' : Q' \rightarrow \mathbb{N}$  on stalks of ghost sheaves. Indeed, it is sufficient to construct a morphism  $\mathrm{Spec} \kappa^\dagger \rightarrow \mathrm{Spec}(Q' \rightarrow \kappa)$ , which is equivalent to giving a local homomorphism  $Q' \rightarrow \mathbb{N}$  and a homomorphism  $Q' \rightarrow \kappa^\times$ . We take the local homomorphism  $Q' \rightarrow \mathbb{N}$  to be given by  $q'$ .

The chosen morphism can also be viewed as arising in a commutative diagram

$$(3.3) \quad \begin{array}{ccc} \mathrm{Spec} \kappa^\dagger & \longrightarrow & W'_2 \\ \downarrow & & \downarrow \\ W'_1 & \longrightarrow & X' \end{array}$$

At the level of ghost sheaves, this diagram is given by

$$\begin{array}{ccc} \mathbb{N} & \xleftarrow{q'_2} & Q'_2 \\ \uparrow q'_1 & & \uparrow \theta'_2 \\ Q'_1 & \xleftarrow{\theta'_1} & P' \end{array}$$

and the morphisms  $\text{Spec } \kappa^\dagger \rightarrow W'_i$  are then determined by additional data of maps  $\psi'_i : Q'_i \rightarrow \kappa^\times$ . Commutativity of (3.3) then comes down to the equality

$$(3.4) \quad (\psi'_1 \circ \theta'_1) \cdot \chi_{f'_1} = (\psi'_2 \circ \theta'_2) \cdot \chi_{f'_2}$$

in  $\text{Hom}(P', \kappa^\times)$ .

We now wish to construct an analogous commutative diagram

$$(3.5) \quad \begin{array}{ccc} \text{Spec } \kappa^\dagger & \longrightarrow & W_2 \\ \downarrow & & \downarrow \\ W_1 & \longrightarrow & X \end{array}$$

given at the level of ghost sheaves by the commutative diagram

$$\begin{array}{ccc} \mathbb{N} & \xleftarrow{q_2} & Q_2 \\ q_1 \uparrow & & \uparrow \theta_2 \\ Q_1 & \xleftarrow{\theta_1} & P \end{array}$$

As all homomorphisms in this diagram are local, it is enough to construct analogously  $\psi_i : Q_i \rightarrow \kappa^\times$  such that

$$(3.6) \quad (\psi_1 \circ \theta_1) \cdot \chi_{f_1} = (\psi_2 \circ \theta_2) \cdot \chi_{f_2}$$

By commutativity of (3.1), we have

$$(3.7) \quad \chi_g \cdot (\chi_{f_i} \circ \bar{g}^b) = \chi_{f'_i} \cdot (\chi_{g_i} \circ \theta'_i)$$

in  $\text{Hom}(P', \kappa^\times)$ .

Using the assumed injectivity of  $\bar{g}_i^b$  of condition (3), we choose a lift  $\psi_i : Q_i^{\text{gp}} \rightarrow \kappa^\times$  of  $\chi_{g_i}^{-1} \cdot \psi'_i : \bar{g}_i^b((Q'_i)^{\text{gp}}) \rightarrow \kappa^\times$ . As  $\kappa$  is algebraically closed, this can always be done even if  $\bar{g}_i^b((Q'_i)^{\text{gp}})$  is not saturated in  $Q_i^{\text{gp}}$ . Then for  $p \in P'$ , we have, with the third line by the definition of  $\psi_i$  and (3.7),

$$\begin{aligned} ((\psi_i \circ \theta_i) \cdot \chi_{f_i})(\bar{g}^b(p)) &= \psi_i(\theta_i(\bar{g}^b(p))) \cdot \chi_{f_i}(\bar{g}^b(p)) \\ &= \psi_i(\bar{g}_i^b(\theta'_i(p))) \cdot \chi_{f_i}(\bar{g}^b(p)) \\ &= [\psi'_i(\theta'_i(p)) \cdot \chi_{g_i}(\theta'_i(p))^{-1}] \cdot [\chi_g(p)^{-1} \cdot \chi_{f'_i}(p) \cdot \chi_{g_i}(\theta'_i(p))] \\ &= [\psi'_i(\theta'_i(p)) \cdot \chi_{f'_i}(p)] \cdot \chi_g(p)^{-1}. \end{aligned}$$

By (3.4), this is independent of  $i$ .

Now consider

$$[(\psi_1 \circ \theta_1) \cdot \chi_{f_1}] \cdot [(\psi_2 \circ \theta_2) \cdot \chi_{f_2}]^{-1} \in \text{Hom}(P^{\text{gp}}, \kappa^\times),$$

and note this homomorphism is the identity on  $\bar{g}^b((P')^{\text{gp}})$  by the previous paragraph. Thus it induces an element of  $\text{Hom}(P^{\text{gp}}/\bar{g}^b((P')^{\text{gp}}), \kappa^\times)$ . As  $\kappa$  is algebraically closed,  $\kappa^\times$  is a divisible group and hence by the injectivity of (3.2), we obtain a surjective map

$$\text{Hom}(Q_1^{\text{gp}}/\bar{g}_1^b((Q'_1)^{\text{gp}}), \kappa^\times) \times \text{Hom}(Q_2^{\text{gp}}/\bar{g}_2^b((Q'_2)^{\text{gp}}), \kappa^\times) \rightarrow \text{Hom}(P^{\text{gp}}/\bar{g}^b((P')^{\text{gp}}), \kappa^\times).$$

Thus we may find  $\varphi_i \in \text{Hom}(Q_i^{\text{gp}}/\bar{g}_i^b((Q'_i)^{\text{gp}}), \kappa^\times)$  such that if we replace  $\psi_i$  with  $\varphi_i \cdot \psi_i$ , (3.6) holds.  $\spadesuit$

## 4. GLUING ONE CURVE

We give a first application of the material of the previous subsection. We consider our standard gluing situation as in Notation 2.1.

Now assume given basic punctured maps  $f_i : C_i^\circ/W_i \rightarrow X$  of type  $\tau_i$  for  $1 \leq i \leq r$ , with the  $W_i = \text{Spec}(Q_i \rightarrow \kappa)$  being logarithmic points,  $Q_i$  the basic monoid associated to  $f_i$ , and  $\kappa$  algebraically closed. Suppose further that whenever  $E \in \mathbf{E}$  with vertices  $v_1, v_2$ , with corresponding punctured points  $p_{E, v_i} \in C_{i(v_i)}$ , we have

$$(4.1) \quad f_{i(v_1)}(p_{E, v_1}) = f_{i(v_2)}(p_{E, v_2}),$$

i.e., the maps  $f_i$  will glue schematically. We may then ask how many gluings exist at the logarithmic level. More precisely, we would like to understand the scheme  $W$  defined as follows:

**Definition 4.1.** The *gluing*  $f : C^\circ/W \rightarrow X$  of the punctured maps  $f_i$  is defined by

$$W := \mathcal{M}'(X, \boldsymbol{\tau}) \times_{\prod_{i=1}^r \mathcal{M}'(X, \tau_i)} \prod_{i=1}^r W_i,$$

and  $f : C^\circ/W \rightarrow X$  the pull-back of the universal map over  $\mathcal{M}'(X, \boldsymbol{\tau})$  to  $W$ .

In this situation, we introduce the following notation. For any punctured point  $p_{E, v}$  of  $C_i^\circ$  indexed by a flag  $v \in E \in \mathbf{E}$ , let  $P_{E, v}$  be the stalk of  $\overline{\mathcal{M}}_X$  at  $f_{i(v)}(p_{E, v})$ . By (4.1), we have  $P_{E, v_1} = P_{E, v_2}$  if  $v_1, v_2$  are the vertices of  $E$ , and write both as  $P_E$ . For any irreducible component of  $C_i$  with generic point  $\eta$  corresponding to a vertex  $v$  of  $G_i$ , write  $P_v$  for the stalk of the ghost sheaf at  $f_i(\eta)$ .

For each  $i$ , we have a family of tropical maps  $h_i : \Gamma(G_i, \ell_i) \rightarrow \Sigma(X)$  defined over  $\tau_i$ . If  $\omega_v \in \Gamma(G_i, \ell_i)$  is the cone corresponding to a vertex  $v \in V(G_i)$ , then we obtain by restriction a map

$$(4.2) \quad \text{ev}_v : \omega_v \rightarrow \Sigma(X)$$

mapping into the cone  $P_{v, \mathbb{R}}^\vee \in \Sigma(X)$ . Explicitly this is defined as the transpose of

$$P_v \xrightarrow{\tilde{f}_i} Q_i,$$

where here  $Q_i$  is identified with  $\overline{\mathcal{M}}_{C_i, \bar{\eta}}$ . Hence at the level of groups we also obtain a map

$$\text{ev}_v : Q_i^* \rightarrow P_v^*.$$

**Definition 4.2.** With the notation as above, choose an orientation on each edge  $E \in E(G)$  so that  $E$  has vertices  $v_E, v'_E$  and is oriented from  $v_E$  to  $v'_E$ . We define the *tropical gluing map*

$$\Psi : \prod_{i=1}^r Q_i^* \times \prod_{E \in \mathbf{E}} \mathbb{Z} \rightarrow \prod_{E \in \mathbf{E}} P_E^*$$

by

$$\Psi((q_1, \dots, q_r), (\ell_E)_{E \in \mathbf{E}}) = (\text{ev}_{v_E}(q_{i(v_E)}) + \ell_E \mathbf{u}(E, v_E) - \text{ev}_{v'_E}(q_{i(v'_E)}))_{E \in \mathbf{E}}.$$

We define the *tropical multiplicity* of the gluing situation to be

$$\mu = \mu(\boldsymbol{\tau}, \mathbf{E}) := |(\text{coker } \Psi)_{\text{tors}}|.$$



**Remark 4.3.** The map  $\Psi$  is called the tropical gluing map for the following reason. Suppose given  $s = ((q_i), (\ell_E)_{E \in \mathbf{E}(G)}) \in \ker \Psi$  such that  $q_i \in Q_i^\vee$  for each  $v$  and  $\ell_E > 0$  for each  $E$ . Then we may construct a tropical map  $h_s : G \rightarrow \Sigma(X)$  as follows. First, for each  $i$ , let  $G'_i$  be the subgraph of  $G_i$  obtained by removing legs of the form  $(E, v)$  for  $E \in \mathbf{E}$ . Then  $G'_i$  is naturally identified with a subgraph of  $G$ , and we may define  $h_s|_{G'_i}$  to agree with  $(h_i)_{q_i}|_{G'_i}$ . On the other hand, if we give each  $E \in \mathbf{E}$  the length  $\ell_E$ , then  $s \in \ker \Psi$  guarantees we can extend  $h_s$  across all edges  $E \in \mathbf{E}$ .

