REMARKS ON GLUING PUNCTURED LOGARITHMIC MAPS

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Abstract. We consider some well-behaved cases of the gluing formalism for punctured log stable maps of [ACGS1, ACGS2]. This gives a gluing formula for log Gromov-Witten invariants in a diverse set of cases; in particular, the gluing formulae of [LR01, Li02, KLR] become an easy special case. The last section gives an application of this gluing formalism to canonical wall structures for K3 surfaces as constructed in [GS8].

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Contents

1. Introduction

One of the key motivations for developing logarithmic Gromov-Witten [AC, Ch, GS4] was to generalize the gluing formulae of Li–Ruan [LR01] and Jun Li [Li02] to more general degenerations. The original gluing formulae consider flat families $\pi : X \to B$, where B is a non-singular curve with a special point $b_0 \in B$, π is a normal crossing morphism with $\pi|_{\pi^{-1}(B\setminus\{0\})}$ smooth and $\pi^{-1}(b_0) =: X_0$ a union of two irreducible divisors Y_1, Y_2 meeting transerversally. The above-mentioned gluing formulae then

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relate the Gromov-Witten invariants of the general fibre of π to relative Gromov-Witten invariants of the pairs (Y_1, D), (Y_2, D), where $D = Y_1 \cap Y_2$. In principal, one would like to allow the fibre X_0 to have many irreducible components and deeper strata, where more than two irreducible components meet.

Logarithmic Gromov-Witten theory defines the notion of a logarithmic stable map to such targets $X \to B$; more generally, this morphism just needs to be toroidal rather than normal crossings, i.e., log smooth. However, completely satisfactory generalizations of the gluing formulae have remained elusive. The work of Abramovich, Chen, Gross and Siebert [ACGS1, ACGS2] sets up a framework for thinking about gluing formulae. This requires, in particular, developing *punctured* stable maps, further generalizing logarithmic stable maps. The essential reason for this is, if given a log smooth curve, the restriction of the log structure to an irreducible component need not yield a log smooth curve. In particular, this requires allowing somewhat more general domains. As a side benefit, this generalization introduces the notion of negative contact order. In turn, this gives a richer set of invariants which have proved invaluable for mirror symmetry constructions, see e.g., [GS7, GS8].

The basic setup for a gluing problem for log or punctured maps with target $X \to B$ is given by the data of a *decorated tropical type*, reviewed in §2. This is data $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$. Here, G is a dual intersection graph for a domain curve, with vertices $V(G)$ corresponding to (unions of) irreducible components, edges $E(G)$ corresponding to nodes and legs $L(G)$ corresponding to marked or punctured points. The map $\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$ records which stratum of X (strata of X being indexed by cones in the tropicalization $\Sigma(X)$ of X) the corresponding curve feature maps into. The data $\mathbf{u}$ records the contact orders associated to edges and legs, and the data $\mathbf{A}$ associates a curve class in X to each vertex of G. Together, this data determines a moduli space $\mathscr{M}(X/B, \boldsymbol{\tau})$ of punctured log maps marked by $\boldsymbol{\tau}$, as defined in [ACGS2]. This moduli space is a proper Deligne-Mumford stack over B, assuming X is projective over B, and carries a virtual fundamental class $[\mathscr{M}(X/B, \boldsymbol{\tau})]^{\mathrm{virt}}$.

The question of gluing is then as follows. Suppose given a decorated tropical type $\boldsymbol{\tau}$ as above, and a subset of edges $\mathbf{E} \subseteq E(G)$. By splitting G at the edges of $\mathbf{E}$, we obtain connected graphs $G_1, \ldots, G_r$. Here, each edge $E \in \mathbf{E}$ with endpoints v_1, v_2 is replaced by two legs with endpoints v_1, v_2 respectively. By restricting $\boldsymbol{\sigma}, \mathbf{u}$ and $\mathbf{A}$ to G_i, we obtain types $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r$. Further, there is a canonical splitting map

$$\mathscr{M}(X/B, \boldsymbol{\tau}) \to \prod_{i=1}^{r} \mathscr{M}(X/B, \boldsymbol{\tau}_i).$$

This splitting map is defined by normalizing the domain curves at the nodes corresponding to edges of $\mathbf{E}$. The key question is then: can we relate $[\mathscr{M}(X/B, \boldsymbol{\tau})]^{\mathrm{virt}}$ in terms of $\prod_{i=1}^{r} [\mathscr{M}(X/B, \boldsymbol{\tau}_i)]^{\mathrm{virt}}$?

There have been a number of approaches to this question. First, Kim, Lho and Ruddat [KLR] proved the Li–Ruan and Jun Li degeneration formula in the context of logarithmic Gromov-Witten theory. Yixian Wu, in [Wu], gave a very general gluing formula, under the hypothesis that all gluing strata are toric. This has already proven to be very useful in [GS8], but doesn't give a proof of the Li–Ruan/Jun Li formula (unless the divisor D is in fact a point).

A very different approach has been pursued by Dhruv Ranganathan, using expanded degenerations, in [Ra]. There, the theoretical difficulties of gluing are removed by allowing target expansions in such a way that the gluing always happens along codimension one strata. However, this can result in an explosion in combinatorial complexity of the problem. It is not clear whether such an explosion can be avoided in any general approach to the gluing problem.

The basic problem is that the description of the glued moduli space $\mathscr{M}(X/B, \boldsymbol{\tau})$ in terms of the moduli spaces $\mathscr{M}(X/B, \boldsymbol{\tau}_i)$ involves a fibre product in the category of fs log schemes; this is roughly encapsulated in one of the main gluing theorems of [ACGS2], quoted here in Theorem 2.5. One of the basic difficulties in log geometry is that the underlying scheme of an fs fibre product can be quite far from the underlying scheme of the ordinary fibre product.

Here, we consider a case of gluing in which, at the virtual level, the ordinary fibre product and fs fibre product are not wildly divergent. Understanding fs fibre products even of log points is non-trivial, and in §3, we give some general results about such fibre products helpful for our situation. There, we give criteria for non-emptiness of an fs fibre product of log points, as well as a computation for the number of connected components of this fs fibre product.

Happily, these criteria for non-emptiness have a simple tropical interpretation. In §4, we consider gluing a single curve. In other words, we consider a type $\boldsymbol{\tau}$ as described above, a set of splitting edges $\mathbf{E}$, and the types $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r$ obtained from splitting at the edges of $\mathbf{E}$. We consider log points W_i and punctured log maps $f_i : C_i^\circ/W_i \to X$ of type $\boldsymbol{\tau}_i$, $1 \le i \le r$. If $E \in \mathbf{E}$ is an edge with vertices v_1, v_2, $v_j \in V(G_{i_j})$, $j = 1, 2$, let $p_{v_j,E} \in C_{i_j}^\circ$ be the punctured point corresponding to the leg of G_{i_j} indexed by the flag $v_j \in E$. Assume that we have for each E the equality $f_{i_1}(p_{v_1,E}) = f_{i_2}(p_{v_2,E})$. Then the maps f_i may be glued schematically. The question is then: how many logarithmic gluings of the f_i's are there?

The existence of a logarithmic gluing is a difficult question, but the number of such gluings, assuming there is at least one, is an easy question given the four-point lemmas of §3. In particular, in Definition 4.2, we define a map of lattices, the *tropical gluing map* Ψ, which depends only on the tropical data of $\boldsymbol{\tau}, \mathbf{E}$, and define the *tropical multiplicity* $\mu(\boldsymbol{\tau}, \mathbf{E})$ as the order of the torsion part of $\operatorname{coker}\Psi$. This lattice map gives the obstruction to gluing tropical maps of types $\tau_1, \ldots, \tau_q$ to obtain a tropical map of type τ. In addition, let $f : C^\circ/W \to X$ be the universal gluing of the punctured maps f_i. Then the first result, Theorem 4.4, is:

Theorem 1.1. *If W is non-empty, then it has $\mu(\boldsymbol{\tau}, \mathbf{E})$ connected components.*

We remark that this is not complete information about W, as it may have some non-reduced structure. However, when gluing questions are set up properly, this becomes unimportant, as is seen in [Wu] or §5.

We say the gluing situation is *tropically transverse* if $\operatorname{coker}\Psi$ is in fact finite. In this case, we have Theorem 4.8, again from the four-point lemmas of §3:

Theorem 1.2. *If the gluing situation is tropically transverse, then W is non-empty.*

These results can be viewed as a generalization of the more hands-on constructions of log stable maps beginning with work of Nishinou–Siebert [NS], Argüz [Ar], Cheung–Fantini–Park–Ulirsch [CFPU], and [ACGS1, §4.2]. In fact, versions of tropical gluing maps already appeared in [NS].

In §5, we now apply these observations to gluing moduli spaces, with an aim to describe $[\mathscr{M}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}$ in terms of $\prod_{i=1}^r[\mathscr{M}(X/B,\boldsymbol{\tau}_i)]^{\mathrm{virt}}$. We define an intermediate moduli space $\mathscr{M}^{\mathrm{sch}}(X,\boldsymbol{\tau})$. A point in this moduli space is represented by a point in $\prod_{i=1}^r \mathscr{M}(X/B,\boldsymbol{\tau}_i)$ corresponding to a collection of punctured maps $f_i : C_i^\circ/W_i \to X$, such that the f_i glue schematically. This moduli stack can be defined via a Cartesian diagram in the category of ordinary stacks, see Theorem 5.1 for details. The results of §4 then apply to give us some information about the natural map $\phi' : \mathscr{M}(X/B,\boldsymbol{\tau}) \to \mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})$. Unfortunately, in general it is difficult to extract useful results from this. However, tropical transversality of the gluing situation, along with a flatness hypothesis which can also be tested tropically, implies virtual surjectivity of this map, with degree given by the tropical multiplicity, so that

$$\phi'_*[\mathscr{M}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}} = \mu(\boldsymbol{\tau},\mathbf{E})[\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}.$$

On the other hand, if the gluing strata are sufficiently nice (e.g., smooth) and certain other conditions hold, then $[\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}$ can be calculated as an ordinary Gysin pull-back of $\prod_{i=1}^r[\mathscr{M}(X/B,\boldsymbol{\tau}_i)]^{\mathrm{virt}}$. See Remark 5.2 for details.

In §5.2, we specialize to the degeneration situation considered in [ACGS1]. Here, B is a curve or spectrum of a DVR over $\Bbbk$, with divisorial log structure coming from a closed point $b_0 \in B$. Let X_0 be the fibre over b_0. In [ACGS1], we showed that virtual irreducible components of $\mathscr{M}(X_0/b_0)$ were indexed by rigid tropical curves. We recast the earlier discussion in the gluing situation provided by a rigid tropical curve.

While these results are not yet the dreamed-of general gluing formula, they in fact appear to be strong enough to be useful in many circumstances. In particular, in §7, we give a very short proof of the Li-Ruan/Jun Li degeneration formula. This is not new even in the logarithmic setup: [KLR] first obtained this result. However, it is pleasant to see that the more general setup proves this special case without pain. The reason this works easily is that the gluing situation is always tropically transverse in this case.

Along the way, in §6, we first prove a more generally useful comparison result between punctured and log invariants for irreducible components of degenerations. Explicitly, given $X \to B$ as in the degeneration situation, often one needs to look at moduli spaces of punctured maps into strata of X_0. In general, this may involve additional information, but if the stratum is an irreducible component $Y \subseteq X_0$, life becomes simpler. In particular, Y carries two possible log structures, one induced from X, and one the divisorial log structure coming from the union of substrata of Y. We write this latter log structure as $\overline{Y}$. We then obtain in Theorem 6.1 an isomorphism of underlying stacks $\underline{\mathscr{M}(Y/b_0,\boldsymbol{\tau})} \cong \underline{\mathfrak{M}(\overline{Y},\bar{\boldsymbol{\tau}})}$, where the type $\bar{\boldsymbol{\tau}}$ is derived from the type $\boldsymbol{\tau}$. Happily, the latter type does not involve punctures, and hence gives a more familiar moduli space. We note these stacks are not isomorphic as log stacks, but they do carry the same virtual fundamental class.

The final section is an extended application of the gluing techniques given in this paper. We study the genus zero punctured Gromov-Witten theory of maximally unipotent degenerations of K3 surfaces, giving an inductive description of the so-called canonical wall structure of [GS8]. This, along with a number of other results proved using our gluing technology, will be of use in [?], which explores mirror symmetry for K3 surfaces and uses the mirror construction of [GS8] to build geoetrically meaningful compactifications of the moduli space of K3 surfaces.

One of the key reasons our approach to gluing is applicable in this case is that the degenerate varieties being considered only have at worst triple points. It was originally observed by Brett Parker in [Pa] that this was a particularly amenable situation for gluing.

Conventions: All logarithmic schemes and stacks are defined over an algebraically closed field $\Bbbk$ of characteristic 0. We follow the convention that if X is a log scheme or stack, then $\underline{X}$ is the underlying scheme or stack. We almost always write $\mathcal{M}_X$ for the sheaf of monoids on X and $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ for the structure map. If P is a monoid, we write $P^\vee := \mathrm{Hom}(P, \mathbb{N})$ and $P^* = \mathrm{Hom}(P, \mathbb{Z})$.

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2. Preliminaries

2.1. **Tropical maps and moduli of punctured curves.** We will work with a relative target space $X \to B$, a proper log smooth morphism. We further assume that the log structure on X is Zariski, and that X satisfies assumptions required to guarantee finite type moduli spaces of punctured curves. At the moment, [ACGS2] requires that $\overline{\mathcal{M}}_X^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be generated by global sections, so what follows will be written with this assumption. However, see [J22] for finiteness results without this condition. Typically B itself is taken to be an affine scheme and B is either log smooth over $\mathrm{Spec}\,\Bbbk$ or a log point $\mathrm{Spec}(Q_B \to B)$.

We briefly review notation from [ACGS1, ACGS2] for tropical maps to $\Sigma(X)$ and punctured log maps to X as developed in [ACGS1, §2.5] and [ACGS2, §2.2].

In what follows, **Cones** denotes the category of rational polyhedral cones with integral structure, i.e., objects are rational polyhedral cones $\omega \subseteq N_\omega \otimes_{\mathbb{Z}} \mathbb{R}$ for N_ω the lattice of integral tangent vectors to ω. Morphisms are maps of cones induced by maps of the corresponding lattices. We write $\omega_{\mathbb{Z}} = \omega \cap N_\omega$ for the set of integral points of ω.

A *generalized cone complex* is a topological space with a presentation as the colimit of an arbitrary diagram in the category **Cones** with all morphisms being face morphisms.

If Σ is such a generalized cone complex, we write $\sigma \in \Sigma$ if σ is a cone in the presentation and $|\Sigma|$ for the underlying topological space. A morphism of generalized cone complexes is a continuous map $f : |\Sigma| \to |\Sigma'|$ such that for each $\sigma \in \Sigma$, the induced map $\sigma \to |\Sigma'|$ factors through a cone map $\sigma \to \sigma' \in \Sigma'$.

There is a functor [U] from fine saturated log schemes to generalized cone complexes, written as $X \mapsto \Sigma(X)$. There is a one-to-one correspondence between elements in the presentation $\Sigma(X)$ and logarithmic strata of X. If $\mathcal{M}_X$ denotes the log structure on X with ghost sheaf $\overline{\mathcal{M}}_X$, and $\bar\eta$ is a geometric generic point of a log stratum, then the corresponding cone is $\mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar\eta}, \mathbb{R}_{\geq 0})$. If $\sigma \in \Sigma(X)$, we write $X_\sigma \subseteq X$ for the corresponding (closed) stratum.

We consider graphs G, with sets of vertices $V(G)$, edges $E(G)$ and legs $L(G)$. In what follows, we will frequently confuse G with its topological realisation $|G|$. Legs will correspond to marked or punctured points of punctured curves, and are rays in the marked case and compact line segments in the punctured case. We view a compact leg as having only one vertex. An abstract tropical curve over $\omega \in \mathbf{Cones}$ is data $(G, \mathbf{g}, \ell)$ where $\mathbf{g} : V(G) \to \mathbb{N}$ is a genus function and $\ell : E(G) \to \mathrm{Hom}(\omega_{\mathbb{Z}}, \mathbb{N}) \setminus \{0\}$ determines edge lengths.

Associated to the data (G, ℓ) is a generalized cone complex $\Gamma(G, \ell)$ along with a morphism of cone complexes $\Gamma(G, \ell) \to \omega$ with fibre over $s \in \mathrm{Int}(\omega)$ being a tropical curve, i.e., a metric graph, with underlying graph G and affine edge length of $E \in E(G)$ being $\ell(E)(s) \in \mathbb{R}_{\geq 0}$. Associated to each vertex $v \in V(G)$ of G is a copy ω_v of ω in $\Gamma(G, \ell)$. Associated to each edge or leg $E \in E(G) \cup L(G)$ is a cone $\omega_E \in \Gamma(G, \ell)$ with $\omega_E \subseteq \omega \times \mathbb{R}_{\geq 0}$ and the map to ω given by projection onto the first coordinate. This projection fibres ω_E in compact intervals or rays over ω (rays only in the case of a leg representing a marked point).

A *family of tropical maps* to $\Sigma(X)$ over $\omega \in \mathbf{Cones}$ is a morphism of cone complexes

$$h : \Gamma(G, \ell) \to \Sigma(X).$$

If $s \in \mathrm{Int}(\omega)$, we may view G as the fibre of $\Gamma(G, \ell) \to \omega$ over s as a metric graph, and write

$$h_s : G \to \Sigma(X)$$

for the corresponding tropical map with domain G. The *type* of such a family consists of the data $\tau := (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ where

$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$$

associates to $x \in V(G) \cup E(G) \cup L(G)$ the minimal cone of $\Sigma(X)$ containing $h(\omega_x)$. Further, $\mathbf{u}$ associates to each (oriented) edge or leg $E \in E(G) \cup L(G)$ the corresponding *contact order* $\mathbf{u}(E) \in N_{\boldsymbol{\sigma}(E)}$, the image of the tangent vector $(0, 1) \in N_{\omega_E} = N_\omega \oplus \mathbb{Z}$ under the map h.

As we shall only consider tropicalizations of pre-stable punctured curves (see [ACGS2, Def. 2.5], following [ACGS2, Prop. 2.21] we may assume that for $L \in L(G)$ with adjacent vertex $v \in V(G)$ giving $\omega_L, \omega_v \subseteq \Gamma(G, \ell)$, we have

$$h(\omega_L) = (h(\omega_v) + \mathbb{R}_{\geq 0}\mathbf{u}(L)) \cap \boldsymbol{\sigma}(L) \subseteq N_{\boldsymbol{\sigma}(L),\mathbb{R}}. \tag{2.1}$$

In other words, the images of legs extend as far as possible inside their cones.

A *decorated type* is data $\boldsymbol{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ where $\mathbf{A} : V(G) \to H_2(X)$ associates a curve class to each vertex of G. The *total curve class* of $\mathbf{A}$ is $A = \sum_{v \in V(G)} \mathbf{A}(v)$.[1]

We also have a notion of a contraction morphism of types $\phi : \tau \to \tau'$, see [ACGS1, Def. 2.24]. This is a contraction of edges on the underlying graphs, and the additional data satisfies some relations as follows. If $x \in V(G) \cup E(G) \cup L(G)$, then $\boldsymbol{\sigma}'(\phi(x)) \subseteq \boldsymbol{\sigma}(x)$ (if x is an edge, it may be contracted to a vertex by ϕ). Further, if $E \in E(G) \cup L(G)$ then $\mathbf{u}(E) = \mathbf{u}'(\phi(E))$ under the inclusion $N_{\boldsymbol{\sigma}'(\phi(E))} \subseteq N_{\boldsymbol{\sigma}(E)}$, provided that E is not an edge contracted by ϕ.

We say a type τ is *realizable* if there exists a family of tropical maps to $\Sigma(X)$ of type τ. We also say $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is realizable if τ is realizable. In this paper, we will only deal with realizable types. As a consequence, we will not need the more general notion of global type discussed in [ACGS2, §3]. However, in the case that there are no punctures, but only marked points, we will use the notion of a *class of logarithmic map* β, consisting of data of a genus g, a curve class A, and contact orders $u_i \in \mathrm{Int}(\sigma_i)$ with $\sigma_i \in \Sigma(X)$ for $1 \le i \le k$ for k marked points. This may be viewed as a decorated type $(G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$ where the underlying graph G has only one vertex v and no edges, $\mathbf{g}(v) = g$, $\boldsymbol{\sigma}(v) = \{0\}$, $\boldsymbol{\sigma}(L_i) = \sigma_i$, $\mathbf{u}(L_i) = u_i$, and $\mathbf{A}(v) = A$.

If a type τ is realizable, then there is a universal family of tropical maps of type τ, parameterized by an object of **Cones**. Hopefully without confusion, we will generally write this cone as τ. Hence we have a cone complex $\Gamma(G, \ell)$ equipped with a map to τ and a map of cone complexes $h = h_\tau : \Gamma(G, \ell) \to \Sigma(X)$. Generally we write h rather than h_τ when unambiguous. Note that for each $x \in E(G) \cup L(G) \cup V(G)$, we thus obtain $\tau_x \in \Gamma(G, \ell)$ the corresponding cone.

We write $\mathcal{A}_X$ for the Artin fan of X, see [ACMW], as well as [ACGS1, §2.2] for a summary. With $X \to B$ log smooth with X Zariski, we obtain a morphism of Artin fans $\mathcal{A}_X \to \mathcal{A}_B$ and define

$$\mathcal{X} := \mathcal{A}_X \times_{\mathcal{A}_B} B.$$

We refer to [ACGS2, Defs. 2.10, 2.13, 2.14] for the notion of a family $\pi : C^\circ \to W$ of punctured curves and pre-stable or stable punctured log maps $f : C^\circ/W \to X$ or $f : C^\circ/W \to \mathcal{X}$ defined over B.

Given a punctured log map with domain $C^\circ \to W$ and $W = \mathrm{Spec}(Q \to \kappa)$ for κ an algebraically closed field and target X or $\mathcal{X}$, we obtain by functoriality of tropicalizations a family of tropical maps

$$\begin{tikzcd} \Sigma(C) = \Gamma(G,\ell) \arrow[r] \arrow[d] & \Sigma(X) \arrow[d] \\ \Sigma(W) = \omega = Q^\vee_{\mathbb{R}} \arrow[r] & \Sigma(B) \end{tikzcd} \tag{2.2}$$

parameterized by W. The *type* of the punctured map is then the type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ of this family of tropical maps. We recall that the punctured map $f : C^\circ/W \to X$ is *basic* if (2.2) is the universal family of tropical maps of type τ.

[1] We recall that $H_2(X)$ represents some choice of group of curve classes. It could be integral homology of X, or the group of curve classes modulo algebraic or numerical equivalence, but other choices are also possible. See e.g., [ACGS1, §2.3.8] for a discussion.

Given a type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$, [ACGS2, Def. 3.7] defines the notion of a *marking* or *weak marking* of a punctured map by τ.[2] Roughly, a *weak marking* of a punctured map $f : C^\circ/W \to X$ involves the following information. (1) A marking of the underlying domain curve $\underline{C}$ by G. In other words, we have a pre-stable curve $\underline{C}_v$ for each $v \in V(G)$ of genus $\mathbf{g}(v)$, a marked point $p_L \in \underline{C}_v$ for each leg $L \in L(G)$ adjacent to v, and marked points $q_{E,1}, q_{E,2}$ in $\underline{C}_{v_1}, \underline{C}_{v_2}$ for each edge E connecting v_1 to v_2. Further, $\underline{C}$ is obtained as a marked curve by identifying pairs $q_{E,1}, q_{E,2}$ for all $E \in E(G)$. (2) For each subcurve or punctured or nodal section Z of $\underline{C}$, indexed by an element $x \in V(G) \cup L(G) \cup E(G)$, the morphism $\underline{f}|_Z$ factors through the closed stratum $X_{\boldsymbol{\sigma}(x)}$ of X. (3) For any geometric point $\bar{w} \to W$ giving a curve of type $\tau_{\bar{w}} = (G_{\bar{w}}, \mathbf{g}_{\bar{w}}, \boldsymbol{\sigma}_{\bar{w}}, \mathbf{u}_{\bar{w}})$, the contraction morphism $G_{\bar{w}} \to G$ induced by the marking of the domain yields a contraction morphism of types $\tau_{\bar{w}} \to \tau$.

A *marking* of a punctured map $f : C^\circ/W \to X$ is a weak marking satisfying an additional requirement that a certain natural monoid ideal on $\mathcal{M}_W$ defines an idealized log structure on W. See [ACGS2, Def. 3.7] for full details.

In either case, if further $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is a decoration of τ, then $f : C^\circ/W \to X$ is (weakly) $\boldsymbol{\tau}$-marked if in addition to being (weakly) τ-marked, for each $v \in V(G)$, the curve class associated the the stable map $\underline{f}$ restricted to the subcurve indexed by v is $\mathbf{A}(v)$.

In particular, this gives rise to the following moduli spaces:

(1) $\mathscr{M}(X, \boldsymbol{\tau})$ (resp. $\mathscr{M}'(X, \boldsymbol{\tau})$) the moduli space of (weakly) $\boldsymbol{\tau}$-marked stable punctured maps.
(2) $\mathfrak{M}(\mathcal{X}, \tau)$ (resp. $\mathfrak{M}'(\mathcal{X}, \tau)$) the moduli space of (weakly) $\boldsymbol{\tau}$-marked punctured maps to $\mathcal{X}$.
(3) $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ (resp. $\mathfrak{M}'(\mathcal{X}, \boldsymbol{\tau})$) the moduli space of (weakly) $\boldsymbol{\tau}$-marked punctured maps to $\mathcal{X}$. Note here that while curve classes in $\mathcal{X}$ are meaningless, the decoration $\mathbf{A}$ on τ affects the notion of isomorphism in the categories $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ or $\mathfrak{M}'(\mathcal{X}, \boldsymbol{\tau})$, and there is an étale morphism $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}) \to \mathfrak{M}(\mathcal{X}, \tau)$.

In general, we are always working over B, but when we need to be more precise, we write $\mathscr{M}(X/B, \boldsymbol{\tau})$. We say type τ is *realisable over* B if $\mathfrak{M}(\mathcal{X}/B, \tau)$ has a geometric point corresponding to a map of type τ, see [ACGS2, Def. 3.26]. By [ACGS2, Prop. 3.27], this is equivalent to a combinatorial condition requiring τ to be realisble by a family of tropical maps defined over $\Sigma(B)$ with an additional condition which is easily checked.

By [ACGS2, Thm. 3.10], all of the above moduli spaces are algebraic stacks and $\mathscr{M}(X, \boldsymbol{\tau})$ and $\mathscr{M}'(X, \boldsymbol{\tau})$ are Deligne-Mumford. Further, there are natural morphisms

$$\begin{aligned} \varepsilon &: \mathscr{M}(X, \boldsymbol{\tau}) \to \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}) \\ \varepsilon &: \mathscr{M}'(X, \boldsymbol{\tau}) \to \mathfrak{M}'(\mathcal{X}, \boldsymbol{\tau}) \end{aligned} \tag{2.3}$$

given by composing a punctured log map $C^\circ \to X$ with the canonical map $X \to \mathcal{X}$. [ACGS2, §4] then gives a perfect relative obstruction theory for ε.

[2]The cited reference applies to global types, but by [ACGS2, Lem. 3.5], giving a realizable global type is the same as giving a realizable type. Since all our types are realizable here, we ignore the notion of global type.

In general, it appears that the $\boldsymbol{\tau}$-marked moduli spaces are more important than the weakly $\boldsymbol{\tau}$-marked moduli spaces. In particular, [ACGS2, Prop. 3.31] shows that $\mathfrak{M}(\mathcal{X},\tau)$ is a closed substack of $\mathfrak{M}'(\mathcal{X},\tau)$ defined by a nilpotent ideal. Further, in the cases of greatest interest for this paper ($\boldsymbol{\tau}$ realisable, B log smooth over $\operatorname{Spec}\Bbbk$ or $B = \operatorname{Spec}\Bbbk^\dagger$, the standard log point), $\mathfrak{M}(\mathcal{X},\tau)$ is in fact reduced and pure-dimensional, see [ACGS2, Prop. 3.28]. In fact, while $\mathfrak{M}(\mathcal{X},\tau)$ may be quite poorly behaved globally, it has a simple local structure coming from the fact that it is idealized log smooth over B, see [ACGS2, Thm. 3.24, Rem. 3.25]. The main point for including the weakly marked moduli spaces is that they naturally occur in the gluing formalism.

If there is a contraction morphism between decorated global types $\phi:\boldsymbol{\tau}\to\boldsymbol{\tau}'$, we obtain a forgetful map $\mathscr{M}(X,\boldsymbol{\tau})\to\mathscr{M}(X,\boldsymbol{\tau}')$. This gives rise to a stratified description of these moduli spaces, see [ACGS2, Rem. 3.29].

If $\boldsymbol{\tau}=(G,\boldsymbol{\sigma},\mathbf{u},\mathbf{A})$ denotes a choice of decorated type, and $I\subseteq E(G)\cup L(G)$ is a collection of edges and legs, then we write

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau})=\mathfrak{M}^{\mathrm{ev}(I)}(\mathcal{X},\boldsymbol{\tau}):=\mathfrak{M}(\mathcal{X},\boldsymbol{\tau})\times_{\underline{\mathcal{X}}^I}\underline{X}^I.$$

Here $\underline{\mathcal{X}}^I$ denotes the product of $|I|$ copies of $\underline{\mathcal{X}}$ over $\underline{S}$, and similarly $\underline{X}^I$; the morphism $\mathfrak{M}(\mathcal{X},\boldsymbol{\tau})\to\underline{\mathcal{X}}^I$ is given by evaluation at the nodes and punctured points indexed by elements of I, and $\underline{X}^I\to\underline{\mathcal{X}}^I$ is induced by the canonical smooth map $\underline{X}\to\underline{\mathcal{X}}$. The map ε then factors as

$$\mathscr{M}(X,\boldsymbol{\tau})\xrightarrow{\varepsilon^{\mathrm{ev}}}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau})\longrightarrow\mathfrak{M}(\mathcal{X},\boldsymbol{\tau}).\tag{2.4}$$

The second morphism is smooth, while $\varepsilon^{\mathrm{ev}}$ also possesses a relative obstruction theory compatible with the morphism ε of (2.3), see [ACGS2, §4.2].

We end by reviewing a basic result which encodes a generalisation of the tropical balancing condition.

Proposition 2.1. *Suppose given a punctured map $f: C^\circ/W\to X$ with W a log point. Let $\tau=(G,\boldsymbol{\sigma},\mathbf{u})$ be the corresponding tropical type. If $v\in V(G)$, let $\underline{C}_v\subseteq\underline{C}$ be the corresponding irreducible component, and let $E_1,\ldots,E_n$ be the edges and legs adjacent to v, oriented away from v. Let $s\in\Gamma(X,\overline{\mathcal{M}}_X^{\mathrm{gp}})$, and let $\mathcal{L}_s$ be the corresponding line bundle, i.e., the line bundle associated to the torsor given by the inverse image of s under the quotient map $\mathcal{M}_X\to\overline{\mathcal{M}}_X$. For an integral tangent vector v of $\sigma=\operatorname{Hom}(\overline{\mathcal{M}}_{X,x},\mathbb{R}_{\geq 0})\in\Sigma(X)$, we write $\langle v,s\rangle$ for the evaluation of v, as an element of $\operatorname{Hom}(\overline{\mathcal{M}}_{X,x},\mathbb{Z})$, on the germ of s at x. Then*

$$\deg(\underline{f}^*\mathcal{L}_s)|_{\underline{C}_v}=-\sum_{i=1}^n\langle\mathbf{u}(E_i),s\rangle.$$

Proof. This is [ACGS2, Prop. 2.27], applied after splitting the punctured map at all nodes contained in $\underline{C}_v$ via [ACGS2, Prop. 5.2] and restricting the punctured map to the component indexed by v. ♠

In the case that $\underline{X}$ is non-singular and the log structure on X comes from a simple normal crossings divisor $D=D_1+\cdots+D_s$ such that all intersections of irreducible components of D are connected, then this has a particularly nice interpretation as saying that the contact orders determine intersection numbers of the curve class with the divisors D_i. More precisely, in this case we may view $\Sigma(X)$ as in [GS7, Ex. 1.4] as follows. Let $\mathrm{Div}_D(X) = \bigoplus_i \mathbb{Z}D_i$ be the group of divisors supported on D, and write $\mathrm{Div}_D(X)^* = \mathrm{Hom}(\mathrm{Div}_D(X),\mathbb{Z}) = \bigoplus_i \mathbb{Z}D_i^*$ for the dual lattice and $\mathrm{Div}_D(X)^*_{\mathbb{R}} = \mathrm{Div}_D(X)^* \otimes_{\mathbb{Z}} \mathbb{R}$. Then

$$\Sigma(X) = \left\{ \sum_{i\in I} \mathbb{R}_{\geq 0} D_i^* \,\middle|\, I \subseteq \{1,\ldots,s\} \text{ an index set with } \textstyle\bigcap_{i\in I} D_i \neq \emptyset \right\}.$$

Thus a contact order u may be written as $\sum_i a_i D_i^*$, with $a_i \in \mathbb{Z}$, with a_i denoting the order of tangency with D_i. We then have the immediate corollary

Corollary 2.2. *Let X, D as above, and $f, \tau, E_1 \ldots, E_n$ as in Proposition 2.1, write $\mathbf{u}(E_i) = \sum_j a_{ij} D_j^*$. Then*

$$\deg(\underline{f}^*\mathcal{O}_X(D_j))|_{\underline{C}_v} = \sum_{i=1}^n a_{ij}.$$

2.2. **The gluing formalism.** We may now describe the key gluing formalism of [ACGS2]. We begin with the *standard gluing situation.*

Notation 2.3 (The standard gluing situation)**.** We fix a target $X \to B$, a proper log smooth morphism. We further assume that the log structure on X is Zariski, and that X satisfies assumptions required to guarantee finite type moduli spaces of punctured curves. Further we assume that B is either log smooth over $\mathrm{Spec}\,\Bbbk$ or $B = \mathrm{Spec}\,\Bbbk^\dagger$, the standard log point. Fix a realisable type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ of tropical map to $\Sigma(X)/\Sigma(B)$. We select a set of splitting edges $\mathbf{E} \subseteq E(G)$, and let $G_1, \ldots, G_r$ be the connected components of the graph obtained by splitting G at the edges of $\mathbf{E}$, i.e., replacing each edge $E \in \mathbf{E}$ with endpoints v_1, v_2 with two legs with endpoints v_1, v_2 respectively. We write these two legs as flags (E, v_1) and (E, v_2). We then let $\tau_1, \ldots, \tau_r$ be the induced set of decorated types with underlying graphs $G_1, \ldots, G_r$. Let $\mathbf{L} \subseteq \bigcup_{i=1}^r L(G_i)$ be the subset of all legs obtained from splitting edges, and $\mathbf{L}_i = \mathbf{L} \cap L(G_i)$. For $v \in V(G)$, let $i(v) \in \{1, \ldots, r\}$ denote the connected component G_i containing v.