Thus it is reasonable to think of  $\ker \Psi$  as the integral tangent space to the family of glued tropical curves.

**Theorem 4.4.** *Suppose we are in the above situation. If the gluing  $W$  is non-empty, then  $W$  has  $\mu(\tau, \mathbf{E})$  connected components.*

*Proof.* By Theorem 2.3, we have an fs Cartesian diagram

$$(4.3) \quad \begin{array}{ccc} \widetilde{W} & \longrightarrow & \prod_{i=1}^r \widetilde{W}_i \\ \downarrow & & \downarrow \text{ev}_L \\ \prod_{E \in \mathbf{E}} X & \xrightarrow{\Delta} & \prod_{E \in \mathbf{E}} X \times X \end{array}$$

Here

$$\widetilde{W}_i := W_i \times_{\mathcal{M}'(X, \tau_i)} \widetilde{\mathcal{M}}'(X, \tau_i)$$

and

$$\widetilde{W} := W \times_{\mathcal{M}'(X, \tau)} \widetilde{\mathcal{M}}'(X, \tau).$$

A priori all fibre products are over  $B$ , but by the assumption (4.1), the composition of  $f_i$  with the structure map  $X \rightarrow B$  are all constant with the same image, so we may replace  $B$  by a suitable affine neighbourhood of this image and apply Proposition 2.4. Thus we may replace  $B$  with  $\text{Spec } \mathbb{k}$  in the above discussion and thus assume that all products in (4.3) are defined over  $\text{Spec } \mathbb{k}$ .

By Proposition 2.2, the underlying schemes of  $\widetilde{W}$  and  $W$  agree, and the reduction of the underlying scheme of  $\widetilde{W}_i$  agrees with the underlying scheme of  $W_i$  (being a point). Thus we need to calculate the number of connected components of  $\widetilde{W}$ . Further, by (4.1), for any edge  $E \in \mathbf{E}$  with endpoints  $v_1, v_2$ , the evaluation maps  $\widetilde{W}_{i(v_1)} \rightarrow X$ ,  $\widetilde{W}_{i(v_2)} \rightarrow X$  both factor through the strict closed point  $f_{i(v_1)}(p_{E, v_1}) = f_{i(v_2)}(p_{E, v_2})$ . Thus we may replace, for each edge  $E \in \mathbf{E}$ , the target  $X$  with the corresponding log point, and hence obtain an fs Cartesian diagram

$$(4.4) \quad \begin{array}{ccc} \widetilde{W} & \longrightarrow & \prod_{i=1}^r \widetilde{W}_i \\ \downarrow & & \downarrow \text{ev}_L \\ \prod_{E \in \mathbf{E}} \text{Spec}(P_E \rightarrow \mathbb{k}) & \xrightarrow{\Delta} & \prod_{E \in \mathbf{E}} \text{Spec}(P_E \rightarrow \mathbb{k})^2 \end{array}$$

Further, by Lemma 3.2, we may replace  $\widetilde{W}_i$  by its reduction without changing the number of connected components of  $\widetilde{W}$ .

Note that for  $L_j = (E, v) \in \mathbf{L}_i$ ,  $W_i \times_{\mathfrak{M}(X, \tau_i)} \mathfrak{M}_{L_j}^{\vee}(X, \tau_i)$  has ghost sheaf  $Q_i^{L_j} \subseteq Q_i \oplus \mathbb{Z}$ , with an equality on the level of groups, and with the induced evaluation map  $f_i \circ p_{E,v} : W_i \rightarrow X$  yielding a map at the tropical level of

$$((f_i \circ p_{E,v})^b)^t : (Q_i^{L_j})^\vee \rightarrow P_E^\vee$$

taking the value on  $(s, r) \in (Q_i^{L_j})^\vee \subseteq Q_i^\vee \oplus \mathbb{Z}$  given by

$$\text{ev}_v(s) + r\mathbf{u}(L_j).$$

Further, from the fibre product description (2.5) we see that

$$\overline{\mathcal{M}}_{\widetilde{W}_i} \subseteq Q_i \oplus \bigoplus_{(E,v) \in L(G_i)} \mathbb{Z},$$

again with an equality on groups.

Of course  $\Delta$  induces a map  $P_E^* \rightarrow P_E^* \times P_E^*$  given by the diagonal. Putting this together, the homomorphism  $\theta^t$  of Lemma 3.3 then takes the form

$$(4.5) \quad \theta^t : \prod_E P_E^* \times \prod_{i=1}^r Q_i^* \times \prod_{v \in E \in \mathbf{E}} \mathbb{Z} \rightarrow \prod_{v \in E \in \mathbf{E}} P_E^*$$

given by

$$\theta^t((n_E)_{E \in \mathbf{E}(G)}, (s_i)_{1 \leq i \leq r}, (\ell_{E,v})_{v \in E \in \mathbf{E}}) = (\text{ev}_v(s_i(v)) + \ell_{E,v}\mathbf{u}(E, v) - n_E)_{v \in E \in \mathbf{E}}.$$

If  $\widetilde{W}$  is non-empty, then by Lemma 3.3 the number of its connected components is the order of the torsion part of  $\text{coker } \theta^t$ . We next compare this with the order of the torsion part of  $\text{coker } \Psi$ .

We have a diagram

$$(4.6) \quad \begin{array}{ccccc} \prod_{E \in \mathbf{E}} \mathbb{Z} \times \prod_{E \in \mathbf{E}} P_E^* & \xrightarrow{\alpha} & \prod_{E \in \mathbf{E}} P_E^* \times \prod_i Q_i^* \times \prod_{v \in E \in \mathbf{E}} \mathbb{Z} & \xrightarrow{\gamma} & \prod_i Q_i^* \times \prod_{E \in \mathbf{E}} \mathbb{Z} \\ \pi \downarrow & & \theta^t \downarrow & & \Psi \downarrow \\ \prod_{E \in \mathbf{E}} P_E^* & \xrightarrow{\beta} & \prod_{v \in E \in \mathbf{E}} P_E^* & \xrightarrow{\delta} & \prod_{E \in \mathbf{E}} P_E^* \\ & & \downarrow & & \downarrow \\ & & \text{coker } \theta^t & \xrightarrow{\cong} & \text{coker } \Psi \end{array}$$

Here  $\pi$  is the surjective map given by

$$\pi((\ell_E)_{E \in \mathbf{E}}, (n_E)_{E \in \mathbf{E}}) = (\ell_E \mathbf{u}(E, v_E) - n_E)_{E \in \mathbf{E}}.$$

Here we use the chosen vertices  $v_E, v'_E$  of Definition 4.2. Further,  $\alpha, \beta$  are injections defined by

$$\begin{aligned} \alpha((\ell_E)_{E \in \mathbf{E}}, (n_E)_{E \in \mathbf{E}}) &= ((n_E)_{E \in \mathbf{E}}, (0)_{1 \leq i \leq r}, ((-1)^{\delta_{v, v_E}} \ell_E)_{v \in E \in \mathbf{E}}), \\ \beta((n_E)_{E \in \mathbf{E}}) &= ((n_E)_{v \in E \in \mathbf{E}}). \end{aligned}$$

while  $\gamma$  and  $\delta$  are surjections defined by

$$\begin{aligned} \gamma((n_E)_{E \in \mathbf{E}}, (s_i)_{1 \leq i \leq r}, (\ell_{E,v})_{v \in E \in \mathbf{E}}) &= ((q_i)_{1 \leq i \leq r}, (\ell_{E, v_E} + \ell_{E, v'_E})_{E \in \mathbf{E}}), \\ \delta((n_{E,v})_{v \in E \in \mathbf{E}}) &= (n_{E, v_E} - n_{E, v'_E})_{E \in \mathbf{E}}. \end{aligned}$$

Finally,  $\Psi$  is the tropical gluing map of Definition 4.2. One checks that the diagram is commutative with the top two rows exact, and hence the snake lemma implies that  $\text{coker } \theta^t \cong \text{coker } \Psi$ , giving the result. ♠

**Example 4.5.** It is very important to note that the gluing parameter space  $W$  need not be reduced, something which is quite different from gluing ordinary stable maps. This arises via saturation in the fs fibre product. For example, consider a target space  $X$  with  $\Sigma(X) = \mathbb{R}_{\geq 0}^2$ , and a gluing situation where  $\tau$  is a graph with two vertices,  $v_1$  and  $v_2$ , and two edges,  $E_1$  and  $E_2$ , each connecting  $v_1$  with  $v_2$ . We split at the two edges, getting types  $\tau_1, \tau_2$ . We assume  $\mathbf{u}(E_1, v_1) = -\mathbf{u}(E_1, v_2) = (-w_1, w_1)$  and  $\mathbf{u}(E_2, v_1) = -\mathbf{u}(E_2, v_2) = (-w_2, w_2)$  with  $\gcd(w_1, w_2) = 1$  and  $w_1 < w_2$ . Finally, we assume  $\sigma(v_1) = \mathbb{R}_{\geq 0} \times 0$  and  $\sigma(v_2) = 0 \times \mathbb{R}_{\geq 0}$ .