For each $E \in \mathbf{E}$ denote by $\mathfrak{M}'_E(\mathcal{X}, \tau)$ the image of the nodal section $s_E : \underline{\mathfrak{M}'}(\mathcal{X}, \tau) \to \underline{\mathfrak{C}'^\circ}(\mathcal{X}, \tau)$ with the restriction of the log structure on the universal domain $\mathfrak{C}'^\circ(\mathcal{X}, \tau)$. Denote further by $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau)$ the fs fiber product

$$\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) = \mathfrak{M}'_{E_1}(\mathcal{X}, \tau) \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X},\tau)} \cdots \times^{\mathrm{fs}}_{\mathfrak{M}'(\mathcal{X},\tau)} \mathfrak{M}'_{E_r}(\mathcal{X}, \tau), \tag{2.5}$$

where $E_1, \ldots, E_r \in E(G)$ are the edges in $\mathbf{E}$. With this enlarged log structure, the pull-back $\widetilde{\mathfrak{C}}'^\circ(\mathcal{X}, \tau) \to \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau)$ of the universal domain has sections $\tilde{s}_E$, $E \in \mathbf{E}$, in the category of log stacks. Moreover, $\underline{\mathrm{ev}_{\mathbf{E}}}$ lifts to a logarithmic evaluation morphism

$$\mathrm{ev}_{\mathbf{E}} : \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) \longrightarrow \textstyle\prod_{E\in\mathbf{E}} \mathcal{X}, \tag{2.6}$$

with E-component equal to $\tilde{f} \circ \tilde{s}_E$ for $\tilde{f} : \widetilde{\mathfrak{C}}'^\circ(\mathcal{X}, \tau) \to \mathcal{X}$ the universal punctured morphism.

Similarly, for each of the types $\tau_i = (G_i, \mathbf{g}_i, \boldsymbol{\sigma}_i, \mathbf{u}_i)$ obtained by splitting and $L \in L(G_i)$, denote by $\mathfrak{M}'_L(\mathcal{X}, \tau_i)$ the image of the punctured section $s_L : \underline{\mathfrak{M}'}(\mathcal{X}, \tau_i) \to$

$\underline{\mathfrak{C}}'^{\circ}(\mathcal{X}, \tau_0)$ defined by L, again endowed with the pull-back of the log structure on $\mathfrak{C}'^{\circ}(\mathcal{X}, \tau_0)$. With $L_1, \ldots, L_s$ the legs of $\mathbf{L}_i$, define the stack

$$\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) = \left(\mathfrak{M}'_{L_1}(\mathcal{X}, \tau_i) \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X},\tau_i)} \cdots \times^{\mathrm{f}}_{\mathfrak{M}'(\mathcal{X},\tau_i)} \mathfrak{M}'_{L_s}(\mathcal{X}, \tau_i)\right)^{\mathrm{sat}},$$

where sat denotes saturation. This stack differs from $\mathfrak{M}'(\mathcal{X}, \tau_i)$ by adding the pull-back of the log structure of each puncture, so that the pull-back $\widetilde{\mathfrak{C}}'^{\circ}(\mathcal{X}, \tau_i) \to \widetilde{\mathfrak{M}}'\mathcal{X}, \tau_i)$ of the universal curve now has punctured sections in the category of log stacks. We define the evaluation morphism

$$\mathrm{ev}_{\mathbf{L}} : \textstyle\prod_{i=1}^r \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) \longrightarrow \prod_{E \in \mathbf{E}} \mathcal{X} \times \mathcal{X}, \tag{2.7}$$

by taking as E-component the evaluation at the corresponding two sections s_{E,v_1}, s_{E,v_2}, where v_1, v_2 are the endpoints of E. Note that this involves a choice of ordering of the endpoints of E, i.e., a choice of orientation of E.

It is worth noting the following (see [ACGS2, Prop. 5.5]).

Proposition 2.4. *The canonical map $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau) \to \mathfrak{M}'(\mathcal{X}, \tau)$ induces an isomorphism of underlying stacks, while the canonical maps $\widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i) \to \mathfrak{M}'(\mathcal{X}, \tau_i)$ induces an isomorphism on reductions.*

There are evaluation space versions of this. We set

$$\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}, \tau_i) = \mathfrak{M}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) \times_{\mathfrak{M}'(\mathcal{X},\tau_i)} \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i)$$

and

$$\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X}, \tau) = \mathfrak{M}'^{\mathrm{ev}(\mathbf{E})}(\mathcal{X}, \tau_i) \times_{\mathfrak{M}'(\mathcal{X},\tau_i)} \widetilde{\mathfrak{M}}'(\mathcal{X}, \tau_i).$$

There is a natural splitting map

$$\delta_{\mathfrak{M}} : \mathfrak{M}'(\mathcal{X}, \tau) \to \prod_{i=1}^r \mathfrak{M}'(\mathcal{X}, \tau_i) \tag{2.8}$$

as defined in [ACGS2, Prop. 5.4], taking a τ-marked map and splitting it at the nodes marked by the edges in $\mathbf{E}$. This lifts to

$$\delta_{\mathfrak{M}} : \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{E})}(\mathcal{X}, \tau) \to \prod_{i=1}^r \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i), \tag{2.9}$$

and is shown to be a finite morphism in [ACGS2, Cor. 5.13]. Passing to the ev-spaces is key here: the morphism in (2.8) is rarely finite or even proper. This also gives an upgrading of the evaluation morphisms:

$$\begin{aligned} \mathrm{ev}_{\mathbf{E}} : & \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{E})}(\mathcal{X}, \tau) \to \prod_{E \in \mathbf{E}} X, \\ \mathrm{ev}_{\mathbf{L}} : & \prod_{i=1}^r \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}, \tau_i) \to \prod_{E \in \mathbf{E}} X \times X. \end{aligned} \tag{2.10}$$

We review the key gluing results of [ACGS2]. We first describe the glued moduli space as an fs fibre product:

Theorem 2.5. *Suppose given a gluing situation as in Notation 2.3. Then the commutative diagram*

$$\begin{tikzcd} \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{L})}(\mathcal{X},\tau) \arrow[r, "\delta_{\mathfrak{M}}"] \arrow[d, "\mathrm{ev}_{\mathbf{E}}"'] & \prod_{i=1}^r \widetilde{\mathfrak{M}}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X},\tau_i) \arrow[d, "\mathrm{ev}_{\mathbf{L}}"] \\ \prod_{E\in\mathbf{E}} X \arrow[r, "\Delta"] & \prod_{E\in\mathbf{E}} X\times X \end{tikzcd}$$

with Δ the product of diagonal embeddings and the other arrows defined in (2.9) *and* (2.10)*, is cartesian in the category of fs log stacks. We remind the reader that all products in this square are taken over B.*

An analogous statement holds for τ replaced by a decorated global type $\boldsymbol{\tau}=(\tau,\mathbf{A})$, or replacing $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X},\tau)$, $\widetilde{\mathfrak{M}}'^{\mathrm{ev}}(\mathcal{X},\tau_i)$ with the analogous moduli spaces of stable maps to X/B, $\widetilde{\mathscr{M}}'(X,\tau)$, $\widetilde{\mathscr{M}}'(X,\tau_i)$.

We remark that in this theorem, the fact that all fibre products are over B can make the actual calculation of fibre products more difficult than they need to be. But it is often enough to work over $\operatorname{Spec}\Bbbk$, as the following proposition (see [ACGS2, Prop. 5.11]) shows:

Proposition 2.6. *Let B be an affine log scheme equipped with a global chart $P\to\mathcal{M}_B$ inducing an isomorphism $P\cong\Gamma(B,\overline{\mathcal{M}}_B)$. Let τ be a type of tropical map for X/B, with underlying graph connected. Then there are isomorphisms $\mathfrak{M}(\mathcal{X},\tau)\cong\mathfrak{M}(\mathcal{X}/\operatorname{Spec}\Bbbk,\tau)$ and $\mathscr{M}(X,\tau)\cong\mathscr{M}(X/\operatorname{Spec}\Bbbk,\tau)$.*[3]

Finally, to make contact with stable punctured maps to X, we have [ACGS2, Prop. 5.15]:

Theorem 2.7. *In the situation of Theorem 2.5 there is a cartesian diagram*

$$\begin{tikzcd} \mathscr{M}(X,\tau) \arrow[r, "\delta"] \arrow[d, "\varepsilon^{\mathrm{ev}}"'] & \prod_{i=1}^r \mathscr{M}(X,\tau_i) \arrow[d, "\hat\varepsilon=\prod_i \varepsilon_i^{\mathrm{ev}}"] \\ \mathfrak{M}^{\mathrm{ev}(\mathbf{E})}(\mathcal{X},\tau) \arrow[r, "\delta^{\mathrm{ev}}"'] & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X},\tau_i) \end{tikzcd} \tag{2.11}$$

with horizontal arrows the canonical splitting maps (see [ACGS2, Prop. 5.4]*, and the vertical arrows the canonical strict morphisms of* (2.4)*.*

Analogous statements hold for decorated and for weakly marked versions of the moduli stacks, see [ACGS2, Def. 3.8].

The evaluation spaces $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)$ etc. play a crucial role here. First, if instead one used the spaces $\mathfrak{M}(\mathcal{X},\tau)$, there would be no way to obtain a Cartesian diagram, as the splitting map on the level of punctured maps to Artin fans has no way to impose a matching condition at the schematic level. Using the evaluation spaces allows δ^{ev} to impose both schematic and logarithmic matching conditions.

Second, the splitting map at the level of the spaces of punctured maps to Artin fans is very poorly behaved, being neither representable nor proper. However, δ^{ev}, as this theorem states, is in fact finite and representable. Hence we may use it to push-forward

[3] We recall our convention that $\mathscr{M}(X,\tau)=\mathscr{M}(X/B,\tau)$.

Chow classes, and in particular, as [ACGS2, Thm. 5.17] points out, the compatibility of obstruction theories then allows a calculation

$$\delta_*([\mathscr{M}(X,\tau)]^{\mathrm{virt}}) = \hat{\varepsilon}^! \delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)]$$

of the virtual fundamental class of $\mathscr{M}(X,\tau)$.

Thus the main task is finding a useful expression for $\delta_*^{\mathrm{ev}}[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\boldsymbol{\tau})]$. While the Artin stacks of the type $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)$ may seem very forbidding, in a certain sense they are very well-behaved: they are idealized log smooth over B, see [ACGS2, Thm. 3.24]. This means that there are local descriptions of these stacks as unions of strata of toric varieties. Further, these local descriptions can be determined very explicitly from the tropical description of the types τ and τ_i.

This effective description of these moduli spaces has led Yixian Wu [Wu], in the case that all gluing strata are toric, to give an effective formula for the Chow class $\delta_*[\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau)]$ as a weighted sum of strata of $\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\tau_i)$. This formula has already proved to be very useful, see e.g., [GS8].

Here we wish to develop a gluing formalism in a complementary direction, which, although very far from general, is also useful.

3. Four-point lemmas: fibre products of log points

The key point is to understand the gluing fibre diagram of Theorem 2.5 by studying fibre products of log points. We carry this study out in this section; unfortunately, this is rather dry. An fs fibre product of log points can be quite subtle. Even determining whether such a fibre product is non-empty is difficult, as a "four-point lemma" does not hold widely in log geometry, see [Og, III Prop. 2.2.3] for some results. Here we will generally be interested in the number of connected components of a fibre product for application to specific gluing situations.

We start with a small lemma:

Lemma 3.1. *Let $W := \mathrm{Spec}(Q \to \kappa)$ be a log point with Q a sharp fine monoid. Then W^{sat} is a disjoint union of possibly non-reduced points, and the number of connected components of W^{sat} is $|(Q^{\mathrm{gp}})_{\mathrm{tors}}|$.*

Proof. By construction of the saturation, [Og, III Prop. 2.1.5],

$$W^{\mathrm{sat}} = W \times_{\mathrm{Spec}\,\kappa[Q]} \mathrm{Spec}\,\kappa[Q^{\mathrm{sat}}],$$

where Q^{sat} is the saturation of Q inside Q^{gp}. Note that the morphism $W \to \mathrm{Spec}\,\kappa[Q]$ identifies W with the closed point of $\mathrm{Spec}\,\kappa[Q]$ corresponding to the maximal monomial ideal $\mathfrak{m} = \langle z^q \,|\, q \in Q \setminus \{0\}\rangle$.

The monomial ideal $I \subseteq \kappa[Q^{\mathrm{sat}}]$ generated by the image of $\mathfrak{m}$ then satisfies $\sqrt{I} = \mathfrak{m}' = \langle z^q \,|\, q \in Q^{\mathrm{sat}} \setminus (Q^{\mathrm{sat}})^\times\rangle$. Indeed, for any element $q \in Q^{\mathrm{sat}} \setminus (Q^{\mathrm{sat}})^\times$, there exists a positive integer n such that $nq \in Q$. Further, $nq \neq 0$ since otherwise q is torsion in Q^{sat} and hence invertible. Thus z^{nq} is a generator of $\mathfrak{m}$, so $z^{nq} \in I$. This shows that $\mathfrak{m}' \subseteq \sqrt{I}$. The converse holds as $I \subseteq \mathfrak{m}'$ and $\mathfrak{m}'$ is a radical ideal, as $\kappa[Q^{\mathrm{sat}}]/\mathfrak{m}' = \kappa[(Q^{\mathrm{gp}})_{\mathrm{tors}}]$ is reduced.

Thus we see that $(W^{\mathrm{sat}})_{\mathrm{red}} = \mathrm{Spec}\,\kappa[Q^{\mathrm{sat}}]/\mathfrak{m}' = \mathrm{Spec}\,\kappa[(Q^{\mathrm{gp}})_{\mathrm{tors}}]$. However, the latter consists of $|(Q^{\mathrm{gp}})_{\mathrm{tors}}|$ points. ♠

Lemma 3.2. *Let W_1, W_2, X be fs log schemes with morphisms $W_1, W_2 \to X$. Then there is a canonical isomorphism*

$$(W_{1,\mathrm{red}} \times^{\mathrm{fs}}_{X_{\mathrm{red}}} W_{2,\mathrm{red}})_{\mathrm{red}} \cong (W_1 \times^{\mathrm{fs}}_X W_2)_{\mathrm{red}}.$$

Proof. Considering the strict closed immersion $(W_1)_{\mathrm{red}} \to W_1$, we obtain via base change a strict closed immersion $(W_1)_{\mathrm{red}} \times^{\mathrm{fs}}_X W_2 \to W_1 \times^{\mathrm{fs}}_X W_2$. Repeating with W_2, we obtain a strict closed immersion $(W_1)_{\mathrm{red}} \times^{\mathrm{fs}}_X (W_2)_{\mathrm{red}} \to W_1 \times^{\mathrm{fs}}_X W_2$. Since the morphisms $(W_i)_{\mathrm{red}} \to X$ factor through X_{red}, the former log scheme is isomorphic to $(W_1)_{\mathrm{red}} \times^{\mathrm{fs}}_{X_{\mathrm{red}}} (W_2)_{\mathrm{red}}$.

Thus we obtain a strict closed immersion $(W_{1,\mathrm{red}} \times^{\mathrm{fs}}_{X_{\mathrm{red}}} W_{2,\mathrm{red}})_{\mathrm{red}} \to (W_1 \times^{\mathrm{fs}}_X W_2)_{\mathrm{red}}$, which we must now prove is an isomorphism, which we do by showing this map induces a bijection on geometric points. Indeed, the set of geometric points of $W_1 \times^{\mathrm{fs}}_X W_2$ and $(W_1 \times^{\mathrm{fs}}_X W_2)_{\mathrm{red}}$ are the same, and a strict geometric point point $\mathrm{Spec}(Q \to \kappa) \to W_1 \times^{\mathrm{fs}}_X W_2$ clearly induces a strict closed point $\mathrm{Spec}(Q \to \kappa) \to (W_1)_{\mathrm{red}} \times^{\mathrm{fs}}_{X_{\mathrm{red}}} (W_2)_{\mathrm{red}}$. This shows the claim. ♠

Lemma 3.3. *Let W_1, W_2, X be finite length connected fs log schemes over* $\mathrm{Spec}\,\kappa$ *with κ an algebraically closed field. Write the ghost sheaf monoids as Q_1, Q_2 and P respectively. Suppose given morphisms $f_i : W_i \to X$ inducing $\theta_i = \bar{f}_i^\flat : P \to Q_i$. Set*

$$\theta := (\theta_1^{\mathrm{gp}}, -\theta_2^{\mathrm{gp}}) : P^{\mathrm{gp}} \to Q_1^{\mathrm{gp}} \oplus Q_2^{\mathrm{gp}}.$$

Then

(1) *If the fs fibre product $W_1 \times^{\mathrm{fs}}_X W_2$ is non-empty, it has $|\operatorname{coker}(\theta)_{\mathrm{tors}}|$ connected components.*
(2) $|\operatorname{coker}(\theta)_{\mathrm{tors}}| = |\operatorname{coker}(\theta^t)_{\mathrm{tors}}|$.

Proof. (1) By Lemma 3.2, we may assume W_i and X are (reduced) log points. According to the construction of the fs fibre product in [Og, III, §2.1], we proceed in a couple of steps. Let W, W^{int} and W^{fs} denote the fibre product $W_1 \times_X W_2$ in the category of log schemes, fine log schemes, and fs log schemes. Then W^{int} is the integralization of W and W^{fs} is the saturation of W^{int}.

First, $\underline{W}$ agrees with the fibre product $\underline{W}_1 \times_{\underline{X}} \underline{W}_2 = \mathrm{Spec}\,\kappa$. Integralization involves passing to a closed subscheme of $\mathrm{Spec}\,\kappa$. Thus either $\underline{W}^{\mathrm{int}} = \mathrm{Spec}\,\kappa$ or is the empty scheme. We rule out the latter as we have assumed that the fs fibre product is non-empty. In this case, $W^{\mathrm{int}} = \mathrm{Spec}(Q \to \kappa)$, where

$$Q = Q_1 \oplus^{\mathrm{fine}}_P Q_2$$

and $\oplus^{\mathrm{fine}}$ denotes push-out in the category of fine monoids. This is constructed (see [Og, I, Prop. 1.3.4] and its proof) as the fine submonoid of

$$\operatorname{coker}\theta = Q_1^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q_2^{\mathrm{gp}} = Q^{\mathrm{gp}}$$

generated by the images of Q_1 and Q_2.