A slightly tedious calculation shows that  $\underline{W} = \text{Spec } \mathbb{k}[t]/(t^{w_1})$  in this case. Morally, one can think of this as follows. If the glued curve smooths, there are smoothing parameters  $u_1, u_2$  for the two nodes, in the sense that the local structure at each node is of the form  $xy = u_1$  or  $xy = u_2$ . The relation  $u_1^{w_2} = u_2^{w_1}$  is then forced by the logarithmic geometry of the situation. Saturation normalizes this curve, and the inverse image of the point with ideal  $(u_1, u_2)$  under this normalization is the non-reduced gluing.

**Remark 4.6.** Suppose instead we are given a gluing situation of maps to  $\mathcal{X}$  equipped with evaluation maps at the punctured points, i.e., the maps are determined by morphisms  $W_i \rightarrow \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i)$  rather than  $W_i \rightarrow \mathcal{M}'(X, \tau_i)$ , still with  $\underline{W}_i = \text{Spec } \kappa$ . In other words, we are given pre-stable punctured maps  $f_v : C_v^\circ/W_v \rightarrow \mathcal{X}$  along with compatible morphisms  $\underline{W}_i \rightarrow \underline{X}$  for each punctured point  $p_{E,v}$ . We assume further, as before, that the images of  $p_{E,v_1}, p_{E,v_2}$  under these maps to  $\underline{X}$  agree for each edge  $E$ . Then similarly, the number of connected components of the glued space  $W$  is  $|(\text{coker } \Psi)_{\text{tors}}|$ . Indeed, the gluing is controlled by precisely the same Cartesian diagram as in the situation of Theorem 4.4.

Note that Theorem 4.4 says nothing about whether  $W$  is non-empty. Here is one often useful criterion:

**Definition 4.7.** We say a gluing situation is *tropically transverse* if the map  $\Psi$  of Definition 4.2 has finite cokernel.

**Theorem 4.8.** *In the situation of Theorem 4.4,  $W$  is non-empty if the gluing situation is tropically transverse.*

*Proof.* We apply the non-emptiness criterion of Lemma 3.4 to the product of (4.4). In that lemma, we take  $W'_i = X' = \text{Spec } \mathbb{k}$  so that the first condition of the lemma is trivially satisfied, and the third condition is equivalent to the injectivity of the map  $\theta$  whose transpose  $\theta^t$  is given in (4.5). However,  $\theta^t$  having a finite cokernel implies  $\theta$  is injective, and (4.6) implies  $\text{coker } \theta^t \cong \text{coker } \Psi$ , and hence tropical transversality implies the third condition. Finally,  $\tau$  realizable implies the second condition. ♠

5.1. **The general situation.** We continue with a standard gluing situation for  $X/B$ , as in Notation 2.1.

**Theorem 5.1.** *There is a diagram*

$$\begin{array}{ccccc}
\mathcal{M}(X/B, \tau) & \xrightarrow{\phi'} & \mathcal{M}^{\text{sch}}(X/B, \tau) & \longrightarrow & \prod_{i=1}^r \mathcal{M}(X/B, \tau_i) \\
\downarrow \varepsilon^{\text{ev}} & & \downarrow \varepsilon^{\text{sch}} & & \downarrow \hat{\varepsilon} \\
\mathfrak{M}^{\text{ev}(\mathbf{L})}(\mathcal{X}/B, \tau) & \xrightarrow{\phi} & \mathfrak{M}^{\text{sch, ev}(\mathbf{L})}(\mathcal{X}/B, \tau) & \xrightarrow{\Delta'} & \prod_{i=1}^r \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i) \\
& & \downarrow & & \downarrow \underline{\text{ev}} \\
& & \prod_{E \in \mathbf{E}} \underline{X}_{\sigma(E)} & \xrightarrow{\Delta} & \prod_{v \in E \in \mathbf{E}} \underline{X}_{\sigma(E)}
\end{array}$$

with all squares Cartesian in all categories, defining the moduli spaces  $\mathcal{M}^{\text{sch}}(X/B, \tau)$  and  $\mathfrak{M}^{\text{sch, ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$ . Further,  $\phi$  is a finite morphism.

*Proof.* We define the morphism  $\prod_{i=1}^r \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i) \rightarrow \prod_{v \in E \in \mathbf{E}} \underline{X}_{\sigma(E)}$  as follows. For flag  $v \in E \in \mathbf{E}$ , consider the composition

$$\prod_{i=1}^r \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i) \rightarrow \mathfrak{M}^{\text{ev}(\mathbf{L}_{i(v)})}(\mathcal{X}/B, \tau_{i(v)}) = \mathfrak{M}(\mathcal{X}/B, \tau_i) \times_{\prod \mathcal{X}} \prod \underline{X} \rightarrow \underline{X}$$

where the first arrow is projection and the second is further projection onto the factor  $\underline{X}$  indexed by the leg  $(E, v) \in \mathbf{L}_i$ . Necessarily, this morphism factors through  $\underline{X}_{\sigma(E)}$  by the definition of a  $\tau_i$ -marked curve, see [ACGS2, Def. 3.7,(1)]. This gives a morphism

$$\underline{\text{ev}}_{(E,v)} : \prod_{i=1}^r \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i) \rightarrow \underline{X}_{\sigma(E)},$$

and we define

$$\underline{\text{ev}} := \prod_{v \in E \in \mathbf{E}} \underline{\text{ev}}_{(E,v)}.$$

The morphism  $\Delta$  is the product of diagonals. It is then clear that the Cartesian diagram of Theorem 2.5 factors as stated, as  $\mathfrak{M}^{\text{sch, ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$  captures those curves which glue schematically. Further,  $\phi$  is a finite and representable morphism as  $\psi = \Delta' \circ \phi$  is finite and representable by Theorem 2.5.  $\spadesuit$

**Remark 5.2.** As stated, this is not a significant improvement over Theorem 2.5: it merely separates the gluing into two steps, the first step being schematic gluing and the second step taking into account only the gluing at the logarithmic level.

For the first step, we are in luck if (1)  $\Delta$  is lci, so that the Gysin pull-back  $\Delta^!$  exists, and (2)  $\underline{\text{ev}}$  is flat, so that  $\Delta^!$  and  $(\Delta')^!$  induce the same map

$$\Delta^! = (\Delta')^! : A_* \left( \prod_i \mathcal{M}(X/B, \tau_i) \right) \rightarrow A_* (\mathcal{M}^{\text{sch}}(X/B, \tau)).$$

In any event, each  $\mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i)$  is pure-dimensional by [ACGS2, Prop. 3.28], so each  $\mathcal{M}(X/B, \tau_i)$  carries a virtual fundamental class. Further, the pull-back of the relative obstruction theory for  $\hat{\varepsilon}$  (the product over  $i$  of the relative obstruction theories for  $\mathcal{M}(X/B, \tau_i) \rightarrow \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i)$ ) yields a relative obstruction theory for  $\varepsilon_{\text{sch}}$ .

By the flatness assumption for  $\underline{\text{ev}}$ ,  $\mathfrak{M}^{\text{sch, ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$  is then also pure-dimensional, yielding a virtual fundamental class  $[\mathcal{M}^{\text{sch}}(X/B, \tau)]^{\text{virt}}$ , and if  $\Delta$  is lci, we have

$$[\mathcal{M}^{\text{sch}}(X/B, \tau)]^{\text{virt}} = \Delta^! \left( \prod_i [\mathcal{M}(X/B, \tau_i)]^{\text{virt}} \right).$$

Note that  $\Delta$  is lci if and only if strata  $X_{\sigma(E)}$  are non-singular. However, a deepest stratum of  $X$  is always non-singular, as is a stratum of dimension one more than a deepest stratum of  $X$ . If instead the log structure on  $X$  arises from an snc divisor, all strata are non-singular.

On the other hand, flatness of  $\underline{\text{ev}}$  is only automatic when the strata  $X_{\sigma(E)}$  are always deepest strata. Otherwise, more care needs to be taken.

The next step is to understand  $\phi$ . In general, we don't yet know how to say much about it, save for the next theorem, which gives us the degree of  $\phi$  onto its image. When the image is a proper closed substack, this tells us there are logarithmic obstructions to gluing, and it is still not understood how to deal with these obstructions in general. However, in the tropically transverse case,  $\phi$  is surjective.

**Theorem 5.3.** *The degree of  $\phi$  onto its image is the tropical multiplicity  $\mu(\tau, \mathbf{E})$  defined in Definition 4.2. If the gluing situation is tropically transverse, then  $\phi$  is dominant.*

*Proof.* We note that as  $\tau$  is realisable,  $\mathfrak{M}^{\text{ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$  has a non-empty dense open stratum  $U$ , whose geometric points are precisely those geometric points corresponding to a punctured map whose tropicalization is a family of tropical maps of type  $\tau$ . Similarly,  $\mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i)$  has a non-empty dense stratum  $U_i$ , with each geometric point in this stratum corresponding to a punctured map whose tropicalization is a family of curves of type  $\tau_i$ . Certainly then  $\Delta' \circ \phi(U) \subseteq \prod_i U_i$ . Further,  $(\phi \circ \Delta)^{-1}(\prod_i U_i) = U$  as the type of punctured map obtained by gluing together punctured maps of type  $\tau_i$  is necessarily  $\tau$ . It is thus sufficient to determine the degree of  $\phi|_U$  onto its image in  $(\Delta')^{-1}(\prod_i U_i)$ . To this end, pick a strict geometric point  $W' \rightarrow \mathfrak{M}^{\text{sch, ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau)$  in the image of  $\phi|_U$ . Because  $\Delta'$  is strict, we can decompose  $W' = \prod_i W_i$  as a product over  $B$ , with  $W_i \rightarrow \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i)$  strict geometric points. Thus we may view this as a situation of §4. In particular, the corresponding gluing is  $W' \times_{\prod_i \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}/B, \tau_i)} \mathfrak{M}^{\text{ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$ . By Lemma 3.2 and [ACGS2, Prop. 3.31], the reduction of this gluing agrees with the reduction of

$$W' \times_{\prod_i \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i)} \mathfrak{M}^{\text{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau) \cong W' \times_{\mathfrak{M}^{\text{sch, ev}(\mathbf{L})}(\mathcal{X}/B, \tau)} \mathfrak{M}^{\text{ev}(\mathbf{L})}(\mathcal{X}/B, \tau).$$

By Remark 4.6, this fibre product has  $\mu(\tau, \mathbf{E})$  connected components. Since this number is independent of the choice of geometric point in the image of  $\phi|_U$  and  $\mathfrak{M}^{\text{ev}}(\mathcal{X}/B, \tau)$  is reduced, it follows this fibre must be always reduced. This shows the degree.

The surjectivity in the tropically transverse case similarly follows from Theorem 4.8. ♠

**5.2. Gluing from rigid tropical curves.** One of the standard situations for gluing is the degeneration situation studied in [ACGS1]. Here, one considers a base  $B$  a smooth curve or spectrum of a DVR, with log structure the divisorial log structure induced by a closed point  $b_0 \in B$ . As usual, we assume  $X \rightarrow B$  is projective and log smooth. We

denote by  $X_0$  the fibre over  $b_0$ , and we view  $b_0$  with its induced log structure, i.e.,  $b_0$  is a standard log point.

In this case, the morphism  $X \rightarrow B$  tropicalizes to a morphism  $\delta : \Sigma(X) \rightarrow \Sigma(B) = \mathbb{R}_{\geq 0}$ . This gives rise to a polyhedral cell complex  $\Delta(X) = \delta^{-1}(1)$ . Given a type  $\beta$  of log curve for  $X/B$ , we have the logarithmic decomposition formula, see [ACGS1, Thm. 1.2]:

$$[\mathcal{M}(X_0/b_0, \beta)]^{\text{virt}} = \sum_{\tau=(\tau, \mathbf{A})} \frac{m_\tau}{|\text{Aut}(\tau)|} j_{\tau*} [\mathcal{M}(X_0/b_0, \tau)]^{\text{virt}}.$$

Here  $\tau$  runs over isomorphism classes of decorated *rigid tropical types*. These are realisable tropical types  $\tau$  such that the moduli space of tropical maps of type  $\tau$  is one-dimensional, with a unique member of this one-dimensional family factoring through  $\Delta(X)$ , hence the term rigid. The quantity  $m_\tau \in \mathbb{N}$  is the smallest integer such that scaling  $\Delta(X)$  by  $m_\tau$  leads to a tropical map with integral vertices and edge lengths; put another way,

$$m_\tau = |\text{coker}(N_\tau \rightarrow N_{\Sigma(B)} = \mathbb{Z})|.$$

We now explore how Theorem 5.1 may be applied in this situation. Unfortunately, the morphisms  $\phi, \phi'$  of Theorem 5.1 need not be surjective, but here we determine the length of the inverse image of a point in the image of  $\phi$ .

We first introduce the following notation. For each  $\sigma \in \Sigma(X)$ , the morphism  $\delta : \Sigma(X) \rightarrow \Sigma(B)$  induces homomorphisms  $\delta_* : N_\sigma \rightarrow \mathbb{Z}$ , and we define  $\overline{N}_\sigma$  to be the kernel of this homomorphism. Provided  $\sigma$  surjects onto  $\Sigma(B) = \mathbb{R}_{\geq 0}$ , this kernel can be identified with the space of integral tangent vectors of the corresponding polyhedron in  $\Delta(X)$ .

**Definition 5.4.** Let  $\tau$  be a type of rigid tropical curve in  $\Sigma(X)$ . Define

$$\overline{\Psi} : \bigoplus_{v \in V(G)} \overline{N}_{\sigma(v)} \oplus \bigoplus_{E \in E(G)} \mathbb{Z} \rightarrow \bigoplus_{E \in E(G)} \overline{N}_{\sigma(E)}$$

to be the homomorphism

$$((n_v)_{v \in V(G)}, (\ell_E)_{E \in E(G)}) \mapsto (n_{v_E} + \ell_E \mathbf{u}(E) - n_{v'_E})_{E \in E(G)}$$

where for each  $E \in E(G)$ , we orient  $E$  from a vertex  $v_E$  to a vertex  $v'_E$  to determine the sign of  $\mathbf{u}(E)$ , which necessarily lies in  $\overline{N}_{\sigma(E)}$ . We define the *tropical multiplicity* of  $\tau$  to be

$$\mu(\tau) := |(\text{coker } \overline{\Psi})_{\text{tors}}|.$$

**Theorem 5.5.** *Let  $X \rightarrow B$  be as above. Fix a rigid tropical type  $\tau = (G, \sigma, \mathbf{u})$ , and let  $\{\tau_v \mid v \in V(G)\}$  be the decorated tropical types obtained by splitting  $\tau$  at all edges. This gives a standard gluing situation, and hence a diagram as in Theorem 5.1. With notation as in that theorem,  $\phi$  is degree  $\mu(\tau)/m_\tau$  onto its image.*

*Proof.* By Theorem 5.3, it is sufficient to compare the multiplicity  $|\text{coker}(\Psi)_{\text{tors}}|$  as defined in Definition 4.2 and the multiplicity  $|\text{coker}(\overline{\Psi})_{\text{tors}}|$ . Note that as each split type  $\tau_v$  consists of a single vertex with a number of adjacent edges, the only moduli of

tropical maps of type  $\tau_v$  is given by the location of  $v$  in  $\sigma(v)$ . Thus one sees that the basic monoid associated to the type  $\tau_v$  is

$$(5.1) \quad Q_v = P_{\sigma(v)}.$$

The map  $\Psi$  of Definition 4.2 now becomes

$$\Psi : \bigoplus_{v \in V(G)} N_{\sigma(v)} \oplus \bigoplus_{E \in E(G)} \mathbb{Z} \rightarrow \bigoplus_{E \in E(G)} N_{\sigma(E)},$$

and the degree of  $\phi$  onto its image is  $|(\text{coker } \Psi)_{\text{tors}}|$ .

We now have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_v \overline{N}_{\sigma(v)} \times \prod_E \mathbb{Z} & \longrightarrow & \prod_v N_{\sigma(v)} \times \prod_E \mathbb{Z} & \xrightarrow{\delta_*} & \prod_v \mathbb{Z} \longrightarrow 0 \\ & & \overline{\Psi} \downarrow & & \Psi \downarrow & & \downarrow \partial \\ 0 & \longrightarrow & \prod_E \overline{N}_{\sigma(E)} & \longrightarrow & \prod_E N_{\sigma(E)} & \xrightarrow{\delta_*} & \prod_E \mathbb{Z} \longrightarrow 0 \end{array}$$

Here the two maps labelled  $\delta_*$  are induced by  $\delta_* : N_{\sigma(v)} \rightarrow \mathbb{Z}$  and  $\delta_* : N_{\sigma(E)} \rightarrow \mathbb{Z}$  for each vertex  $v$  and edge  $E$ . The map  $\overline{\Psi}$  is as defined in Definition 5.4. The map  $\partial$  is defined by

$$\partial((n_v)_{v \in V(G)}) = (n_{v_E} - n_{v'_E})_{E \in E(G)}.$$

In particular,  $\partial : \prod_v \mathbb{Z} \rightarrow \prod_E \mathbb{Z}$  is the complex calculating the simplicial cohomology of  $G$ , and thus  $\ker \partial = H^0(G, \mathbb{Z}) = \mathbb{Z}$ , as  $G$  is assumed connected. Note  $\ker \partial$  is generated by  $(1)_{v \in V(G)}$ . Also  $\text{coker } \partial = H^1(G, \mathbb{Z}) = \mathbb{Z}^{b_1(G)}$ . Thus the snake lemma gives a long exact sequence

$$0 \rightarrow \ker \overline{\Psi} \rightarrow \ker \Psi \rightarrow H^0(G, \mathbb{Z}) \rightarrow \text{coker } \overline{\Psi} \rightarrow \text{coker } \Psi \rightarrow H^1(G, \mathbb{Z}) \rightarrow 0.$$

Note that  $\ker \overline{\Psi}$  and  $\ker \Psi$  can be interpreted as the space of integral tangent vectors to the moduli space of maps of type  $\tau$  in  $\Delta(X)$  and  $\Sigma(X)$  respectively. By the assumption of rigidity of  $\tau$ , we thus have  $\ker \overline{\Psi} = 0$  and  $\ker \Psi = \mathbb{Z}$ . Further, the map  $\mathbb{Z} \cong \ker \Psi \rightarrow H^0(G, \mathbb{Z}) \cong \mathbb{Z}$  is multiplication by  $m_\tau$  by definition of the latter number. This gives an exact sequence

$$0 \rightarrow \mathbb{Z}/m_\tau \mathbb{Z} \rightarrow \text{coker } \overline{\Psi} \rightarrow \text{coker } \Psi \rightarrow H^1(G, \mathbb{Z}) \rightarrow 0.$$

Since  $H^1(G, \mathbb{Z})$  is torsion-free and  $\mathbb{Z}/m_\tau \mathbb{Z}$  is torsion, we easily obtain a short exact sequence

$$0 \rightarrow \mathbb{Z}/m_\tau \mathbb{Z} \rightarrow (\text{coker } \overline{\Psi})_{\text{tors}} \rightarrow (\text{coker } \Psi)_{\text{tors}} \rightarrow 0.$$

This shows that the multiplicity  $\mu(\tau, \mathbf{E})$  defined in Definition 4.2 agrees with  $\mu(\tau)/m_\tau$ , and the result follows.  $\spadesuit$

Again, we get better behaviour in the tropically transverse case, where now we have a slightly weaker definition for tropical transversality.