As W^{fs} is the saturation of W^{int}, (1) now follows from Lemma 3.1.

(2) is easy homological algebra: $0 \to \ker\theta \to P^{\mathrm{gp}} \to Q_1^{\mathrm{gp}} \oplus Q_2^{\mathrm{gp}} \to \operatorname{coker}(\theta) \to 0$ is a free resolution of $\operatorname{coker}(\theta)$, and $\mathrm{Ext}^1(\operatorname{coker}(\theta), \mathbb{Z})$ is isomorphic to the torsion part of $\operatorname{coker}(\theta)$. However, this Ext group is calculated as the middle cohomology of the

complex $Q_1^* \oplus Q_2^* \to P^* \to \ker(\theta)^*$, and as the kernel of $P^* \to \ker(\theta)^*$ is a saturated sublattice of P^*, the torsion part of $\mathrm{coker}(\theta^t)$ agrees with $\mathrm{Ext}^1(\mathrm{coker}(\theta), \mathbb{Z})$. ♠

The following is a somewhat technically complicated criterion for non-emptiness for $W_1 \times_X^{\mathrm{fs}} W_2$, which is tailored for our gluing needs.

Lemma 3.4. *Let W_i, X be as in Lemma 3.3, and suppose also given log points W_i', X' over $\mathrm{Spec}\,\kappa$ similarly with maps $f_i' : W_i' \to X'$. Write Q_i', P' for the corresponding monoids, $\theta_i' : P' \to Q_i'$ for the induced maps. Suppose further given a commutative diagram*

$$\begin{tikzcd} W_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & X \arrow[d, "g"'] & W_2 \arrow[l, "f_2"'] \arrow[d, "g_2"] \\ W_1' \arrow[r, "f_1'"'] & X' & W_2' \arrow[l, "f_2'"] \end{tikzcd} \tag{3.1}$$

Suppose that:

(1) *$W_1' \times_{X'}^{\mathrm{fs}} W_2'$ is non-empty.*
(2) *The projections $Q_1^\vee \times_{P^\vee} Q_2^\vee \to Q_i^\vee$ have image intersecting the interior of $Q_i^\vee$.*
(3) *The maps $\bar{g}^\flat : P' \to P$ and $\bar{g}_i^\flat : Q_i' \to Q_i$ are all injective, and the map induced by θ,*

$$P^{\mathrm{gp}}/\bar{g}^\flat((P')^{\mathrm{gp}}) \to Q_1^{\mathrm{gp}}/\bar{g}_1^\flat((Q_1')^{\mathrm{gp}}) \oplus Q_2^{\mathrm{gp}}/\bar{g}_2^\flat((Q_2')^{\mathrm{gp}}) \tag{3.2}$$

is injective.

Then $W_1 \times_X^{\mathrm{fs}} W_2$ is non-empty.

Proof. Given the hypotheses, we will construct a morphism $\mathrm{Spec}\,\kappa^\dagger \to W_1 \times_X^{\mathrm{fs}} W_2$ from the standard log point. Recall that giving a morphism of log points $g : \mathrm{Spec}(Q \to \kappa) \to \mathrm{Spec}(P \to \kappa)$ is equivalent to giving $g^\flat : P \times \kappa^\times \to Q \times \kappa^\times$ written as

$$g^\flat(p, t) = (\bar{g}^\flat(p), \chi_g(p) \cdot t)$$

for $\bar{g}^\flat : P \to Q$ a local homomorphism and $\chi_g : P \to \kappa^\times$ an arbitrary homomorphism. Note also that $\theta : P \to Q$ being a local homomorphism can be characterized dually by the statement that $\theta^t(Q^\vee)$ intersects the interior of $P^\vee$.

First, let $Q = (Q_1 \oplus_P^{\mathrm{fs}} Q_2)/\mathrm{tors}$. Note that Q would be the stalk of the ghost sheaf of any point of $W_1 \times_X^{\mathrm{fs}} W_2$ if this log scheme were non-empty. Then $Q^\vee = Q_1^\vee \times_{P^\vee} Q_2^\vee$ ([ACGS1, Proposition 6.3.5]), with a similar expression for $(Q')^\vee$.

Choose an element $q \in \mathrm{Int}(Q^\vee)$. Then necessarily the image q_i of q in $Q_i^\vee$ lies in $\mathrm{Int}(Q_i^\vee)$ by condition (2). Further, q_1 and q_2 have the same image in $P^\vee$. Let $q_i' = (\bar{g}_i^\flat)^t(q_i)$. Necessarily $q_i' \in \mathrm{Int}((Q_i')^\vee)$ as $\bar{g}_i^\flat : Q_i' \to Q_i$ is a local homomorphism. Further, q_1' and q_2' have the same image in $(P')^\vee$ because of commutativity of (3.1). This gives an element $q' := (q_1', q_2')$ of $(Q')^\vee = (Q_1')^\vee \times_{(P')^\vee} (Q_2')^\vee$. In particular q' lies in the interior of $(Q')^\vee$.

Hence we obtain a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "q"] & \mathbb{N} \\ Q' \arrow[u] \arrow[ur, "q'"'] & \end{tikzcd}$$

with q, q' local homomorphisms. Note that Q' is the stalk of the ghost sheaf at any point of $W_1' \times^{\mathrm{fs}}_{X'} W_2'$. By condition (1) this latter log scheme is non-empty, and so there is a morphism $\operatorname{Spec}\kappa^\dagger \to W_1' \times^{\mathrm{fs}}_{X'} W_2'$ which induces the map $q' : Q' \to \mathbb{N}$ on stalks of ghost sheaves. Indeed, it is sufficient to construct a morphism $\operatorname{Spec}\kappa^\dagger \to \operatorname{Spec}(Q' \to \kappa)$, which is equivalent to giving a local homomorphism $Q' \to \mathbb{N}$ and a homomorphism $Q' \to \kappa^\times$. We take the local homorphism $Q' \to \mathbb{N}$ to be given by q'.

The chosen morphism can also be viewed as arising in a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}\kappa^\dagger \arrow[r] \arrow[d] & W_2' \arrow[d] \\ W_1' \arrow[r] & X' \end{tikzcd} \tag{3.3}$$

At the level of ghost sheaves, this diagram is given by

$$\begin{tikzcd} \mathbb{N} & Q_2' \arrow[l, "q_2'"'] \\ Q_1' \arrow[u, "q_1'"] & P' \arrow[l, "\theta_1'"] \arrow[u, "\theta_2'"'] \end{tikzcd}$$

and the morphisms $\operatorname{Spec}\kappa^\dagger \to W_i'$ are then determined by additional data of maps $\psi_i' : Q_i' \to \kappa^\times$. Commutativity of (3.3) then comes down to the equality

$$(\psi_1' \circ \theta_1') \cdot \chi_{f_1'} = (\psi_2' \circ \theta_2') \cdot \chi_{f_2'} \tag{3.4}$$

in $\operatorname{Hom}(P', \kappa^\times)$.

We now wish to construct an analogous commutative diagram

$$\begin{tikzcd} \operatorname{Spec}\kappa^\dagger \arrow[r] \arrow[d] & W_2 \arrow[d] \\ W_1 \arrow[r] & X \end{tikzcd} \tag{3.5}$$

given at the level of ghost sheaves by the commutative diagram

$$\begin{tikzcd} \mathbb{N} & Q_2 \arrow[l, "q_2"'] \\ Q_1 \arrow[u, "q_1"] & P \arrow[l, "\theta_1"] \arrow[u, "\theta_2"'] \end{tikzcd}$$

As all homomorphisms in this diagram are local, it is enough to construct analogously $\psi_i : Q_i \to \kappa^\times$ such that

$$(\psi_1 \circ \theta_1) \cdot \chi_{f_1} = (\psi_2 \circ \theta_2) \cdot \chi_{f_2} \tag{3.6}$$

By commutativity of (3.1), we have

$$\chi_g \cdot (\chi_{f_i} \circ \bar{g}^\flat) = \chi_{f'_i} \cdot (\chi_{g_i} \circ \theta'_i) \tag{3.7}$$

in $\mathrm{Hom}(P', \kappa^\times)$.

Using the assumed injectivity of $\bar{g}_i^\flat$ of condition (3), we choose a lift $\psi_i : Q_i^{\mathrm{gp}} \to \kappa^\times$ of $\chi_{g_i}^{-1} \cdot \psi'_i : \bar{g}_i^\flat((Q'_i)^{\mathrm{gp}}) \to \kappa^\times$. As κ is algebraically closed, this can always be done even if $\bar{g}_i^\flat((Q'_i)^{\mathrm{gp}})$ is not saturated in Q_i^{gp}. Then for $p \in P'$, we have, with the third line by the definition of ψ_i and (3.7),

$$\begin{aligned}
\big((\psi_i \circ \theta_i) \cdot \chi_{f_i}\big)(\bar{g}^\flat(p)) &= \psi_i(\theta_i(\bar{g}^\flat(p))) \cdot \chi_{f_i}(\bar{g}^\flat(p)) \\
&= \psi_i(\bar{g}_i^\flat(\theta'_i(p))) \cdot \chi_{f_i}(\bar{g}^\flat(p)) \\
&= \big[\psi'_i(\theta'_i(p)) \cdot \chi_{g_i}(\theta'_i(p))^{-1}\big] \cdot \big[\chi_g(p)^{-1} \cdot \chi_{f'_i}(p) \cdot \chi_{g_i}(\theta'_i(p))\big] \\
&= \big[\psi'_i(\theta'_i(p)) \cdot \chi_{f'_i}(p)\big] \cdot \chi_g(p)^{-1}.
\end{aligned}$$

By (3.4), this is independent of i.

Now consider

$$[(\psi_1 \circ \theta_1) \cdot \chi_{f_1}] \cdot [(\psi_2 \circ \theta_2) \cdot \chi_{f_2}]^{-1} \in \mathrm{Hom}(P^{\mathrm{gp}}, \kappa^\times),$$

and note this homomorphism is the identity on $\bar{g}^\flat((P')^{\mathrm{gp}})$ by the previous paragraph. Thus it induces an element of $\mathrm{Hom}(P^{\mathrm{gp}}/\bar{g}^\flat((P')^{\mathrm{gp}}), \kappa^\times)$. As κ is algebraically closed, $\kappa^\times$ is a divisible group and hence by the injectivity of (3.2), we obtain a surjective map

$$\mathrm{Hom}(Q_1^{\mathrm{gp}}/\bar{g}_1^\flat((Q'_1)^{\mathrm{gp}}), \kappa^\times) \times \mathrm{Hom}(Q_2^{\mathrm{gp}}/\bar{g}_2^\flat((Q'_2)^{\mathrm{gp}}), \kappa^\times) \to \mathrm{Hom}(P^{\mathrm{gp}}/\bar{g}^\flat((P')^{\mathrm{gp}}), \kappa^\times).$$

Thus we may find $\varphi_i \in \mathrm{Hom}(Q_i^{\mathrm{gp}}/\bar{g}_i^\flat((Q'_i)^{\mathrm{gp}}), \kappa^\times)$ such that if we replace ψ_i with $\varphi_i \cdot \psi_i$, (3.6) holds. ♠

4. Gluing one curve

We give a first application of the material of the previous subsection. We consider our standard gluing situation as in Notation 2.3.

Now assume given basic punctured maps $f_i : C_i^\circ/W_i \to X$ of type $\boldsymbol{\tau}_i$ for $1 \leq i \leq r$, with the $W_i = \mathrm{Spec}(Q_i \to \kappa)$ being logarithmic points, Q_i the basic monoid associated to f_i, and κ algebraically closed. Suppose further that whenever $E \in \mathbf{E}$ with vertices v_1, v_2, with corresponding punctured points $p_{E,v_i} \in C_{i(v_i)}$, we have

$$f_{i(v_1)}(p_{E,v_1}) = f_{i(v_2)}(p_{E,v_2}), \tag{4.1}$$

i.e., the maps f_i will glue schematically. We may then ask how many gluings exist at the logarithmic level. More precisely, we would like to understand the scheme W defined as follows:

Definition 4.1. The *gluing* $f : C^\circ/W \to X$ of the punctured maps f_i is defined by

$$W := \mathscr{M}'(X, \boldsymbol{\tau}) \times_{\prod_{i=1}^r \mathscr{M}'(X, \boldsymbol{\tau}_i)} \prod_{i=1}^r W_i,$$

and $f : C^\circ/W \to X$ the pull-back of the universal map over $\mathscr{M}'(X, \boldsymbol{\tau})$ to W.

In this situation, we introduce the following notation. For any punctured point $p_{E,v}$ of C_i° indexed by a flag $v \in E \in \mathbf{E}$, let $P_{E,v}$ be the stalk of $\overline{\mathcal{M}}_X$ at $f_{i(v)}(p_{E,v})$. By (4.1), we have $P_{E,v_1} = P_{E,v_2}$ if v_1, v_2 are the vertices of E, and write both as P_E. For any irreducible component of C_i with generic point η corresponding to a vertex v of G_i, write P_v for the stalk of the ghost sheaf at $f_i(\eta)$.

For each i, we have a family of tropical maps $h_i : \Gamma(G_i, \ell_i) \to \Sigma(X)$ defined over τ_i. If $\omega_v \in \Gamma(G_i, \ell_i)$ is the cone corresponding to a vertex $v \in V(G_i)$, then we obtain by restriction a map

$$\mathrm{ev}_v : \omega_v \to \Sigma(X) \tag{4.2}$$

mapping into the cone $P_{v,\mathbb{R}}^\vee \in \Sigma(X)$. Explicitly this is defined as the transpose of

$$P_v \xrightarrow{\bar f_i^\flat} Q_i,$$

where here Q_i is identified with $\overline{\mathcal{M}}_{C_i,\bar\eta}$. Hence at the level of groups we also obtain a map

$$\mathrm{ev}_v : Q_i^* \to P_v^*.$$

Definition 4.2. With the notation as above, choose an orientation on each edge $E \in E(G)$ so that E has vertices v_E, v_E' and is oriented from v_E to v_E'. We define the *tropical gluing map*

$$\Psi : \prod_{i=1}^r Q_i^* \times \prod_{E \in \mathbf{E}} \mathbb{Z} \to \prod_{E\in\mathbf{E}} P_E^*$$

by

$$\Psi\big((q_1, \ldots, q_r), (\ell_E)_{E\in\mathbf{E}}\big) = \big(\mathrm{ev}_{v_E}(q_{i(v_E)}) + \ell_E \mathbf{u}(E) - \mathrm{ev}_{v_E'}(q_{i(v_E')})\big)_{E\in\mathbf{E}}.$$

We define the *tropical multiplicity* of the gluing situation to be

$$\mu = \mu(\tau, \mathbf{E}) := |(\operatorname{coker}\Psi)_{\mathrm{tors}}|.$$

Remark 4.3. The map Ψ is called the tropical gluing map for the following reason. Suppose given $s = \big((q_i), (\ell_E)_{E\in E(G)}\big) \in \ker\Psi$ such that $q_i \in Q_i^\vee$ for each v and $\ell_E > 0$ for each E. Then we may construct a tropical map $h_s : G \to \Sigma(X)$ as follows. First, for each i, let G_i' be the subgraph of G_i obtained by removing legs of the form (E, v) for $E \in \mathbf{E}$. Then G_i' is naturally identified with a subgraph of G, and we may define $h_s|_{G_i'}$ to agree with $(h_i)_{q_i}|_{G_i'}$. On the other hand, if we give each $E \in \mathbf{E}$ the length ℓ_E, then $s \in \ker\Psi$ guarantees we can extend h_s across all edges $E \in \mathbf{E}$.

Thus it is reasonable to think of $\ker\Psi$ as the integral tangent space to the family of glued tropical curves.

Theorem 4.4. *Suppose we are in the above situation. If the gluing W is non-empty, then W has $\mu(\tau, \mathbf{E})$ connected components.*

Proof. By Theorem 2.5, we have an fs Cartesian diagram

$$\begin{tikzcd} \widetilde{W} \arrow[r] \arrow[d] & \prod_{i=1}^r \widetilde{W}_i \arrow[d, "\mathrm{ev}_{\mathbf{L}}"] \\ \prod_{E\in\mathbf{E}} X \arrow[r, "\Delta"'] & \prod_{E\in\mathbf{E}} X \times X \end{tikzcd} \tag{4.3}$$

Here

$$\widetilde{W}_i := W_i \times_{\mathscr{M}'(X,\boldsymbol{\tau}_i)} \widetilde{\mathscr{M}'}(X,\boldsymbol{\tau}_i)$$

and

$$\widetilde{W} := W \times_{\mathscr{M}'(X,\boldsymbol{\tau})} \widetilde{\mathscr{M}'}(X,\boldsymbol{\tau}).$$

A priori all fibre products are over B, but by the assumption (4.1), the composition of f_i with the structure map $X \to B$ are all constant with the same image, so we may replace B by a suitable affine neighbourhood of this image and apply Proposition 2.6. Thus we may replace B with $\operatorname{Spec}\Bbbk$ in the above discussion and thus assume that all products in (4.3) are defined over $\operatorname{Spec}\Bbbk$.