**Definition 5.6.** Let  $\tau$  be the combinatorial type of a rigid tropical curve in  $\Delta(X)$ . We say that  $\tau$  is *tropically transverse* if the image of the map  $\overline{\Psi}$  of Definition 5.4 has finite index.

**Theorem 5.7.** *In the situation of Theorem 5.5, suppose that the rigid decorated type  $\tau$  is tropically transverse and the central fibre  $\underline{X}_0$  is reduced. Then  $\phi$  is a finite dominant morphism of degree  $\mu(\tau)/m_\tau$ . Further, suppose that (1)  $\underline{\text{ev}}$  is flat; and (2) each  $\underline{X}_{\sigma(E)}$  is non-singular (so that  $\Delta$  is an lci morphism and the Gysin map  $\Delta^!$  exists). Then*

$$\phi'_*[\mathcal{M}(X_0/b_0, \tau)]^{\text{virt}} = \frac{\mu(\tau)}{m_\tau} \Delta^! \left( \prod_{v \in V(G)} [\mathcal{M}(X_0/b_0, \tau_v)]^{\text{virt}} \right).$$

*Proof.* The statement concerning the degree of  $\phi$  follows from Theorem 5.5 provided that we know  $\phi$  is dominant. To show this, we follow the argument of Theorem 5.3 and modify the argument of Theorem 4.4. In particular, we may assume given a strict geometric point  $W' \rightarrow \mathfrak{M}^{\text{sch, ev}}(\mathcal{X}_0/b_0, \tau)$ , with an induced decomposition with  $W' = \prod_v W_v$  and  $W_v \rightarrow \mathfrak{M}^{\text{ev}}(\mathcal{X}_0/b_0, \tau_v)$  strict geometric points with image in the dense open strata of the latter moduli spaces. We need to show that the gluing  $W' \times_{\mathfrak{M}^{\text{sch, ev}}(\mathcal{X}_0/b_0, \tau)} \mathfrak{M}^{\text{ev}}(\mathcal{X}_0/b_0, \tau)$  is non-empty for general choice of strict morphism  $W' \rightarrow \mathfrak{M}^{\text{sch, ev}}(\mathcal{X}_0/b_0, \tau)$  from a log point. To show this non-emptiness, we return to the setup and notation of the proof of Theorems 4.4 and use the non-emptiness criterion of Lemma 3.4, applied to the commutative diagram obtained from the fact that all spaces involved are defined over  $b_0$ :<sup>2</sup>

$$(5.2) \quad \begin{array}{ccccc} \prod_{E \in E(G)} X_{\sigma(E)} & \longrightarrow & \prod_{v \in E \in E(G)} X_{\sigma(E)} & \longleftarrow & \prod_{v \in V(G)} \widetilde{W}_v \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{E \in E(G)} b_0 & \longrightarrow & \prod_{v \in E \in E(G)} b_0 & \longleftarrow & \prod_{v \in V(G)} b_0 \end{array}$$

Now  $\prod_E b_0 \times_{\prod_{v \in E} b_0} \prod_v b_0$  is easily seen to be non-empty: indeed, there is a morphism from  $b_0$  to this fibre product induced by the diagonal morphisms  $b_0 \rightarrow \prod_E b_0$  and  $b_0 \rightarrow \prod_v b_0$ . (In fact, the induced morphism is an isomorphism, so the fibre product is  $b_0$ , but we don't need this.) This gives condition (1) of the hypotheses of Lemma 3.4.

Condition (2) follows from realizability the tropical type  $\tau$ . Indeed, at the level of dual of monoids, the fibre product involving the top row of (5.2) gives a Cartesian diagram of monoids

$$\begin{array}{ccc} \widetilde{Q}^\vee & \longrightarrow & \prod_{v \in V(G)} \widetilde{Q}_v^\vee \\ \downarrow & & \downarrow \Sigma(\text{ev}) \\ \prod_{E \in E(G)} P_{\sigma(E)}^\vee & \xrightarrow{\Sigma(\Delta)} & \prod_{v \in E \in E(G)} P_{\sigma(E)}^\vee \end{array}$$

Here  $\widetilde{Q}_v$  is the stalk of the ghost sheaf of  $\widetilde{W}_v$ , with  $\widetilde{Q}_v \subseteq Q_v \oplus \bigoplus_{v \in E \in E(G)} \mathbb{Z}$ . Recall (5.1) that  $Q_v = P_{\sigma(v)}$ . We need to show that the image of  $\widetilde{Q}^\vee$  in  $\prod_E P_{\sigma(E)}^\vee$  or  $\prod_v \widetilde{Q}_v^\vee$  intersects the interior of that monoid.

<sup>2</sup>We note here we work with fibre products over  $\text{Spec } \mathbb{k}$  rather than  $B = b_0$ , as we did in Theorem 4.4. Here we apply Proposition 2.4 to see that we may just as well work with the moduli spaces over  $\text{Spec } \mathbb{k}$ .



First we describe  $\Sigma(\Delta)$  and  $\Sigma(\text{ev})$ . Indeed,  $\Sigma(\Delta)$  is just the diagonal, while

$$\Sigma(\text{ev}) \left( (n_v)_{v \in V(G)}, (\ell_{v,E})_{v \in E \in E(G)} \right) = (n_v + \ell_{v,E} \mathbf{u}_v(E))_{v \in E \in E(G)}$$

where  $\mathbf{u}_v(E)$  is the contact order of the leg  $(v, E)$  of  $G_v$ . Thus the above fibre product diagram allows us to view  $\tilde{Q}^\vee$  as a submonoid of  $\prod_{v \in v(G)} \tilde{Q}_v^\vee$  consisting of those tuples  $((n_v)_{v \in V(G)}, (\ell_{v,E})_{v \in E \in E(G)})$  such that, for every edge  $E \in E(G)$  with vertices  $v_1, v_2$ , we have

$$(5.3) \quad n_{v_1} + \ell_{v_1,E} \mathbf{u}_{v_1}(E) = n_{v_2} + \ell_{v_2,E} \mathbf{u}_{v_2}(E).$$

We may then interpret a point  $s \in \tilde{Q}^\vee$  as giving the data of:

- (1) The choice of a tropical map  $h_s : G \rightarrow \Sigma(X)$  of type  $\tau$ . This tropical map is determined by  $h_s(v) = n_v \in P_{\sigma(v)}^\vee$ , and the condition (5.3) then guarantees that the edge  $E$  is still mapped to an edge in  $\sigma(E)$  with tangent vector  $\mathbf{u}(E)$ .
- (2) For each edge  $E \in E(G)$ , a choice of decomposition of the length of  $E$  as a sum  $\ell_{v_1,E} + \ell_{v_2,E}$ . This is given by (5.3): since  $\mathbf{u}_{v_2}(E) = -\mathbf{u}_{v_1}(E)$ , (5.3) can be rewritten as  $n_{v_2} - n_{v_1} = (\ell_{v_1,E} + \ell_{v_2,E}) \mathbf{u}_{v_1}(E)$ , and thus  $\ell_E = \ell_{v_1,E} + \ell_{v_2,E}$ .

However, since  $\tau$  is a type of rigid curve, there is a unique tropical curve  $h_s : \Gamma \rightarrow \Sigma(X)$  of type  $\tau$ , up to scaling. By scaling this curve sufficiently so it is integrally defined and all edge lengths are at least 2, we may also choose non-trivial integral decompositions of the edge lengths. This yields an interior point of  $\tilde{Q}^\vee$ , which necessarily maps to the interior of  $\prod_v \tilde{Q}_v^\vee$  and to an interior point of  $\prod_E P_{\sigma(E)}^\vee$ . This shows condition (2) of Lemma 3.4.

Abusing notation, we denote by  $\delta$  the generator of the ghost sheaf  $\mathbb{N}$  of  $b_0$ , and also denote by  $\delta$  its image in any of the monoids  $P_{\sigma(E)}$  under the structure map  $X_0 \rightarrow b_0$ . Note this is largely compatible with our previous use of the notation  $\delta : \Sigma(X) \rightarrow \Sigma(B)$ . To verify condition (3) of the lemma, using the above description of the monoids involved in the fibre product, we need to show injectivity of

$$\bigoplus_{v \in E \in E(G)} P_{\sigma(E)}^{\text{gp}} / \mathbb{Z}\delta \rightarrow \left[ \bigoplus_{E \in E(G)} P_{\sigma(E)}^{\text{gp}} / \mathbb{Z}\delta \right] \times \left[ \bigoplus_{v \in V(G)} P_{\sigma(v)}^{\text{gp}} / \mathbb{Z}\delta \times \bigoplus_{v \in E \in E(G)} \mathbb{Z} \right]$$

Because of the hypothesis that the central fibre  $X_0$  is reduced, all groups  $P_{\sigma(E)}^{\text{gp}} / \mathbb{Z}\delta$  are torsion-free, and hence injectivity is equivalent to the transpose map having finite cokernel. However, we have a variant of diagram (4.6) given by replacing each  $N_{\sigma(E)}, N_{\sigma(v)}$  with  $\overline{N}_{\sigma(E)}, \overline{N}_{\sigma(v)}$ , which shows that the cokernel of the transpose of the above map is finite if and only if the cokernel of  $\overline{\Psi}$  is finite, which is the tropically transverse condition.