By Proposition 2.4, the underlying schemes of $\widetilde{W}$ and W agree. Thus we need to calculate the number of connected components of $\widetilde{W}$. Further, by (4.1), for any edge $E \in \mathbf{E}$ with endpoints v_1, v_2, the evaluation maps $\widetilde{W}_{i(v)} \to X$, $\widetilde{W}_{i(v')} \to X$ both factor through the strict closed point $f_{i(v_1)}(p_{E,v_1}) = f_{i(v_2)}(p_{E,v_2})$. Thus we may replace, for each edge $E \in \mathbf{E}$, the target X with the corresponding log point, and hence obtain an fs Cartesian diagram

$$\begin{tikzcd}
\widetilde{W} \arrow[r] \arrow[d] & \prod_{i=1}^r \widetilde{W}_i \arrow[d, "\mathrm{ev}_{\mathbf{L}}"] \\
\prod_{E\in\mathbf{E}} \operatorname{Spec}(P_E \to \Bbbk) \arrow[r, "\Delta"'] & \prod_{E\in\mathbf{E}} \operatorname{Spec}(P_E \to \Bbbk)^2
\end{tikzcd} \tag{4.4}$$

Further, by Lemma 3.2, we may replace $\widetilde{W}_i$ by its reduction without changing the number of connected components of $\widetilde{W}$. Again by Proposition 2.4, the reduction of the underlying scheme of $\widetilde{W}_i$ agrees with the underlying scheme of W_i (being a point). Thus now $\widetilde{W}$ is a fibre product of log points.

Note that for $L_j = (E,v) \in \mathbf{L}_i$, $W_i \times_{\mathfrak{M}'(X,\boldsymbol{\tau}_i)} \mathfrak{M}'_{L_j}(X,\boldsymbol{\tau}_i)$ has ghost sheaf $Q_i^{L_j} \subseteq Q_i \oplus \mathbb{Z}$, with an equality on the level of groups, and with the induced evaluation map $f_i \circ p_{E,v} : W_i \to X$ yielding a map at the tropical level of

$$((f_i \circ p_{E,v})^\flat)^t : (Q_i^{L_j})^\vee \to P_E^\vee$$

taking the value on $(s,r) \in (Q_i^{L_j})^\vee \subseteq Q_i^\vee \oplus \mathbb{Z}$ given by

$$\mathrm{ev}_v(s) + r\mathbf{u}(L_i).$$

Further, from the fibre product description (2.5) we see that

$$\overline{\mathcal{M}}_{\widetilde{W}_i} \subseteq Q_i \oplus \bigoplus_{(E,v)\in L(G_i)} \mathbb{Z},$$

again with an equality on groups.

Of course Δ induces a map $P_E^* \to P_E^* \times P_E^*$ given by the diagonal. Putting this together, the homorphism θ^t of Lemma 3.3 then takes the form

$$\theta^t : \prod_E P_E^* \times \prod_{i=1}^r Q_i^* \times \prod_{v\in E\in\mathbf{E}} \mathbb{Z} \to \prod_{v\in E\in\mathbf{E}} P_E^* \tag{4.5}$$

given by

$$\theta^t\big((n_E)_{E\in E(G)}, (s_i)_{1\le i\le r}, (\ell_{E,v})_{v\in E\in\mathbf{E}}\big) = (\mathrm{ev}_v(s_{i(v)}) + \ell_{E,v}\mathbf{u}(E,v) - n_E)_{v\in E\in\mathbf{E}}.$$

If $\widetilde{W}$ is non-empty, then by Lemma 3.3 the number of its connected components is the order of the torsion part of $\operatorname{coker}\theta^t$. We next compare this with the order of the torsion part of $\operatorname{coker}\Psi$.

We have a diagram

(4.6)

$$\begin{tikzcd}
\prod_{E\in\mathbf{E}}\mathbb{Z}\times\prod_{E\in\mathbf{E}}P_E^* \arrow[r, "\alpha"] \arrow[d, "\pi"] & \prod_{E\in\mathbf{E}}P_E^*\times\prod_i Q_i^*\times\prod_{v\in E\in\mathbf{E}}\mathbb{Z} \arrow[r, "\gamma"] \arrow[d, "\theta^t"] & \prod_i Q_i^*\times\prod_{E\in\mathbf{E}}\mathbb{Z} \arrow[d, "\Psi"] \\
\prod_{E\in\mathbf{E}}P_E^* \arrow[r, "\beta"] & \prod_{v\in E\in\mathbf{E}}P_E^* \arrow[r, "\delta"] \arrow[d] & \prod_{E\in\mathbf{E}}P_E^* \arrow[d] \\
& \operatorname{coker}\theta^t \arrow[r, "\cong"'] & \operatorname{coker}\Psi
\end{tikzcd}$$

Here π is the surjective map given by

$$\pi\big((\ell_E)_{E\in\mathbf{E}},(n_E)_{E\in\mathbf{E}}\big)=\big(\ell_E\mathbf{u}(E,v_E)-n_E)_{E\in\mathbf{E}}.$$

Here we use the chosen vertices v_E, v_E' of Definition 4.2. Further, α,β are injections defined by

$$\begin{aligned}
\alpha\big((\ell_E)_{E\in\mathbf{E}},(n_E)_{E\in\mathbf{E}}\big) &= \big((n_E)_{E\in\mathbf{E}},(0)_{1\le i\le r},((-1)^{\delta_{v,v_E}}\ell_E)_{v\in E\in\mathbf{E}}\big),\\
\beta\big((n_E)_{E\in\mathbf{E}}\big) &= \big((n_E)_{v\in E\in\mathbf{E}}\big).
\end{aligned}$$

while γ and δ are surjections defined by

$$\begin{aligned}
\gamma\big((n_E)_{E\in\mathbf{E}},(s_i)_{1\le i\le r},(\ell_{E,v})_{v\in E\in\mathbf{E}}\big) &= \big((s_i)_{1\le i\le r},(\ell_{E,v_E}+\ell_{E,v_E'})_{E\in\mathbf{E}}\big),\\
\delta\big((n_{E,v})_{v\in E\in\mathbf{E}}\big) &= \big(n_{E,v_E}-n_{E,v_E'}\big)_{E\in\mathbf{E}}.
\end{aligned}$$

Finally, Ψ is the tropical gluing map of Definition 4.2. One checks that the diagram is commutative with the top two rows exact, and hence the snake lemma implies that $\operatorname{coker}\theta^t\cong\operatorname{coker}\Psi$, giving the result. ♠

Example 4.5. It is very important to note that the gluing parameter space W need not be reduced, something which is quite different from gluing ordinary stable maps. This arises via saturation in the fs fibre product. For example, consider a target space X with $\Sigma(X)=\mathbb{R}^2_{\ge 0}$, and a gluing situation where τ is a graph with two vertices, v_1 and v_2, and two edges, E_1 and E_2, each connecting v_1 with v_2. We split at the two edges, getting types τ_1,τ_2. We assume $\mathbf{u}(E_1,v_1)=-\mathbf{u}(E_1,v_2)=(-w_1,w_1)$ and $\mathbf{u}(E_2,v_1)=-\mathbf{u}(E_2,v_2)=(-w_2,w_2)$ with $\gcd(w_1,w_2)=1$ and $w_1<w_2$. Finally, we assume $\boldsymbol{\sigma}(v_1)=\mathbb{R}_{\ge 0}\times 0$ and $\boldsymbol{\sigma}(v_2)=0\times\mathbb{R}_{\ge 0}$.

A slightly tedious calculation shows that $\underline{W}=\operatorname{Spec}\Bbbk[t]/(t^{w_1})$ in this case. Morally, one can think of this as follows. If the glued curve smooths, there are smoothing parameters u_1,u_2 for the two nodes, in the sense that the local structure at each node is of the form $xy=u_1$ or $xy=u_2$. The relation $u_1^{w_2}=u_2^{w_1}$ is then forced by the logarithmic geometry of the situation. Saturation normalizes this curve, and the inverse image of the point with ideal (u_1,u_2) under this normalization is the non-reduced gluing.

Remark 4.6. Suppose instead we are given a gluing situation of maps to $\mathcal{X}$ equipped with evaluation maps at the punctured points, i.e., the maps are determined by morphisms $W_i \to \mathfrak{M}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}, \boldsymbol{\tau}_i)$ rather than $W_i \to \mathscr{M}'(X, \boldsymbol{\tau}_i)$, still with $\underline{W}_i = \operatorname{Spec}\kappa$. In other words, we are given pre-stable punctured maps $f_v : C_v^\circ/W_v \to \mathcal{X}$ along with compatible morphisms $\underline{W}_i \to \underline{X}$ for each punctured point $p_{E,v}$. We assume further, as before, that the images of p_{E,v_1}, p_{E,v_2} under these maps to $\underline{X}$ agree for each edge E. Then similarly, the number of connected components of the glued space W is $\mu(\tau, \mathbf{E})$. Indeed, the gluing is controlled by precisely the same Cartesian diagram as in the situation of Theorem 4.4.

Note that Theorem 4.4 says nothing about whether W is non-empty. Here is one often useful criterion:

Definition 4.7. We say a gluing situation is *tropically transverse* if the map Ψ of Definition 4.2 has finite cokernel.

Theorem 4.8. *In the situation of Theorem 4.4, W is non-empty if the gluing situation is tropically transverse.*

Proof. We apply the non-emptiness criterion of Lemma 3.4 to the product of (4.4). In that lemma, we take $W_i' = X' = \operatorname{Spec}\Bbbk$ so that the first condition of the lemma is trivially satisfied, and the third condition is equivalent to the injectivity of the map θ whose transpose θ^t is given in (4.5). However, θ^t having a finite cokernel implies θ is injective, and (4.6) implies $\operatorname{coker}\theta^t \cong \operatorname{coker}\Psi$, and hence tropical transversality implies the third condition. Finally, τ realizable implies the second condition. ♠

5. Gluing moduli spaces

5.1. **The general situation.** We continue with a standard gluing situation for X/B, as in Notation 2.3.

Theorem 5.1. *There is a diagram*

$$\begin{tikzcd}
\mathscr{M}(X/B,\boldsymbol{\tau}) \arrow[r, "\phi'"] \arrow[d, "\varepsilon^{\mathrm{ev}}"'] & \mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau}) \arrow[r] \arrow[d, "\varepsilon^{\mathrm{sch}}"'] & \prod_{i=1}^r \mathscr{M}(X/B,\boldsymbol{\tau}_i) \arrow[d, "\hat{\varepsilon}"] \\
\mathfrak{M}^{\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau) \arrow[r, "\phi"'] & \mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau) \arrow[r, "\Delta'"'] \arrow[d, "\underline{\mathrm{ev}}'"'] & \prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i) \arrow[d, "\underline{\mathrm{ev}}"] \\
 & \prod_{E\in\mathbf{E}} \underline{X}_{\boldsymbol{\sigma}(E)} \arrow[r, "\Delta"'] & \prod_{v\in E\in\mathbf{E}} \underline{X}_{\boldsymbol{\sigma}(E)}
\end{tikzcd}$$

with all squares Cartesian in all categories, defining the moduli spaces $\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})$ and $\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$. Further, ϕ is a finite morphism.

Proof. We define the morphism $\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i) \to \prod_{v\in E\in\mathbf{E}} \underline{X}_{\boldsymbol{\sigma}(E)}$ as follows. For flag $v \in E \in \mathbf{E}$, consider the composition

$$\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i) \to \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_{i(v)})}(\mathcal{X}/B,\tau_{i(v)}) = \mathfrak{M}(\mathcal{X}/B,\tau_i) \times_{\prod \underline{\mathcal{X}}} \prod \underline{X} \to \underline{X}$$

where the first arrow is projection and the second is further projection onto the factor $\underline{X}$ indexed by the leg $(E,v)\in \mathbf{L}_i$. Necessarily, this morphism factors through $\underline{X}_{\boldsymbol{\sigma}(E)}$ by the definition of a τ_i-marked curve, see [ACGS2, Def. 3.7,(1)]. This gives a morphism

$$\underline{\mathrm{ev}}_{(E,v)}:\prod_{i=1}^r \mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)\to \underline{X}_{\boldsymbol{\sigma}(E)},$$

and we define

$$\underline{\mathrm{ev}}:=\prod_{v\in E\in\mathbf{E}}\underline{\mathrm{ev}}_{(E,v)}.$$

The morphism Δ is the product of diagonals. It is then clear that the Cartesian diagram of Theorem 2.7 factors as stated, as $\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$ captures those curves which glue schematically. Further, ϕ is a finite and representable morphism as $\psi=\Delta'\circ\phi$ is finite and representable by Theorem 2.7. ♠

Remark 5.2. As stated, this is not a significant improvement over Theorem 2.7: it merely separates the gluing into two steps, the first step being schematic gluing and the second step taking into account only the gluing at the logarithmic level.

For the first step, we are in luck if (1) Δ is lci, so that the Gysin pull-back $\Delta^!$ exists, and (2) $\underline{\mathrm{ev}}$ is flat, so that $\Delta^!$ and $(\Delta')^!$ induce the same map

$$\Delta^!=(\Delta')^!:A_*\left(\prod_i \mathscr{M}(X/B,\boldsymbol{\tau}_i)\right)\to A_*\left(\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})\right).$$

In any event, each $\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)$ is pure-dimensional by [ACGS2, Prop. 3.28], so each $\mathscr{M}(X/B,\boldsymbol{\tau}_i)$ carries a virtual fundamental class. Further, the pull-back of the relative obstruction theory for $\hat{\varepsilon}$ (the product over i of the relative obstruction theories for $\mathscr{M}(X/B,\boldsymbol{\tau}_i)\to\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)$) yields a relative obstruction theory for $\varepsilon_{\mathrm{sch}}$. If $\underline{\mathrm{ev}}$ is flat, then $\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$ is also pure-dimensional, yielding a virtual fundamental class $[\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}$, and if Δ is lci, we have

$$[\mathscr{M}^{\mathrm{sch}}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}=\Delta^!\left(\prod_i[\mathscr{M}(X/B,\boldsymbol{\tau}_i)]^{\mathrm{virt}}\right). \tag{5.1}$$

Note that Δ is lci if and only if the strata $\underline{X}_{\boldsymbol{\sigma}(E)}$ are non-singular. However, a deepest stratum of X is always non-singular, as is a stratum of dimension one more than a deepest stratum of X. If instead the log structure on X arises from an snc divisor, all strata are non-singular.

On the other hand, flatness of $\underline{\mathrm{ev}}$ is only automatic when the strata $X_{\boldsymbol{\sigma}(E)}$ are always deepest strata. Otherwise, more care needs to be taken. Again, when the gluing strata are non-singular, there is a tropical characterization of flatness.

Theorem 5.3. *Let $\tau=(G,\mathbf{g},\boldsymbol{\sigma},\mathbf{u})$ be a realizable type of punctured map over B, $\mathbf{L}\subseteq L(G)$ a subset of legs, and let*

$$\underline{\mathrm{ev}}:\mathfrak{M}(\mathcal{X}/B,\tau)\to\prod_{L\in\mathbf{L}}\underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$$

be the schematic evaulation map at the punctured points indexed by $\mathbf{L}$. *Suppose the strata* $\underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$ *for* $L \in \mathbf{L}$ *are non-singular. Then* $\underline{\mathrm{ev}}$ *is flat if and only if, for every realizable type* $\tau' = (G', \mathbf{g}', \boldsymbol{\sigma}', \mathbf{u}')$ *with a contraction morphism* $\phi : \tau' \to \tau$, *the inequality*

$$\dim \tau' \geq \dim \tau + \sum_{L \in \mathbf{L}} (\dim \boldsymbol{\sigma}'(L) - \dim \boldsymbol{\sigma}(L))$$

holds.

Proof. Since $\mathfrak{M}^{\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B, \tau)$ is equi-dimensional (see [ACGS2, Prop. 3.28]), if $\underline{\mathrm{ev}}$ is flat then all stack-theoretic fibres $\underline{\mathrm{ev}}^{-1}(\bar{x})$ of $\underline{\mathrm{ev}}$ are equi-dimensional with dimension independent of the choice of a geometric point $\bar{x} \to \prod_{L \in \mathbf{L}} \underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$. Conversely, it follows from [ACGS2, Thm. 3.24] that $\mathfrak{M}(\mathcal{X}/B, \tau)$ is idealized log smooth over $\operatorname{Spec} \Bbbk$ (as $\mathbf{M}(G, \mathbf{g}) \times B$ appearing in that theorem is idealized log smooth over $\operatorname{Spec} \Bbbk$). However, from the specific description of the idealized structure on $\mathfrak{M}(\mathcal{X}/B, \tau)$ in [ACGS2, Prop. 3.23], it follows from [ACGS2, Prop. B.4] that $\mathfrak{M}(\mathcal{X}/B, \tau)$ smooth locally is a stratum of a toric variety, and hence in particular smooth locally is toric, hence Cohen-Macaulay. Thus, by [Ma89, Thm. 23.1], if the fibres of $\underline{\mathrm{ev}}$ are all equi-dimensional of dimension $\dim \mathfrak{M}(\mathcal{X}/B, \tau) - \dim \prod_{L \in \mathbf{L}} \underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$, then $\underline{\mathrm{ev}}$ is flat. Thus the question reduces to calculating the fibre dimension.

Note that because we assumed τ was realisable over B, $\mathfrak{M}(\mathcal{X}/B, \tau)$ contains a dense open stratum consisting of punctured maps to $\mathcal{X}$ of type τ. Further, we have by [ACGS2, Prop. 3.28] that

$$\dim \mathfrak{M}(\mathcal{X}/B, \tau) = 3|\mathbf{g}| - 3 + |L(G)| - \dim \tau + \delta_B$$

where

$$\delta_B := \begin{cases} 0 & B = \operatorname{Spec} \Bbbk \\ 1 & B = \operatorname{Spec} \Bbbk^\dagger \end{cases}$$

and $|\mathbf{g}| = b_1(G) + \sum_{v \in V(G)} \mathbf{g}(v)$.

Let $\bar{x}$ be a geometric point of the open stratum of $\prod_{L \in \mathbf{L}} \underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$. Note this open stratum is isomorphic, as an algebraic stack, to is $B\mathbb{G}_m^N$ where

$$N = -\dim \prod_{L \in \mathbf{L}} \underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)} = \sum_{L \in \mathbf{L}} (\dim \boldsymbol{\sigma}(L) - \delta_B),$$

and the dimension of the fibre of $\underline{\mathrm{ev}}$ over $\bar{x}$ is then

$$\dim \mathfrak{M}(\mathcal{X}/B, \tau) + N = 3|\mathbf{g}| - 3 + |L(G)| - \dim \tau + \delta_B + \sum_{L \in \boldsymbol{\sigma}(L)} (\dim \boldsymbol{\sigma}(L) - \delta_B). \tag{5.2}$$

Thus we need to show all fibres, when non-empty, have this dimension. So now choose some other geometric point $\bar{x} = (\bar{x}_L) \to \prod_{L \in \mathbf{L}} \underline{\mathcal{X}}_{\boldsymbol{\sigma}(L)}$, and suppose $\bar{x}_L$ lies in the stratum of $\mathcal{X}$ indexed by a cone σ_L. Let τ' be a type of punctured map which appears in $\underline{\mathrm{ev}}^{-1}(\bar{x})$, so that there is a contraction map $\phi : \tau' \to \tau$. Note that $\sigma_L = \boldsymbol{\sigma}'(L)$ for $L \in \mathbf{L}$. Then the dimension of the stratum of the fibre over x with of punctured maps with this type is similarly calculated as

$$\begin{aligned} &3|\mathbf{g}'| - 3 + |L(G')| - \dim\tau' + \delta_B + \sum_{L\in\boldsymbol{\sigma}(L)} (\dim\sigma_L - \delta_B) \\ &= 3|\mathbf{g}| - 3 + |L(G)| - \dim\tau' + \delta_B + \sum_{L\in\boldsymbol{\sigma}(L)} (\dim\boldsymbol{\sigma}'(L) - \delta_B). \end{aligned} \tag{5.3}$$

Now since fibre dimension is upper semi-continuous, it is sufficient to show that the quantity of (5.2) is greater than or equal to the quantity of (5.3). But this is the inequality of the theorem. ♠

Remark 5.4. The criterion of Theorem 5.3 may seem imposing to check as it in theory involves an arbitrary number of type τ'. But we may always replace $\mathfrak{M}(\mathcal{X}/B,\tau)$ with an open subset obtained by deleting those strata with type τ' (equipped with a contraction $\tau'\to\tau$) such that there does not exist a punctured map in $\mathscr{M}(X/B,\boldsymbol{\tau})$ of type τ'' such that the induced contraction map $\tau''\to\tau$ factors through $\tau'\to\tau$. Thus for any application, it is sufficient to restrict attention to tropical types which are contractions of types which actually occur.

The next step is to understand ϕ. In general, we don't yet know how to say much about it, save for the next theorem, which gives us the degree of ϕ onto its image. When the image is a proper closed substack, this tells us there are logarithmic obstructions to gluing, and it is still not understood how to deal with these obstructions in general. However, in the tropically transverse case, ϕ is surjective.

Theorem 5.5. *The degree of ϕ onto its image is the tropical multiplicity $\mu(\tau,\mathbf{E})$ defined in Definition 4.2. If the gluing situation is tropically transverse and $\underline{\mathrm{ev}}$ is flat, then ϕ is dominant.*

Proof. We note that as τ is realisable, $\mathfrak{M}^{\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$ has a non-empty dense open stratum U, whose geometric points are precisely those geometric points corresponding to a punctured map whose tropicalization is a family of tropical maps of type τ. Similarly, $\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)$ has a non-empty dense stratum U_i, with each geometric point in this stratum corresponding to a punctured map whose tropicalization is a family of curves of type τ_i. Certainly then $\Delta'\circ\phi(U)\subseteq\prod_i U_i$. Further, $(\phi\circ\Delta)^{-1}(\prod_i U_i)=U$ as the type of punctured map obtained by gluing together punctured maps of type τ_i is necessarily τ. It is thus sufficient to determine the degree of $\phi|_U$ onto its image in $(\Delta')^{-1}(\prod_i U_i)$. To this end, pick a strict geometric point $W'\to\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau)$ in the image of $\phi|_U$. Because Δ' is strict, we can decompose $W'=\prod_i W_i$ as a product over B, with $W_i\to\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)$ strict geometric points. Thus we may view this as a situation of §4. In particular, the corresponding gluing is $W'\times_{\prod_i\mathfrak{M}'^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X}/B,\tau_i)}\mathfrak{M}'^{\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$. By Lemma 3.2 and [ACGS2, Prop. 3.31], the reduction of this gluing agrees with the reduction of

$$W'\times_{\prod_i\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X},\tau_i)}\mathfrak{M}^{\mathrm{ev}(\mathbf{L}_i)}(\mathcal{X},\tau)\cong W'\times_{\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)}\mathfrak{M}^{\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau).$$

By Remark 4.6, this fibre product has $\mu(\tau,\mathbf{E})$ connected components. Since this number is independent of the choice of geometric point in the image of $\phi|_U$ and $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\tau)$ is reduced, it follows this fibre must be always reduced. This shows the degree.

If $\underline{\mathrm{ev}}$ is flat, then so is $\underline{\mathrm{ev}}'$, and hence every irreducible component of $\mathfrak{M}^{\mathrm{sch},\mathrm{ev}(\mathbf{L})}(\mathcal{X}/B,\tau)$ dominates $\prod_{E\in\mathbf{E}}\underline{X}_{\boldsymbol{\sigma}(E)}$. The domination of ϕ then similarly follows from Theorem 4.8. ♠

5.2. **Gluing from rigid tropical curves.** One of the standard situations for gluing is the degeneration situation studied in [ACGS1]. Here, one considers a base B a smooth curve or spectrum of a DVR, with log structure the divisorial log structure induced by a closed point $b_0\in B$. As usual, we assume $X\to B$ is projective and log smooth. We denote by X_0 the fibre over b_0, and we view b_0 with its induced log structure, i.e., b_0 is a standard log point.

In this case, the morphism $X\to B$ tropicalizes to a morphism $\delta:\Sigma(X)\to\Sigma(B)=\mathbb{R}_{\geq 0}$. This gives rise to a polyhedral cell complex $\Delta(X)=\delta^{-1}(1)$. Given a class β of log curve for X/B, we have the logarithmic decomposition formula, see [ACGS1, Thm. 1.2]:

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}}=\sum_{\boldsymbol{\tau}=(\tau,\mathbf{A})}\frac{m_\tau}{|\operatorname{Aut}(\boldsymbol{\tau})|}j_{\tau*}[\mathscr{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}}.$$

Here $\boldsymbol{\tau}$ runs over isomorphism classes of decorated *rigid tropical types.* These are realisable tropical types $\boldsymbol{\tau}$ such that the moduli space of tropical maps of type $\boldsymbol{\tau}$ is one-dimensional, with a unique member of this one-dimensional family factoring through $\Delta(X)$, hence the term rigid. The quantity $m_\tau\in\mathbb{N}$ is the smallest integer such that scaling $\Delta(X)$ by m_τ leads to a tropical map with integral vertices and edge lengths; put another way,

$$m_\tau=|\operatorname{coker}(N_\tau\to N_{\Sigma(B)}=\mathbb{Z})|.$$

We now explore how Theorem 5.1 may be applied in this situation. Unfortunately, the morphisms ϕ,ϕ' of Theorem 5.1 need not be surjective, but here we determine the length of the inverse image of a point in the image of ϕ.

We first introduce the following notation. For each $\sigma\in\Sigma(X)$, the morphism $\delta:\Sigma(X)\to\Sigma(B)$ induces homomorphisms $\delta_*:N_\sigma\to\mathbb{Z}$, and we define $\overline{N}_\sigma$ to be the kernel of this homomorphism. Provided $\delta|_\sigma$ surjects onto $\Sigma(B)=\mathbb{R}_{\geq 0}$, this kernel can be identified with the space of integral tangent vectors of the corresponding polyhedron in $\Delta(X)$.

Definition 5.6. Let τ be a type of rigid tropical curve in $\Delta(X)$. Define

$$\overline{\Psi}:\bigoplus_{v\in V(G)}\overline{N}_{\boldsymbol{\sigma}(v)}\oplus\bigoplus_{E\in E(G)}\mathbb{Z}\to\bigoplus_{E\in E(G)}\overline{N}_{\boldsymbol{\sigma}(E)}$$

to be the homomorphism

$$\big((n_v)_{v\in V(G)},(\ell_E)_{E\in E(G)}\big)\mapsto\big(n_{v_E}+\ell_E\mathbf{u}(E)-n_{v'_E}\big)_{E\in E(G)}$$

where for each $E\in E(G)$, we orient E from a vertex v_E to a vertex v'_E to determine the sign of $\mathbf{u}(E)$, which necessarily lies in $\overline{N}_{\boldsymbol{\sigma}(E)}$. We define the *tropical multiplicity* of τ to be

$$\mu(\tau):=|(\operatorname{coker}\overline{\Psi})_{\mathrm{tors}}|.$$

Theorem 5.7. *Let $X \to B$ be as above. Fix a rigid tropical type $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u})$, and let $\{\boldsymbol{\tau}_v \,|\, v \in V(G)\}$ be the decorated tropical types obtained by splitting $\boldsymbol{\tau}$ at all edges. This gives a standard gluing situation, and hence a diagram as in Theorem 5.1. With notation as in that theorem, ϕ is degree $\mu(\tau)/m_\tau$ onto its image.*

Proof. By Theorem 5.5, it is sufficient to compare the multiplicity $|\operatorname{coker}(\Psi)_{\mathrm{tors}}|$ as defined in Definition 4.2 and the multiplicity $|\operatorname{coker}(\overline{\Psi})_{\mathrm{tors}}|$. Note that as each split type τ_v consists of a single vertex with a number of adjacent edges, the only moduli of tropical maps of type τ_v is given by the location of v in $\boldsymbol{\sigma}(v)$. Thus one sees that the basic monoid associated to the type τ_v is

$$Q_v = P_{\boldsymbol{\sigma}(v)}. \tag{5.4}$$

Thus $Q_v^* = P_{\boldsymbol{\sigma}(v)}^* = N_{\boldsymbol{\sigma}(v)}$ and $P_E^* = N_{\boldsymbol{\sigma}(E)}$ in Definition 4.2. The map Ψ of that definition now becomes

$$\Psi : \bigoplus_{v \in V(G)} N_{\boldsymbol{\sigma}(v)} \oplus \bigoplus_{E \in E(G)} \mathbb{Z} \to \bigoplus_{E \in E(G)} N_{\boldsymbol{\sigma}(E)},$$

and the degree of ϕ onto its image is $|(\operatorname{coker}\Psi)_{\mathrm{tors}}|$.

We now have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \prod_v \overline{N}_{\boldsymbol{\sigma}(v)} \times \prod_E \mathbb{Z} \arrow[r] \arrow[d, "\overline{\Psi}"'] & \prod_v N_{\boldsymbol{\sigma}(v)} \times \prod_E \mathbb{Z} \arrow[r, "\delta_*"] \arrow[d, "\Psi"'] & \prod_v \mathbb{Z} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \prod_E \overline{N}_{\boldsymbol{\sigma}(E)} \arrow[r] & \prod_E N_{\boldsymbol{\sigma}(E)} \arrow[r, "\delta_*"'] & \prod_E \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Here the two maps labelled δ_* are induced by $\delta_* : N_{\boldsymbol{\sigma}(v)} \to \mathbb{Z}$ and $\delta_* : N_{\boldsymbol{\sigma}(E)} \to \mathbb{Z}$ for each vertex v and edge E. The map $\overline{\Psi}$ is as defined in Definition 5.6. The map ∂ is defined by

$$\partial\big((n_v)_{v \in V(G)}\big) = \big(n_{v_E} - n_{v'_E}\big)_{E \in E(G)}.$$

In particular, $\partial : \prod_v \mathbb{Z} \to \prod_E \mathbb{Z}$ is the complex calculating the simplicial cohomology of G, and thus $\ker\partial = H^0(G, \mathbb{Z}) = \mathbb{Z}$, as G is assumed connected. Note $\ker\partial$ is generated by $(1)_{v \in V(G)}$. Also $\operatorname{coker}\partial = H^1(G, \mathbb{Z}) = \mathbb{Z}^{b_1(G)}$. Thus the snake lemma gives a long exact sequence

$$0 \to \ker\overline{\Psi} \to \ker\Psi \to H^0(G, \mathbb{Z}) \to \operatorname{coker}\overline{\Psi} \to \operatorname{coker}\Psi \to H^1(G, \mathbb{Z}) \to 0.$$

Note that $\ker\overline{\Psi}$ and $\ker\Psi$ can be interpreted as the space of integral tangent vectors to the moduli space of maps of type τ in $\Delta(X)$ and $\Sigma(X)$ respectively. By the assumption of rigidity of τ, we thus have $\ker\overline{\Psi} = 0$ and $\ker\Psi = \mathbb{Z}$. Further, the map $\mathbb{Z} \cong \ker\Psi \to H^0(G, \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by m_τ by definition of the latter number. This gives an exact sequence

$$0 \to \mathbb{Z}/m_\tau\mathbb{Z} \to \operatorname{coker}\overline{\Psi} \to \operatorname{coker}\Psi \to H^1(G, \mathbb{Z}) \to 0.$$

Since $H^1(G, \mathbb{Z})$ is torsion-free and $\mathbb{Z}/m_\tau\mathbb{Z}$ is torsion, we easily obtain a short exact sequence

$$0 \to \mathbb{Z}/m_\tau\mathbb{Z} \to (\operatorname{coker}\overline{\Psi})_{\mathrm{tors}} \to (\operatorname{coker}\Psi)_{\mathrm{tors}} \to 0.$$

This shows that the multiplicity $\mu(\tau, \mathbf{E})$ defined in Definition 4.2 agrees with $\mu(\tau)/m_\tau$, and the result follows. ♠

Again, we get better behaviour in the tropically transverse case, where now we have a slightly weaker definition for tropical transversality.

Definition 5.8. Let τ be the combinatorial type of a rigid tropical curve in $\Delta(X)$. We say that τ is *tropically transverse* if the image of the map $\overline{\Psi}$ of Definition 5.6 has finite index.

Theorem 5.9. *In the situation of Theorem 5.7, suppose that the rigid decorated type $\boldsymbol{\tau}$ is tropically transverse, $\underline{\mathrm{ev}}$ is flat, and the central fibre $\underline{X}_0$ is reduced. Then ϕ is a finite dominant morphism of degree $\mu(\tau)/m_\tau$. Further, suppose that each $\underline{X}_{\boldsymbol{\sigma}(E)}$ is non-singular (so that Δ is an lci morphism and the Gysin map $\Delta^!$ exists). Then*

$$\phi'_*[\mathscr{M}(X_0/b_0, \boldsymbol{\tau})]^{\mathrm{virt}} = \frac{\mu(\boldsymbol{\tau})}{m_{\boldsymbol{\tau}}}\Delta^!\left(\prod_{v\in V(G)}[\mathscr{M}(X_0/b_0, \boldsymbol{\tau}_v)]^{\mathrm{virt}}\right).$$

Proof. The statement concerning the degree of ϕ follows from Theorem 5.7 provided that we know ϕ is dominant. To show this, we follow the argument of Theorem 5.5 and modify the argument of Theorem 4.4. In particular, we may assume given a strict geometric point $W' \to \mathfrak{M}^{\mathrm{sch,ev}}(\mathcal{X}_0/b_0, \tau)$, with an induced decomposition with $W' = \prod_v W_v$ and $W_v \to \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0, \tau_v)$ strict geometric points with image in the dense open strata of the latter moduli spaces. We need to show that the gluing $W' \times_{\mathfrak{M}^{\mathrm{sch,ev}}(\mathcal{X}_0/b_0,\tau)} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}_0/b_0, \tau)$ is non-empty for general choice of strict morphism $W' \to \mathfrak{M}^{\mathrm{sch,ev}}(\mathcal{X}_0/b_0, \tau)$ from a log point. To show this non-emptiness, we return to the setup and notation of the proof of Theorems 4.4 and use the non-emptiness criterion of Lemma 3.4, applied to the commutative diagram obtained from the fact that all spaces involved are defined over b_0:[4]

$$\begin{tikzcd} \prod_{E\in E(G)} X_{\boldsymbol{\sigma}(E)} \arrow[r] \arrow[d] & \prod_{v\in E\in E(G)} X_{\boldsymbol{\sigma}(E)} \arrow[d] & \prod_{v\in V(G)} \widetilde{W}_v \arrow[l] \arrow[d] \\ \prod_{E\in E(G)} b_0 \arrow[r] & \prod_{v\in E\in E(G)} b_0 & \prod_{v\in V(G)} b_0 \arrow[l] \end{tikzcd} \tag{5.5}$$

Now $\prod_E b_0 \times_{\prod_{v\in E} b_0} \prod_v b_0$ is easily seen to be non-empty: indeed, there is a morphism from b_0 to this fibre product induced by the diagonal morphisms $b_0 \to \prod_E b_0$ and $b_0 \to \prod_v b_0$. (In fact, the induced morphism is an isomorphism, so the fibre product is b_0, but we don't need this.) This gives condition (1) of the hypotheses of Lemma 3.4.