For the last statement, condition (2) implies  $\Delta$  is lci, hence  $\Delta^!$  makes sense. If condition (1) holds, i.e.,  $\text{ev}$  is flat, then the Gysin pull-backs  $\Delta^!$  and  $(\Delta')^!$  agree. The result then follows from Theorem 5.5 and properties of virtual pull-backs [Ma12, Thm. 4.1].  $\spadesuit$

## 6. PUNCTURED VERSUS RELATIVE

In a gluing situation arising from a degeneration situation  $X \rightarrow B$ , one has the moduli spaces  $\mathcal{M}(X, \tau_v)$ , classifying punctured maps marked by  $\tau_v$ . In particular,

such maps will factor through a stratum  $X_{\sigma(v)}$ , with its induced log structure. On the other hand,  $X_{\sigma(v)}$  comes with a divisorial log structure induced by  $\partial X_{\sigma(v)} \subseteq X_{\sigma(v)}$ , the union of lower dimensional strata of  $X$  contained in  $X_{\sigma(v)}$ . We write this different log scheme as  $\overline{X}_{\sigma(v)}$ . It is then natural to compare  $\mathcal{M}(X, \tau_v)$  with a moduli space of stable log maps to  $\overline{X}_{\sigma(v)}$ . In general, this still a non-trivial question, and is likely related to double ramification cycles. Here, however, we deal with a special case where  $X_{\sigma(v)}$  is an irreducible component of  $X_{b_0}$ .

To set this up, let  $\sigma \in \Sigma(X)$  be a non-zero cone, corresponding to a stratum  $X_\sigma$ , and assume  $X_\sigma \subseteq X_{b_0}$ . If  $\dim \sigma = 1$ , then  $X_\sigma$  is an irreducible component of  $X_{b_0}$ . As above, we have the log scheme  $\overline{X}_\sigma$ , and there is a canonical morphism of log schemes

$$\psi : X_\sigma \rightarrow \overline{X}_\sigma$$

induced by the natural inclusion  $\mathcal{M}_{\overline{X}_\sigma} \subseteq \mathcal{M}_{X_\sigma} = \mathcal{M}_X|_{X_\sigma}$ . Indeed, smooth locally at a geometric point  $\bar{x} \in X_\sigma$ ,  $X$  is given as  $\text{Spec } \mathbb{k}[P]$  for  $P = \overline{\mathcal{M}}_{X, \bar{x}}$ , and the monoid  $P$  has a codimension  $\dim \sigma$  face  $F \subseteq P$  such that  $X_\sigma$  is smooth locally  $\text{Spec } \mathbb{k}[F]$  and  $F = \overline{\mathcal{M}}_{X_\sigma, \bar{x}}$ . The log structure on  $X_\sigma$  at  $\bar{x}$  is given by a neat chart  $P \rightarrow \mathcal{O}_{X_\sigma}$ , and the log structure on  $\overline{X}_\sigma$  is given by restricting this chart to  $F$ . Hence we obtain the inclusion of log structures.

Note the inclusion of faces  $F \subseteq P$  dualizes to a generization  $P_{\mathbb{R}}^{\vee} \rightarrow F_{\mathbb{R}}^{\vee} = (P_{\mathbb{R}}^{\vee} + \mathbb{R}\sigma)/\mathbb{R}\sigma$ . Thus it is easy to describe the induced map of cone complexes

$$\Sigma(\psi) : \Sigma(X_\sigma) \rightarrow \Sigma(\overline{X}_\sigma).$$

In particular, a type  $\tau = (G, \sigma, \mathbf{u})$  for tropical map to  $\Sigma(X_\sigma)$  induces a type  $\bar{\tau} = (\bar{G}, \bar{\sigma}, \bar{\mathbf{u}})$  for tropical map to  $\Sigma(\overline{X}_\sigma)$ . Indeed,  $\bar{G}$  will coincide with  $G$ , except for some legs of  $G$  which correspond to punctured points, i.e., are line segments, will be replaced by rays, corresponding to marked points. For  $v \in V(G)$ , we may take  $\bar{\sigma}(v)$  to be the minimal cone of  $\Sigma(\overline{X}_\sigma)$  containing  $\Sigma(\psi)(\sigma(v))$ . For an edge  $E$  of  $G$ , we define  $\bar{\mathbf{u}}(E) = \Sigma(\Psi)_*(\mathbf{u}(E))$ .

We remark that the strict inclusion  $X_\sigma \hookrightarrow X$  induces a map  $\Sigma(X_\sigma) \rightarrow \Sigma(X)$ . Even though  $X$  is assumed to be Zariski, this need not be an inclusion of cone complexes. For example, if  $X_0$  is a union of two  $\mathbb{P}^1$ 's meeting at two points and  $X_\sigma$  is one of these  $\mathbb{P}^1$ 's, we have  $\Sigma(X)$  a union of two quadrants glued along their boundary, but  $\Sigma(X_\sigma)$  consists of two quadrants glued together along one boundary ray. However, the map  $|\Sigma(X_\sigma)| \rightarrow |\Sigma(X)|$  will always be injective in a neighbourhood of  $\sigma$ .

Now suppose given a type  $\tau$  of punctured maps to  $X/B$ , with underlying graph  $G$  having precisely one vertex  $v$  and adjacent legs  $L_1, \dots, L_n$ . Suppose further that  $\sigma(v) = \sigma$ . Then  $\tau$  may also be viewed as a type of punctured map to  $X_\sigma/B$ , and thus we also obtain a type  $\bar{\tau}$  of punctured map to  $\overline{X}_\sigma$ . However, in fact all contact orders are positive. Indeed, because  $\sigma(v) = \sigma$ , any  $L_i$  must have contact order  $\mathbf{u}(L_i)$  lying in the tangent wedge of  $\sigma(L_i)$  along the face  $\sigma$  of  $\sigma(L_i)$ . Hence the image of  $\mathbf{u}(L_i)$  in the tangent space to  $(\sigma(L_i) + \mathbb{R}\sigma)/\mathbb{R}\sigma$  in fact lies in this cone. Therefore we may view the type  $\bar{\tau}$  as a type of logarithmic map to  $\overline{X}_\sigma$ .

In addition, in this situation, we have a commutative diagram

$$(6.1) \quad \begin{array}{ccc} \mathcal{M}(X/B, \boldsymbol{\tau}) & \longrightarrow & \mathcal{M}(\overline{X}_\sigma, \overline{\boldsymbol{\tau}}) \\ \downarrow & & \downarrow \\ \mathfrak{M}(\mathcal{X}/B, \tau) & \longrightarrow & \mathfrak{M}(\overline{\mathcal{X}}_\sigma, \overline{\tau}) \end{array}$$

where  $\overline{\mathcal{X}}_\sigma$  is defined in the same way as  $\overline{X}_\sigma$ . While the vertical arrows are the standard ones, the horizontal ones are as follows. By the definition of a  $\boldsymbol{\tau}$ -marked curve, we have  $\mathcal{M}(X/B, \boldsymbol{\tau}) \cong \mathcal{M}(X_\sigma/B, \boldsymbol{\tau})$ , i.e., any  $\boldsymbol{\tau}$ -marked curve factors through  $X_\sigma$ , by [ACGS2, Def. 3.7,(1)]. Given a stable punctured map  $f : C^\circ/W \rightarrow X_\sigma$ , we have  $\psi \circ f : C^\circ/W \rightarrow \overline{X}_\sigma$ . In general, this is neither a pre-stable nor basic log map, but we may pre-stabilize by [ACGS2, Prop. 2.4] and replace with a basic log map by [ACGS2, Prop. 2.40].<sup>3</sup> This gives the upper horizontal arrow, and a similar discussion gives the lower horizontal arrow.

**Theorem 6.1.** *Let  $X \rightarrow B$ ,  $\boldsymbol{\tau}$ ,  $\sigma$  be as in the above discussion, and suppose further that  $\dim \sigma = 1$ . Then the horizontal arrows of the diagram (6.1) are isomorphisms at the level of underlying stacks (but not at the level of logarithmic stacks). Furthermore these isomorphisms induce an isomorphism of obstruction theories.*

*Proof.* We work with the top horizontal arrow of (6.1), as the Artin fan case is identical. We need to construct the inverse map. Explicitly, given a basic  $\overline{\boldsymbol{\tau}}$ -marked stable log map  $\bar{f} : \overline{C}/\overline{W} \rightarrow \overline{X}_\sigma$ , we need to construct a stable punctured map  $f : C/W \rightarrow X_\sigma$ . The maps  $f, \bar{f}$  will be the same on schemes, but we need to modify the log structures on  $\overline{W}, \overline{C}$ .

To this end, let  $\overline{Q}_{\bar{w}}$  be the stalk of the ghost sheaf at a geometric point  $\bar{w} \in \overline{W}$ , so that  $\bar{\tau}_{\bar{w}} = \overline{Q}_{\bar{w}, \mathbb{R}}^\vee$  parametrizes a universal family of tropical maps  $\bar{h} : \Gamma(\overline{G}_{\bar{w}}, \bar{\ell}) \rightarrow \Sigma(\overline{X}_\sigma)$ , inducing for each  $s \in \bar{\tau}$  a tropical map  $\bar{h}_s : \overline{G}_{\bar{w}} \rightarrow \Sigma(\overline{X}_\sigma)$ .

For  $r \in \mathbb{R}_{\geq 0}$ , write  $\Delta_r$  for the subset  $\Sigma(\psi)(\delta^{-1}(r))$ , where  $\delta : \Sigma(X_\sigma) \rightarrow \Sigma(B)$  is the tropicalization of  $X_\sigma \rightarrow B$ . Of course  $\Delta_1$  agrees with the underlying topological space of  $\Delta(X_\sigma)$ . Now note that we may restrict the map  $\Sigma(\psi)$  to  $\Delta_r$  to obtain an inclusion of topological spaces  $\Delta_r \hookrightarrow |\Sigma(\overline{X}_\sigma)|$ . The spaces  $\Delta_r$  exhaust  $|\Sigma(\overline{X}_\sigma)|$ , i.e., for  $y \in |\Sigma(\overline{X}_\sigma)|$ , there exists an  $r_0 \geq 0$  such that  $y \in \Delta_r$  for all  $r \geq r_0$ .