Condition (2) follows from realizability the tropical type τ. Indeed, at the level of dual of monoids, the fibre product involving the top row of (5.5) gives a Cartesian

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[4] We note here we work with fibre products over $\operatorname{Spec}\Bbbk$ rather than $B = b_0$, as we did in Theorem 4.4. Here we apply Proposition 2.6 to see that we may just as well work with the moduli spaces over $\operatorname{Spec}\Bbbk$.

diagram of monoids

$$\begin{array}{ccc}\widetilde{Q}^\vee & \longrightarrow & \prod_{v\in V(G)}\widetilde{Q}_v^\vee \\ \big\downarrow & & \big\downarrow{\scriptstyle \Sigma(\mathrm{ev})} \\ \prod_{E\in E(G)} P^\vee_{\boldsymbol{\sigma}(E)} & \xrightarrow[\Sigma(\Delta)]{} & \prod_{v\in E\in E(G)} P^\vee_{\boldsymbol{\sigma}(E)}\end{array}$$

Here $\widetilde{Q}_v$ is the stalk of the ghost sheaf of $\widetilde{W}_v$, with $\widetilde{Q}_v \subseteq Q_v \oplus \bigoplus_{v\in E\in E(G)}\mathbb{Z}$. Recall (5.4) that $Q_v = P_{\boldsymbol{\sigma}(v)}$. We need to show that the image of $\widetilde{Q}^\vee$ in $\prod_E P^\vee_{\boldsymbol{\sigma}(E)}$ or $\prod_v \widetilde{Q}_v^\vee$ intersects the interior of that monoid.

First we describe $\Sigma(\Delta)$ and $\Sigma(\mathrm{ev})$. Indeed, $\Sigma(\Delta)$ is just the diagonal, while

$$\Sigma(\mathrm{ev})\left((n_v)_{v\in V(G)}, (\ell_{v,E})_{v\in E\in E(G)}\right) = (n_v + \ell_{v,E}\mathbf{u}_v(E))_{v\in E\in E(G)}$$

where $\mathbf{u}_v(E)$ is the contact order of the leg (v,E) of G_v. Thus the above fibre product diagram allows us to view $\widetilde{Q}^\vee$ as a submonoid of $\prod_{v\in v(G)}\widetilde{Q}_v^\vee$ consisting of those tuples $\left((n_v)_{v\in V(G)}, (\ell_{v,E})_{v\in E\in E(G)}\right)$ such that, for every edge $E\in E(G)$ with vertices v_1, v_2, we have

$$n_{v_1} + \ell_{v_1,E}\mathbf{u}_{v_1}(E) = n_{v_2} + \ell_{v_2,E}\mathbf{u}_{v_2}(E). \tag{5.6}$$

We may then interpret a point $s\in \widetilde{Q}^\vee$ as giving the data of:

(1) The choice of a tropical map $h_s : G\to \Sigma(X)$ of type τ. This tropical map is determined by $h_s(v) = n_v \in P^\vee_{\boldsymbol{\sigma}(v)}$, and the condition (5.6) then guarantees that the edge E is still mapped to an edge in $\boldsymbol{\sigma}(E)$ with tangent vector $\mathbf{u}(E)$.
(2) For each edge $E\in E(G)$, a choice of decomposition of the length of E as a sum $\ell_{v_1,E}+\ell_{v_2,E}$. This is given by (5.6): since $\mathbf{u}_{v_2}(E) = -\mathbf{u}_{v_1}(E)$, (5.6) can be rewritten as $n_{v_2}-n_{v_1} = (\ell_{v_1,E}+\ell_{v_2,E})\mathbf{u}_{v_1}(E)$, and thus $\ell_E = \ell_{v_1,E}+\ell_{v_2,E}$.

However, since τ is a type of rigid curve, there is a unique tropical curve $h_s:\Gamma\to\Sigma(X)$ of type τ, up to scaling. By scaling this curve sufficiently so it is integrally defined and all edge lengths are at least 2, we may also choose non-trivial integral decompositions of the edge lengths. This yields an interior point of $\widetilde{Q}^\vee$, which necessarily maps to the interior of $\prod_v \widetilde{Q}_v^\vee$ and to an interior point of $\prod_E P^\vee_{\boldsymbol{\sigma}(E)}$. This shows condition (2) of Lemma 3.4.

Abusing notation, we denote by δ the generator of the ghost sheaf $\mathbb{N}$ of b_0, and also denote by δ its image in any of the monoids $P_{\boldsymbol{\sigma}(E)}$ under the structure map $X_0\to b_0$. Note this is largely compatible with our previous use of the notation $\delta:\Sigma(X)\to\Sigma(B)$. To verify condition (3) of the lemma, using the above description of the monoids involved in the fibre product, we need to show injectivity of

$$\bigoplus_{v\in E\in E(G)} P^{\mathrm{gp}}_{\boldsymbol{\sigma}(E)}/\mathbb{Z}\delta \to \left[\bigoplus_{E\in E(G)} P^{\mathrm{gp}}_{\boldsymbol{\sigma}(E)}/\mathbb{Z}\delta\right]\times\left[\bigoplus_{v\in V(G)} P^{\mathrm{gp}}_{\boldsymbol{\sigma}(v)}/\mathbb{Z}\delta\times\bigoplus_{v\in E\in E(G)}\mathbb{Z}\right]$$

Because of the hypothesis that the central fibre X_0 is reduced, all groups $P^{\mathrm{gp}}_{\boldsymbol{\sigma}(E)}/\mathbb{Z}\delta$ are torsion-free, and hence injectivity is equivalent to the transpose map having finite cokernel. However, we have a variant of diagram (4.6) given by replacing each $N_{\boldsymbol{\sigma}(E)}, N_{\boldsymbol{\sigma}(v)}$ with $\overline{N}_{\boldsymbol{\sigma}(E)}, \overline{N}_{\boldsymbol{\sigma}(v)}$, which shows that the cokernel of the transpose of the above map is finite if and only if the cokernel of $\overline{\Psi}$ is finite, which is the tropically transverse condition.

For the last statement, the extra condition implies Δ is lci, hence $\Delta^!$ makes sense. By flatness of $\underline{\mathrm{ev}}$, the Gysin pull-backs $\Delta^!$ and $(\Delta')^!$ agree. The result then follows from Theorem 5.7 and properties of virtual pull-backs [Ma12, Thm. 4.1]. ♠

6. Punctured versus relative

In a gluing situation arising from a degeneration situation $X \to B$ as reviewed in §5.2, one has the moduli spaces $\mathscr{M}(X, \boldsymbol{\tau}_v)$, classifying punctured maps marked by $\boldsymbol{\tau}_v$. In particular, such maps will factor through a stratum $X_{\boldsymbol{\sigma}(v)}$, with its induced log structure. On the other hand, $X_{\boldsymbol{\sigma}(v)}$ comes with a divisorial log structure induced by $\partial X_{\boldsymbol{\sigma}(v)} \subseteq X_{\boldsymbol{\sigma}(v)}$, the union of lower dimensional strata of X contained in $X_{\boldsymbol{\sigma}(v)}$. We write this different log scheme as $\overline{X}_{\boldsymbol{\sigma}(v)}$. It is then natural to compare $\mathscr{M}(X, \boldsymbol{\tau}_v)$ with a moduli space of stable log maps to $\overline{X}_{\boldsymbol{\sigma}(v)}$. In general, this still a non-trivial question, and is likely related to double ramification cycles. Here, however, we deal with a special case where $X_{\boldsymbol{\sigma}(v)}$ is an irreducible component of X_{b_0}.

To set this up, let $\sigma \in \Sigma(X)$ be a non-zero cone, corresponding to a stratum X_σ, and assume $X_\sigma \subseteq X_{b_0}$. If $\dim \sigma = 1$, then X_σ is an irreducible component of X_{b_0}. As above, we have the log scheme $\overline{X}_\sigma$, and there is a canonical morphism of log schemes

$$\psi : X_\sigma \to \overline{X}_\sigma$$

induced by the natural inclusion $\mathcal{M}_{\overline{X}_\sigma} \subseteq \mathcal{M}_{X_\sigma} = \mathcal{M}_X|_{X_\sigma}$. Indeed, smooth locally at a geometric point $\bar{x} \in X_\sigma$, X is given as $\operatorname{Spec} \Bbbk[P]$ for $P = \overline{\mathcal{M}}_{X,\bar{x}}$, and the monoid P has a codimension $\dim \sigma$ face $F \subseteq P$ such that X_σ is smooth locally $\operatorname{Spec} \Bbbk[F]$ and $F = \overline{\mathcal{M}}_{\overline{X}_\sigma, \bar{x}}$. The log structure on X_σ at $\bar{x}$ is given by a neat chart $P \to \mathcal{O}_{X_\sigma}$, and the log structure on $\overline{X}_\sigma$ is given by restricting this chart to F. Hence we obtain the inclusion of log structures.

Note the inclusion of faces $F \subseteq P$ dualizes to a generization $P^\vee_\mathbb{R} \to F^\vee_\mathbb{R} = (P^\vee_\mathbb{R} + \mathbb{R}\sigma)/\mathbb{R}\sigma$. Thus yields a description of the induced map of cone complexes

$$\Sigma(\psi) : \Sigma(X_\sigma) \to \Sigma(\overline{X}_\sigma)$$

at the level of cones of $\Sigma(X_\sigma)$. In particular, a type $\tau = (G, \mathbf{g}, \boldsymbol{\sigma}, \mathbf{u})$ for tropical map to $\Sigma(X_\sigma)$ induces a type $\bar{\tau} = (\bar{G}, \bar{\mathbf{g}}, \bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}})$ for tropical map to $\Sigma(\overline{X}_\sigma)$. Indeed, $\bar{G}$ will coincide with G, except for some legs of G which correspond to punctured points, i.e., are line segments, will be replaced by rays, corresponding to marked points. Of course $\bar{\mathbf{g}} = \mathbf{g}$. For $v \in V(G)$, we may take $\bar{\boldsymbol{\sigma}}(v)$ to be the minimal cone of $\Sigma(\overline{X}_\sigma)$ containing $\Sigma(\psi)(\boldsymbol{\sigma}(v))$. For an edge E of G, we define $\bar{\mathbf{u}}(E) = \Sigma(\Psi)_*(\mathbf{u}(E))$.

We remark that the strict inclusion $X_\sigma \hookrightarrow X$ induces a map $\Sigma(X_\sigma) \to \Sigma(X)$. Even though X is assumed to be Zariski, this need not be an inclusion of cone complexes. For example, if X_0 is a union of two $\mathbb{P}^1$'s meeting at two points and X_σ is one of these $\mathbb{P}^1$'s, we have $\Sigma(X)$ a union of two quadrants glued along their boundary, but $\Sigma(X_\sigma)$ consists of two quadrants glued together along one boundary ray. However, the map $|\Sigma(X_\sigma)| \to |\Sigma(X)|$ will always be injective in a neighbourhood of σ.

Now suppose given a type τ of punctured maps to X/B, with underlying graph G having precisely one vertex v and adjacent legs $L_1, \ldots, L_n$. Suppose further that

$\boldsymbol{\sigma}(v) = \sigma$. Then τ may also be viewed as a type of punctured map to X_σ/B, and thus we also obtain a type $\bar\tau$ of punctured map to $\overline{X}_\sigma$. However, in fact all contact orders are positive. Indeed, because $\boldsymbol{\sigma}(v) = \sigma$, any L_i must have contact order $\mathbf{u}(L_i)$ lying in the tangent wedge of $\boldsymbol{\sigma}(L_i)$ along the face σ of $\boldsymbol{\sigma}(L_i)$. Hence the image of $\mathbf{u}(L_i)$ in the tangent space to $(\boldsymbol{\sigma}(L_i) + \mathbb{R}\sigma)/\mathbb{R}\sigma$ in fact lies in this cone. Therefore we may view the type $\bar\tau$ as a type of logarithmic map to $\overline{X}_\sigma$.

In addition, in this situation, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{M}(X/B,\boldsymbol{\tau}) & \longrightarrow & \mathscr{M}(\overline{X}_\sigma,\bar{\boldsymbol{\tau}}) \\ \downarrow & & \downarrow \\ \mathfrak{M}(\mathcal{X}/B,\tau) & \longrightarrow & \mathfrak{M}(\overline{\mathcal{X}}_\sigma,\bar\tau) \end{array} \tag{6.1}$$

where $\overline{\mathcal{X}}_\sigma$ is defined in the same way as $\overline{X}_\sigma$. While the vertical arrows are the standard ones, the horizontal ones are as follows. By the definition of a $\boldsymbol{\tau}$-marked curve, we have $\mathscr{M}(X/B,\boldsymbol{\tau}) \cong \mathscr{M}(X_\sigma/B,\boldsymbol{\tau})$, i.e., any $\boldsymbol{\tau}$-marked curve factors through X_σ, by [ACGS2, Def. 3.7,(1)]. Given a stable punctured map $f : C^\circ/W \to X_\sigma$, we have $\psi\circ f : C^\circ/W \to \overline{X}_\sigma$. In general, this is neither a pre-stable nor basic log map, but we may pre-stabilize by [ACGS2, Prop. 2.4] and replace with a basic log map by [ACGS2, Prop. 2.40].[5] This gives the upper horizontal arrow, and a similar discussion gives the lower horizontal arrow.

Theorem 6.1. *Let $X \to B$, $\boldsymbol{\tau}$, σ be as in the above discussion, and suppose further that* $\dim\sigma = 1$. *Then the horizontal arrows of the diagram* (6.1) *are isomorphisms at the level of underlying stacks (but not at the level of logarithmic stacks). Furthermore these isomorphisms induce an isomorphism of obstruction theories.*

Proof. We work with the top horizontal arrow of (6.1), as the Artin fan case is identical. We need to construct the inverse map. Explicitly, given a basic $\bar{\boldsymbol{\tau}}$-marked stable log map $\bar f : \overline{C}/\overline{W} \to \overline{X}_\sigma$, we need to construct a stable punctured $\boldsymbol{\tau}$-marked map $f : C/W \to X_\sigma$. The maps $f, \bar f$ will be the same on schemes, but we need to modify the log structures on $\overline{W}$, $\overline{C}$.

To this end, let $\overline{Q}_{\bar w}$ be the stalk of the ghost sheaf at a geometric point $\bar w \in \overline{W}$, so that $\bar\tau_{\bar w} = \overline{Q}^\vee_{\bar w,\mathbb{R}}$ parametrizes a universal family of tropical maps $\bar h : \Gamma(\bar G_{\bar w}, \bar\ell) \to \Sigma(\overline{X}_\sigma)$, inducing for each $s \in \bar\tau$ a tropical map $\bar h_s : \bar G_{\bar w} \to \Sigma(\overline{X}_\sigma)$.

For $r \in \mathbb{R}_{\ge 0}$, write $\Delta_r := \delta^{-1}(r)$, where $\delta : \Sigma(X_\sigma) \to \Sigma(B)$ is the tropicalization of $X_\sigma \to B$. Of course Δ_1 agrees with the underlying topological space of $\Delta(X_\sigma)$. Now note that we may restrict the map $\Sigma(\psi)$ to Δ_r to obtain an inclusion of topological spaces $\Delta_r \hookrightarrow |\Sigma(\overline{X}_\sigma)|$. Indeed, it is sufficient to check that for each $\omega \in \Sigma(X_\sigma)$, $\Sigma(\psi)$ induces an inclusion on $\omega_r := \omega \cap \delta^{-1}(r)$. But this follows immediately from the fact that $\Sigma(\psi)|_\omega$ is given by localizing along the ray $\sigma \subseteq \omega$ and $\delta|_\sigma$ surjects onto $\Sigma(B)$. We thus view Δ_r in this way as a subspace of $\Sigma(\overline{X}_\sigma)$.

One also easily sees that the spaces Δ_r exhaust $|\Sigma(\overline{X}_\sigma)|$, i.e., for $y \in |\Sigma(\overline{X}_\sigma)|$, there exists an $r_0 \ge 0$ such that $y \in \Delta_r$ for all $r \ge r_0$.

[5] Note that once we pre-stabilize, all punctured points become marked as all contact orders in $\bar{\mathbf{u}}$ are positive. Hence we could just as well apply the earlier [GS4, Prop. 1.24].

Let $\bar{G}'_{\bar{w}}$ be the subgraph of $\bar{G}_{\bar{w}}$ obtained by deleting all legs of $\bar{G}_{\bar{w}}$. We define a function $\alpha : \bar{\tau}_{\bar{w}} \to \mathbb{R}_{\geq 0}$ by

$$\alpha(s) := \inf\{r \in \mathbb{R}_{\geq 0} \,|\, \bar{h}_s(\bar{G}'_{\bar{w}}) \subseteq \Delta_r\}.$$

The set on the right-hand side is non-empty, and hence this makes sense.

Claim. α is an upper convex piecewise linear function on $\bar{\tau}_{\bar{w}}$ with rational slopes.

Proof. Suppose $\omega \in \Sigma(X_\sigma)$, necessarily containing σ. Let $F^1, \ldots, F^p \subseteq \omega$ be the codimension one faces of ω not containing σ with $\delta|_{F^i}$ surjective. Let $G = \delta^{-1}(0) \cap \omega$. Necessarily G does not contain σ. We then have the Minkowski decomposition

$$\omega_r = \mathrm{Conv}\{F^1_r, \ldots, F^p_r, \sigma_r\} + G$$

where Conv denotes convex hull.

Let $n_{F^i} \in \omega^\vee$ be a rational generator of the dual one-dimensional face to F^i. Since $\sigma \not\subseteq F^i$, n_{F^i} is positive on $\sigma \setminus \{0\}$. So we can normalize n_{F^i} so that $n_{F^i}|_\sigma$ agrees with $\delta|_\sigma$ by rescaling by a rational number. Thus $\delta - n_{F^i}$ vanishes on σ and hence descends to a linear function on $\bar{\omega} = (\omega + \mathbb{R}\sigma)/\mathbb{R}\sigma$. Further, $\delta - n_{F^i}$ takes the value r on F^i_r.