Let  $\overline{G}'_{\bar{w}}$  be the subgraph of  $\overline{G}_{\bar{w}}$  obtained by deleting all legs of  $\overline{G}_{\bar{w}}$ . We define a function  $\alpha : \bar{\tau}_{\bar{w}} \rightarrow \mathbb{R}_{\geq 0}$  by

$$\alpha(s) := \inf\{r \in \mathbb{R}_{\geq 0} \mid h_s(\overline{G}'_{\bar{w}}) \subseteq \Delta_r\}.$$

The set on the right-hand side is non-empty, and hence this makes sense.

*Claim.*  $\alpha$  is an upper convex piecewise linear function on  $\bar{\tau}_{\bar{w}}$  with rational slopes.

*Proof.* For each  $\omega \in \Sigma(X_\sigma)$ , write  $\omega_r = \omega \cap \Delta_r$ . Suppose  $\omega$  contains  $\sigma$ . For each facet  $F$  of  $\omega$  not containing  $\sigma$ , let  $n_F \in \omega^\vee$  be a rational generator of the dual one-dimensional face. Since  $\sigma \not\subseteq F$ ,  $n_F|_\sigma$  is non-vanishing and non-negative, so we can normalize  $n_F$  so that this restriction agrees with  $\delta$  by rescaling by a rational number. Thus  $\delta - n_F$

<sup>3</sup>Note that once we pre-stabilize, all punctured points become marked as all contact orders in  $\bar{\mathbf{u}}$  are positive. Hence we could just as well apply the earlier [GS4, Prop. 1.24].

vanishes on  $\sigma$  and hence descends to a linear function on  $\bar{\omega} = (\omega + \mathbb{R}\sigma)/\mathbb{R}\sigma$ . Further, the value of  $\delta - n_F$  on  $F_r$  is  $r$  and 0 on  $\sigma$ . From this, we see that if  $F_1, \dots, F_p$  is the set of all such facets, then

$$\Sigma(\psi)(\omega_r) = \{m \in \bar{\omega} \mid \langle \delta - n_{F_i}, m \rangle \leq r, 1 \leq i \leq p\}.$$

Now for each vertex  $v \in V(\bar{G}_{\bar{\omega}})$ , we have the evaluation map  $\text{ev}_v : \bar{\tau}_{\bar{\omega}} \rightarrow \bar{\sigma}(v)$  at  $v$  given by (4.2). Note that  $h_s(\bar{G}'_{\bar{\omega}}) \subseteq \Delta_r$  if and only if  $h_s(v) \in \bar{\sigma}(v)_r$  for all  $v \in V(\bar{G}_{\bar{\omega}})$ . In particular, for a given  $v$ ,  $h_s(v) \in \bar{\sigma}(v)_r$  if  $\langle \delta - n_{v, F_i}, \text{ev}_v(s) \rangle \leq r$  for  $\{n_{v, F_i}\}$  the set of normal vectors to facets of  $\bar{\sigma}(v)$  not containing  $\sigma$  as above. Thus  $\alpha$  is given as

$$\alpha(s) = \sup\{\langle \delta - n_{v, F_i}, \text{ev}_v(s) \rangle\},$$

with the supremum running over all  $v \in V(\bar{G}_{\bar{\omega}})$  and normal vectors  $n_{v, F_i}$ , i.e.,  $\alpha$  agrees with the supremum of the linear functionals  $\text{ev}_v^*(\delta - n_{v, F_i})$ . This makes  $\alpha$  piecewise linear and (upper) convex.  $\spadesuit$

We may now define

$$\tau_{\bar{\omega}} := \{(s, r) \in \bar{\tau}_{\bar{\omega}} \mid r \geq \alpha(s)\}.$$

This is a rational polyhedral cone by the claim. From construction,  $\tau_{\bar{\omega}}$  parameterizes lifts of the tropical maps  $\bar{h}_s$  to  $\Sigma(X_\sigma)$ . Indeed, if  $(s, r) \in \tau_{\bar{\omega}}$ ,  $\bar{h}_s$  maps  $\bar{G}'_{\bar{\omega}}$  into  $\Delta_r \subseteq |\Sigma(X_\sigma)|$ . We may then extend this map as far as possible along each leg  $L_i$  to define a domain graph  $G_{\bar{\omega}}$ , which coincides with  $\bar{G}_{\bar{\omega}}$  except that the legs may be replaced with line segments. We then get  $h_{s,r} : G_{\bar{\omega}} \rightarrow \Sigma(X_\sigma)$ . In particular, a leg  $L_i$  turns into a punctured leg, i.e., a line segment, if only a portion of  $h_s(L_i)$  is contained in  $\Delta_r$ .

Note here we may now view  $\tau_{\bar{\omega}}$  as type of tropical map to  $\Sigma(X_\sigma)$ , with  $\bar{\sigma}$  defined in the obvious way from  $\bar{\sigma}$  and  $\mathbf{u}(E)$  a tangent vector of  $\sigma(E)_1$  mapping under  $\Sigma(\psi)_*$  to  $\bar{\mathbf{u}}(E)$ . In particular, we obtain a universal family of tropical maps of type  $\tau_{\bar{\omega}}$ ,

$$(6.2) \quad h_{\bar{\omega}} : \Gamma(G, \ell) \rightarrow \Sigma(X_\sigma).$$

We then have the corresponding basic monoid

$$Q_{\bar{\omega}} := \tau_{\bar{\omega}}^\vee \cap (\Lambda_{\tau_{\bar{\omega}}}^* \oplus \mathbb{Z}).$$

Note that

$$Q_{\bar{\omega}} \subseteq \Lambda_{\tau_{\bar{\omega}}}^* \oplus \mathbb{N}$$

since  $(0, 1) \in \tau_{\bar{\omega}}$ . The monoid  $Q_{\bar{\omega}}$  will be the basic monoid for the point  $\bar{w}$  for punctured log maps to  $X_\sigma$ . Note the projection  $\tau_{\bar{\omega}} \rightarrow \bar{\tau}_{\bar{\omega}}$  dualizes to an inclusion  $\bar{Q}_{\bar{\omega}} \hookrightarrow Q_{\bar{\omega}}$  which identifies  $\bar{Q}_{\bar{\omega}}$  with the facet  $\{(q, 0) \in Q_{\bar{\omega}}\}$  of  $Q_{\bar{\omega}}$ . Further,  $Q_{\bar{\omega}}$  is identified with a submonoid of  $\bar{Q}_{\bar{\omega}}^{\text{gp}} \oplus \mathbb{N}$ .

It is easy to check that if  $\bar{w}'$  is a generization of  $\bar{w}$ , the above construction of  $Q_{\bar{w}}$  is compatible with generization maps, i.e., the generization map  $\bar{Q}_{\bar{w}} \rightarrow \bar{Q}_{\bar{w}'}$  induces a generization map  $Q_{\bar{w}} \rightarrow Q_{\bar{w}'}$ . Indeed, dually, we have an inclusion of faces  $\bar{\tau}_{\bar{w}'} \subseteq \bar{\tau}_{\bar{w}}$ , and  $\alpha : \bar{\tau}_{\bar{w}} \rightarrow \mathbb{R}_{\geq 0}$  restricts to the corresponding map for  $\bar{\tau}_{\bar{w}'}$ . Hence we obtain an inclusion of faces  $\tau_{\bar{w}'} \subseteq \tau_{\bar{w}}$ . From this, we see that the monoids  $Q_{\bar{w}}$  for various  $\bar{w}$  define a fine subsheaf  $\bar{\mathcal{M}}_W$  of  $\bar{\mathcal{M}}_W^{\text{gp}} \oplus \mathbb{N}$ .

Let  $W^\dagger = \bar{W} \times b_0$ , so that  $\bar{\mathcal{M}}_{W^\dagger} = \bar{\mathcal{M}}_{\bar{W}} \oplus \mathbb{N}$ , and set

$$\mathcal{M}_W := \bar{\mathcal{M}}_W \times_{\bar{\mathcal{M}}_{W^\dagger}^{\text{gp}}} \mathcal{M}_{W^\dagger}^{\text{gp}}.$$

We may then define a structure morphism  $\alpha_W : \mathcal{M}_W \rightarrow \mathcal{O}_W$  by taking  $\alpha_W = \alpha_{\overline{W}}$  on  $\mathcal{M}_{\overline{W}} \oplus 0 \subseteq \mathcal{M}_W$  and  $\alpha_W$  taking the value 0 on  $\mathcal{M}_W \setminus (\mathcal{M}_{\overline{W}} \oplus 0)$ . Thus we obtain an fs log scheme  $W$ .

We have morphisms  $W \rightarrow W^\dagger \rightarrow \overline{W}$  by construction, and the log smooth curve  $\overline{C} \rightarrow \overline{W}$  pulls back to give  $C^\dagger \rightarrow W^\dagger$  and  $C \rightarrow W$ . Note further  $\bar{f} : \overline{C} \rightarrow \overline{X}_\sigma$  induces a morphism  $f^\dagger : C^\dagger \rightarrow \overline{X}_\sigma \times b_0$  defined over  $b_0$ . We wish to define a punctured structure  $C^\circ$  on  $C$  yielding a commutative diagram

$$\begin{array}{ccc} C^\circ & \xrightarrow{f} & X_\sigma \\ \downarrow & & \downarrow \\ C^\dagger & \xrightarrow{f^\dagger} & \overline{X}_\sigma \times b_0 \end{array}$$

As  $\mathcal{M}_{W^\dagger}^{\text{gp}} = \mathcal{M}_W^{\text{gp}}$ , so that  $\mathcal{M}_{C^\circ}^{\text{gp}} = \mathcal{M}_{C^\dagger}^{\text{gp}}$ , and  $\mathcal{M}_{X_\sigma}^{\text{gp}} = \mathcal{M}_{\overline{X}_\sigma \times b_0}^{\text{gp}}$ ,  $\bar{f}^\flat$  and  $(f^\dagger)^\flat$  agree at the level of groups, and it is sufficient to construct a pre-stable punctured log structure  $C^\circ$  on  $C$  so that

$$(6.3) \quad (\bar{f}^\dagger)^\flat(\overline{\mathcal{M}}_{X_\sigma}) \subseteq \overline{\mathcal{M}}_{C^\circ}.$$

Note that the saturation of  $\overline{\mathcal{M}}_{C^\circ}$  over  $\bar{w} \in W$ , as well as the map  $(\bar{f}^\dagger)^\flat : f^{-1}\overline{\mathcal{M}}_{X_\sigma} \rightarrow \overline{\mathcal{M}}_{C^\circ}^{\text{sat}}$ , are completely determined locally near  $\bar{w}$  by the data of the tropical family of maps  $h_{\bar{w}}$  of (6.2). Thus we only need to consider the punctured points.

So let  $p \in C_{\bar{w}}$  be a marked point corresponding to a leg  $L$  of  $\overline{G}_{\bar{w}}$  (or the corresponding leg of  $G_{\bar{w}}$ ). We have the monoids  $\overline{P}_{\bar{\sigma}(L)}$  (resp.  $P_{\sigma(L)}$ ) which are the stalks of  $\overline{\mathcal{M}}_{\overline{X}_\sigma}$  (resp.  $\overline{\mathcal{M}}_{X_\sigma}$ ) at the generic point of the strata of  $\overline{X}_\sigma$  (resp.  $X_\sigma$ ) corresponding to  $\bar{\sigma}(L)$  (resp.  $\sigma(L)$ ). Then  $\overline{P}_{\bar{\sigma}(L)}$  is a facet of  $P_{\sigma(L)}$ , and we have a commutative diagram with horizontal arrows induced by  $\bar{f}$ ,  $f^\dagger$  and the family of tropical maps parameterized by  $\tau_{\bar{w}}$ :

$$\begin{array}{ccc} \overline{P}_{\bar{\sigma}(L)} & \longrightarrow & \overline{\mathcal{M}}_{\overline{C},p} = \overline{Q}_{\bar{w}} \oplus \mathbb{N} \\ \downarrow & & \downarrow \\ \overline{P}_{\bar{\sigma}(L)} \oplus \mathbb{N} & \longrightarrow & \overline{\mathcal{M}}_{C^\dagger,p} = (\overline{Q}_{\bar{w}} \oplus \mathbb{N}) \oplus \mathbb{N} \\ \downarrow & & \downarrow \\ P_{\sigma(L)} & \longrightarrow & \overline{\mathcal{M}}_{C^\circ,p} \subseteq Q_{\bar{w}} \oplus \mathbb{Z} \end{array}$$

Here the inclusion  $\overline{P}_{\bar{\sigma}(L)} \oplus \mathbb{N} \rightarrow P_{\sigma(L)}$  identifies  $P_{\sigma(L)}$  with a submonoid of  $\overline{P}_{\bar{\sigma}(L)} \oplus \mathbb{N}$  with the facet  $\overline{P}_{\bar{\sigma}(L)}$  of  $P_{\sigma(L)}$  identified with  $\overline{P}_{\bar{\sigma}(L)} \oplus 0$ . To obtain a pre-stable puncturing, we take the stalk of the ghost sheaf of  $\overline{\mathcal{M}}_{C^\circ}$  at  $p$  to be the submonoid of  $Q_{\bar{w}} \oplus \mathbb{Z}$  generated by  $Q_{\bar{w}} \oplus \mathbb{N}$  and the image of  $P_{\sigma(L)}$  under  $(\bar{f}^\dagger)^\flat$ . We just need to make sure that this defines a puncturing by showing that if  $(\bar{m}_1, \bar{m}_2) \in \overline{\mathcal{M}}_{C^\circ} \setminus \overline{\mathcal{M}}_C$  is in the image of  $P_{\sigma(L)}$ , and  $m_1$  is a lift of  $\bar{m}_1$  to a local section of  $\mathcal{M}_W$ , then  $\alpha_W(m_1) = 0$ . But by construction, the composition of  $P_{\sigma(L)} \rightarrow \overline{\mathcal{M}}_{C^\circ,p}$  with the first projection preserves the second coordinate, i.e.,  $(p, r) \in \overline{P}_{\bar{\sigma}(L)} \oplus \mathbb{N}$  is mapped to  $(q, r) \in \overline{Q}_{\bar{w}}^{\text{gp}} \oplus \mathbb{N}$ . As  $\mathbf{u}(L)$  is positive on  $\overline{P}_{\bar{\sigma}(L)}$ ,  $p$  being a marked point on  $\overline{C}$ , we see  $\bar{m}_1 \in P_{\sigma(L)} \setminus \overline{P}_{\bar{\sigma}(L)}$ , and

hence  $\bar{m}_1$  maps to  $Q_{\bar{w}} \setminus \overline{Q_{\bar{w}}}$ . Hence, by definition of the structure map  $\alpha_W$ , we have  $\alpha_W(m_1) = 0$ , as desired.

It is easy to check that the morphisms constructed between  $\mathcal{M}(X_\sigma/B, \tau)$  and  $\mathcal{M}(\overline{X}_\sigma, \bar{\tau})$  are inverse to each other, and hence isomorphisms. Further, one easily checks the induced obstruction theories are the same, as the construction of these obstruction theories does not depend on the log structure.  $\spadesuit$

## 7. THE CLASSICAL DEGENERATION FORMULA

The classical degeneration situation, originally considered by Li and Ruan in [LR01] and developed in the algebro-geometric context by Jun Li in [Li02] is a special case of the degeneration situation of the previous section. We have already discussed this case in the context of the decomposition formula in [ACGS1, §6.1]. We consider  $X \rightarrow B$  a simple normal crossings degeneration with  $X_0 = Y_1 \cup Y_2$  a reduced union of two irreducible components, with  $Y_1 \cap Y_2 = D$  a smooth divisor in both  $Y_1$  and  $Y_2$ . In this case  $\Sigma(X) = (\mathbb{R}_{\geq 0})^2$  with map  $\Sigma(X) \rightarrow \Sigma(B)$  given by  $(x, y) \mapsto x + y$ , so that  $\Delta(X)$  is a unit interval. We showed in [ACGS1, Prop. 6.1.1] that if  $f : \Gamma \rightarrow \Delta(X)$  is a rigid tropical curve, then all vertices of  $\Gamma$  map to endpoints of  $\Delta(X)$  and all edges of  $\Gamma$  surject onto  $\Delta(X)$ . In this case, giving a rigid tropical curve with target  $\Delta(X)$  is the same information as an admissible triple of [Li02].

Using the setup of §5, we immediately obtain the following restatement of the main result of [Li02]:

**Theorem 7.1.** *In the situation described above, let  $\tau$  be a decorated type of rigid tropical curve in  $\Delta(X)$ . There is a diagram of (non-logarithmic) stacks*

$$\begin{array}{ccc} \mathcal{M}(X_0/b_0, \tau) & \xrightarrow{\phi'} & \mathcal{M}^{\text{sch}}(X_0/b_0, \tau) & \longrightarrow & \prod_{v \in V(G)} \mathcal{M}(\overline{X}_{\sigma(v)}, \bar{\tau}_v) \\ & & \downarrow & & \downarrow \\ & & \prod_{E \in E(G)} \underline{D} & \xrightarrow{\Delta} & \prod_{v \in E \in E(G)} \underline{D} \end{array}$$

with the square Cartesian and defining the space  $\mathcal{M}^{\text{sch}}(X_0/b_0, \tau)$ . Further,  $\phi'$  is finite and

$$\phi'_*[\mathcal{M}(X_0/b_0, \tau)]^{\text{virt}} = m_\tau^{-1} \left( \prod_{E \in E(G)} w(E) \right) \Delta^! \left( \prod_{v \in V(G)} [\mathcal{M}(\overline{X}_{\sigma(v)}, \bar{\tau}_v)]^{\text{virt}} \right),$$

where  $w(E)$  is the index (degree of divisibility) of  $\mathbf{u}(E)$ .

*Proof.* Using Theorem 6.1, the given diagram is a part of the diagram of Theorem 5.1. The result will follow from Theorem 5.7. Thus we first verify the tropical transversality condition, and calculate  $\mu(\tau)$ .

Note that  $\overline{N}_{\sigma(v)} = 0$ , while each  $\overline{N}_{\sigma(E)}$  can be identified with  $\mathbb{Z} \cong \mathbb{Z}(1, -1) \subseteq \mathbb{Z}^2$  for any vertex  $v$ , edge  $E$ . Thus the morphism  $\Psi$  of Definition 5.4 takes the form

$$\Psi : \bigoplus_{E \in E(G)} \mathbb{Z} \rightarrow \bigoplus_{E \in E(G)} \mathbb{Z}$$

given by  $\Psi((\ell_E)_{E \in E(G)}) = (\ell_E \mathbf{u}(E))$ . In particular, the image has finite index, and this index is  $\prod_E w(E)$ .

By Theorem 5.7, it is thus sufficient to show that  $\underline{ev}$  is flat. However, since  $\underline{X}_{\sigma(E)} = \underline{D}$ , and  $\underline{D}$  is a single stratum, it follows that  $\underline{ev}$  is base-changed from a morphism  $\prod_{v \in V(G)} \mathfrak{M}(\mathcal{X}, \tau_v) \rightarrow \prod_{v \in E \in E(G)} \mathcal{X}_{\sigma(E)}$ . But the stratum of  $\mathcal{X}$  corresponding to  $D$  is isomorphic to  $B\mathbb{G}_m$ , and such a morphism is always flat. ♠

## 8. APPLICATIONS TO WALL STRUCTURES FOR TYPE III DEGENERATIONS OF K3 SURFACES

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