We now show that

$$\Sigma(\psi)(\omega_r) = \{m \in \bar{\omega} \,|\, \langle \delta - n_{F^i}, m\rangle \leq r, 1 \leq i \leq p\}. \tag{6.2}$$

To see the forward inclusion, we may write an element w of ω_r as

$$\sum_{i=1}^p a_i f_i + a_{p+1} s + g$$

with $f_i \in F^i_r$, $s \in \sigma_r$, $\sum_{i=1}^{p+1} a_i = 1$, $a_i \geq 0$, and $g \in G$. Then

$$(\delta - n_{F^i})(w) = \sum_{i=1}^p a_i r - \langle n_{F^i}, g\rangle \leq r.$$

Conversely, if m lies in the right-hand side of (6.2), choose a lift m' of m to ω. Write $\sigma_1 = \{s\}$, and set $m'' = m' + s(r - \delta(m'))$. Then m'' is another lift of m to ω^{gp}, but may not lie in ω. However, $\delta(m'') = r$, and we would like to show $m'' \in \omega$. First, $\langle \delta - n_{F^i}, m''\rangle = \langle \delta - n_{F^i}, m\rangle \leq r$, so $\langle n_{F^i}, m''\rangle \geq \delta(m'') - r = 0$. If $H \subseteq \omega$ is a codimension one face of ω containing σ, and n_H is a generator of the dual ray, then $\langle n_H, m''\rangle = \langle n_H, m'\rangle \geq 0$. Finally, if G is a codimension one face of ω, then δ is a generator of the dual ray, $\delta(m'') = r > 0$. Thus we see that m'' is positive on all generators of rays of $\omega^\vee$, and hence lies in ω, hence in ω_r. So m lies in $\Sigma(\psi)(\omega_r)$.

Now for each vertex $v \in V(\bar{G}_{\bar{w}})$, we have the evaluation map $\mathrm{ev}_v : \bar{\tau}_{\bar{w}} \to \bar{\boldsymbol{\sigma}}(v)$ at v given by (4.2). Note that $\bar{h}_s(\bar{G}'_{\bar{w}}) \subseteq \Delta_r$ if and only if $\bar{h}_s(v) \in \Sigma(\psi)(\boldsymbol{\sigma}(v)_r)$ for all $v \in V(\bar{G}_{\bar{w}})$, where $\boldsymbol{\sigma}(v)$ is the cone of $\Sigma(X_\sigma)$ mapping to $\bar{\boldsymbol{\sigma}}(v)$ under $\Sigma(\psi)$. In particular, for a given v, $\bar{h}_s(v) \in \Sigma(\psi)(\boldsymbol{\sigma}(v)_r)$ if and only if $\langle \delta - n_{v,F_i}, \mathrm{ev}_v(s)\rangle \leq r$ for n_{v,F_i} running over the set of normal vectors to codimension one faces of $\boldsymbol{\sigma}(v)$ not containing σ or contained in $\delta^{-1}(0)$ as above. Thus α is given as

$$\alpha(s) = \sup\{\langle \delta - n_{v,F_i}, \mathrm{ev}_v(s)\rangle\},$$

with the supremum running over all $v \in V(\bar{G}_{\bar{w}})$ and normal vectors n_{v,F_i}, i.e., α agrees with the supremum of the linear functionals $\mathrm{ev}_v^*(\delta - n_{v,F_i})$. This makes α piecewise linear and (upper) convex. ♠

We may now define

$$\tau_{\bar{w}} := \{(s,r) \in \bar{\tau}_{\bar{w}} \times \mathbb{R}_{\geq 0} \,|\, r \geq \alpha(s)\}.$$

This is a rational polyhedral cone by the claim. From construction, $\tau_{\bar{w}}$ parameterizes lifts of the tropical maps $\bar{h}_s$ to $\Sigma(X_\sigma)$. Indeed, if $(s,r) \in \tau_{\bar{w}}$, $\bar{h}_s$ maps $\bar{G}'_{\bar{w}}$ into Δ_r, which we now view as a subset of $|\Sigma(X_\sigma)|$. We may then extend this map as far as possible along each leg L_i to define a domain graph $G_{\bar{w}}$, which coincides with $\bar{G}_{\bar{w}}$ except that the legs may be replaced with line segments. We then get $h_{s,r} : G_{\bar{w}} \to \Sigma(X_\sigma)$. In particular, a leg L_i turns into a punctured leg, i.e., a line segment, if only a portion of $h_s(L_i)$ is contained in Δ_r.

Note here we may now view $\tau_{\bar{w}}$ as type of tropical map to $\Sigma(X_\sigma)$, with $\boldsymbol{\sigma}$ defined in the obvious way from $\bar{\boldsymbol{\sigma}}$ and $\mathbf{u}(E)$ a tangent vector of $\boldsymbol{\sigma}(E)_1$ mapping under $\Sigma(\psi)_*$ to $\bar{\mathbf{u}}(E)$. In particular, we obtain a universal family of tropical maps of type $\tau_{\bar{w}}$,

$$h_{\bar{w}} : \Gamma(G,\ell) \to \Sigma(X_\sigma). \tag{6.3}$$

We then have the corresponding basic monoid

$$Q_{\bar{w}} := \tau_{\bar{w}}^\vee \cap (N^*_{\bar{\tau}_{\bar{w}}} \oplus \mathbb{Z}).$$

Note that

$$Q_{\bar{w}} \subseteq N^*_{\bar{\tau}_{\bar{w}}} \oplus \mathbb{N}$$

since $(0,1) \in \tau_{\bar{w}}$. The monoid $Q_{\bar{w}}$ will be the basic monoid for the point $\bar{w}$ for punctured log maps to X_σ. Note the projection $\tau_{\bar{w}} \to \bar{\tau}_{\bar{w}}$ dualizes to an inclusion $\overline{Q}_{\bar{w}} \hookrightarrow Q_{\bar{w}}$ which identifies $\overline{Q}_{\bar{w}}$ with the facet $\{(q,0) \in Q_{\bar{w}}\}$ of $Q_{\bar{w}}$.

It is easy to check that if $\bar{w}'$ is a generization of $\bar{w}$, the above construction of $Q_{\bar{w}}$ is compatible with generization maps, i.e., the generization map $\overline{Q}_{\bar{w}} \to \overline{Q}_{\bar{w}'}$ induces a generization map $Q_{\bar{w}} \to Q_{\bar{w}'}$. Indeed, dually, we have an inclusion of faces $\bar{\tau}_{w'} \subsetneq \bar{\tau}_w$, and $\alpha : \bar{\tau}_w \to \mathbb{R}_{\geq 0}$ restricts to the corresponding map for $\bar{\tau}_{w'}$. Hence we obtain an inclusion of faces $\tau_{\bar{w}'} \subsetneq \tau_{\bar{w}}$. From this, we see that the monoids $Q_{\bar{w}}$ for various $\bar{w}$ define a fine subsheaf $\overline{\mathcal{M}}_W$ of $\overline{\mathcal{M}}^{\mathrm{gp}}_{\overline{W}} \oplus \mathbb{N}$ containing $\overline{\mathcal{M}}_{\overline{W}} \oplus \mathbb{N}$.

Let $W^\dagger = \overline{W} \times b_0$, so that $\overline{\mathcal{M}}_{W^\dagger} = \overline{\mathcal{M}}_{\overline{W}} \oplus \mathbb{N}$, and set

$$\mathcal{M}_W := \overline{\mathcal{M}}_W \times_{\overline{\mathcal{M}}^{\mathrm{gp}}_{W^\dagger}} \mathcal{M}^{\mathrm{gp}}_{W^\dagger}.$$

We may then define a structure morphism $\alpha_W : \mathcal{M}_W \to \mathcal{O}_W$ by taking $\alpha_W = \alpha_{\overline{W}}$ on $\mathcal{M}_{\overline{W}} \oplus 0 \subseteq \mathcal{M}_W$ and α_W taking the value 0 on $\mathcal{M}_W \setminus (\mathcal{M}_{\overline{W}} \oplus 0)$. Thus we obtain an fs log scheme W.

We have morphisms $W \to W^\dagger \to \overline{W}$ by construction, and the log smooth curve $\overline{C} \to \overline{W}$ pulls back to give $C^\dagger \to W^\dagger$ and $C \to W$. Note further $\bar{f} : \overline{C} \to \overline{X}_\sigma$ induces a morphism $f^\dagger : C^\dagger \to \overline{X}_\sigma \times b_0$ defined over b_0. We wish to define a punctured structure

C° on C yielding a commutative diagram

$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d] & X_\sigma \arrow[d] \\ C^\dagger \arrow[r, "f^\dagger"'] & \overline{X}_\sigma \times b_0 \end{tikzcd}$$

As $\mathcal{M}^{\mathrm{gp}}_{W^\dagger} = \mathcal{M}^{\mathrm{gp}}_{W}$, so that $\mathcal{M}^{\mathrm{gp}}_{C^\circ} = \mathcal{M}^{\mathrm{gp}}_{C^\dagger}$, and $\mathcal{M}^{\mathrm{gp}}_{X_\sigma} = \mathcal{M}^{\mathrm{gp}}_{\overline{X}_\sigma \times b_0}$, $\bar{f}^\flat$ and $\overline{(f^\dagger)}^\flat$ agree at the level of groups, and it is sufficient to construct a pre-stable punctured log structure C° on C so that

$$\overline{(f^\dagger)}^\flat(\overline{\mathcal{M}}_{X_\sigma}) \subseteq \overline{\mathcal{M}}_{C^\circ}. \tag{6.4}$$

Note that the saturation of $\overline{\mathcal{M}}_{C^\circ}$ over $\bar{w} \in W$, as well as the map $\overline{(f^\dagger)}^\flat : f^{-1}\overline{\mathcal{M}}_{X_\sigma} \to \overline{\mathcal{M}}^{\mathrm{sat}}_{C^\circ}$, are completely determined locally near $\bar{w}$ by the data of the tropical family of maps $h_{\bar{w}}$ of (6.3). Thus we only need to consider the punctured points.

So let $p \in C_{\bar{w}}$ be a marked point corresponding to a leg L of $\bar{G}_{\bar{w}}$ (or the corresponding leg of $G_{\bar{w}}$). We have the monoids $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)}$ (resp. $P_{\boldsymbol{\sigma}(L)}$) which are the stalks of $\overline{\mathcal{M}}_{\overline{X}_\sigma}$ (resp. $\overline{\mathcal{M}}_{X_\sigma}$) at the generic point of the strata of $\overline{X}_\sigma$ (resp. X_σ) corresponding to $\bar{\boldsymbol{\sigma}}(L)$ (resp. $\boldsymbol{\sigma}(L)$). Then $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)}$ is a codimension one face of $P_{\boldsymbol{\sigma}(L)}$, and we have a commutative diagram with horizontal arrows induced by $\bar{f}$, $f^\dagger$ and the family of tropical maps parameterized by $\tau_{\bar{w}}$:

$$\begin{tikzcd} \overline{P}_{\bar{\boldsymbol{\sigma}}(L)} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\overline{C},p} = \overline{Q}_{\bar{w}} \oplus \mathbb{N} \arrow[d] \\ \overline{P}_{\bar{\boldsymbol{\sigma}}(L)} \oplus \mathbb{N} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{C^\dagger,p} = (\overline{Q}_{\bar{w}} \oplus \mathbb{N}) \oplus \mathbb{N} \arrow[d] \\ P_{\boldsymbol{\sigma}(L)} \arrow[r] & \overline{\mathcal{M}}_{C^\circ,p} \subseteq Q_{\bar{w}} \oplus \mathbb{Z} \end{tikzcd}$$

Here the inclusion $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)} \oplus \mathbb{N} \to P_{\boldsymbol{\sigma}(L)}$ identifies $P_{\boldsymbol{\sigma}(L)}$ with a submonoid of $\overline{P}^{\mathrm{gp}}_{\bar{\boldsymbol{\sigma}}(L)} \oplus \mathbb{N}$ with the facet $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)}$ of $P_{\boldsymbol{\sigma}(L)}$ identified with $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)} \oplus 0$. To obtain a pre-stable puncturing, we take the stalk of the ghost sheaf of $\overline{\mathcal{M}}_{C^\circ}$ at p to be the submonoid of $Q_{\bar{w}} \oplus \mathbb{Z}$ generated by $Q_{\bar{w}} \oplus \mathbb{N}$ and the image of $P_{\boldsymbol{\sigma}(L)}$ under $\overline{(f^\dagger)}^\flat$. We just need to make sure that this defines a puncturing by showing that if $(\bar{m}_1, \bar{m}_2) \in \overline{\mathcal{M}}_{C^\circ} \setminus \overline{\mathcal{M}}_C$ is in the image of $P_{\boldsymbol{\sigma}(L)}$, and m_1 is a lift of $\bar{m}_1$ to a local section of $\mathcal{M}_W$, then $\alpha_W(m_1) = 0$.

Let $\varphi_L : \overline{P}_{\bar{\boldsymbol{\sigma}}(L)} \oplus \mathbb{N} \to \overline{Q}_{\bar{w}} \oplus \mathbb{N}$ be the composition of $\overline{(f^\dagger)}^\flat$ with the projection to $Q_{\bar{w}} \oplus \mathbb{N}$. By construction, φ_L preserves the second component, i.e., $\varphi_L(p, r) = (q, r)$ for some q. Also by construction, φ_L^{gp} coincides with the similarly defined map on groups $P^{\mathrm{gp}}_{\boldsymbol{\sigma}(L)} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{C^\circ,p}$ coming from $\overline{f}^\flat$, and $\varphi_L^{\mathrm{gp}}(P_{\boldsymbol{\sigma}(L)} \setminus \overline{P}_{\bar{\boldsymbol{\sigma}}(L)}) \subseteq Q_{\bar{w}} \setminus \overline{Q}_{\bar{w}}$. Suppose $(\bar{m}_1, \bar{m}_2)$ is in the image of $P_{\boldsymbol{\sigma}(L)}$ and $\bar{m}_2 < 0$. Then $\bar{m}_1 = \varphi_L^{\mathrm{gp}}(\bar{n}_1)$ for some $\bar{n}_1 \in P_{\boldsymbol{\sigma}(L)}$. As $\mathbf{u}(L)$ is positive on $\overline{P}_{\bar{\boldsymbol{\sigma}}(L)}$, p being a marked point on $\overline{C}$, we see $\bar{n}_1 \in P_{\boldsymbol{\sigma}(L)} \setminus \overline{P}_{\bar{\boldsymbol{\sigma}}(L)}$, and hence $\varphi_L^{\mathrm{gp}}(\bar{n}_1) \in Q_{\bar{w}} \setminus \overline{Q}_{\bar{w}}$. Hence, by definition of the structure map α_W, we have $\alpha_W(m_1) = 0$, as desired.

It is easy to check that the morphisms constructed between $\underline{\mathscr{M}}(X_\sigma/B,\boldsymbol{\tau})$ and $\underline{\mathscr{M}}(\overline{X}_\sigma,\bar{\boldsymbol{\tau}})$ are inverse to each other, and hence isomorphisms. Further, one easily checks the induced obstruction theories are the same, as the construction of these obstruction theories does not depend on the log structure. ♠

7. The classical degeneration formula

The classical degeneration situation, originally considered by Li and Ruan in [LR01] and developed in the algebro-geometric context by Jun Li in [Li02] is a special case of the degeneration situation of the previous section. We have already discussed this case in the context of the decomposition formula in [ACGS1, §6.1]. We consider $X\to B$ a simple normal crossings degeneration with $X_0 = Y_1\cup Y_2$ a reduced union of two irreducible components, with $Y_1\cap Y_2 = D$ a smooth divisor in both Y_1 and Y_2. In this case $\Sigma(X) = (\mathbb{R}_{\ge 0})^2$ with map $\Sigma(X)\to\Sigma(B)$ given by $(x,y)\mapsto x+y$, so that $\Delta(X)$ is a unit interval. We showed in [ACGS1, Prop. 6.1.1] that if $f:\Gamma\to\Delta(X)$ is a rigid tropical map, then all vertices of Γ map to endpoints of $\Delta(X)$ and all edges of Γ surject onto $\Delta(X)$. In this case, giving a rigid tropical map with target $\Delta(X)$ is the same information as an admissible triple of [Li02].

Using the setup of §5, we immdiately obtain the logarithmic stable map version of the main result of [Li02], as proved in [KLR]:

Theorem 7.1. *In the situation described above, let $\boldsymbol{\tau}$ be a decorated type of rigid tropical map in $\Delta(X)$. There is a diagram of (non-logarithmic) stacks*

$$\begin{array}{ccccc}
\underline{\mathscr{M}(X_0/b_0,\boldsymbol{\tau})} & \xrightarrow{\phi'} & \underline{\mathscr{M}^{\mathrm{sch}}(X_0/b_0,\boldsymbol{\tau})} & \longrightarrow & \prod_{v\in V(G)}\underline{\mathscr{M}(\overline{X}_{\boldsymbol{\sigma}(v)},\bar{\boldsymbol{\tau}}_v)} \\
 & & \downarrow & & \downarrow \\
 & & \prod_{E\in E(G)}\underline{D} & \xrightarrow[\Delta]{} & \prod_{v\in E\in E(G)}\underline{D}
\end{array}$$

with the square Cartesian and defining the space $\underline{\mathscr{M}^{\mathrm{sch}}(X_0/b_0,\boldsymbol{\tau})}$. Further, ϕ' is finite and

$$\phi'_*[\mathscr{M}(X_0/b_0,\boldsymbol{\tau})]^{\mathrm{virt}} = m_{\boldsymbol{\tau}}^{-1}\left(\prod_{E\in E(G)} w(E)\right)\Delta^!\left(\prod_{v\in V(G)}[\mathscr{M}(\overline{X}_{\boldsymbol{\sigma}(v)},\bar{\boldsymbol{\tau}}_v)]^{\mathrm{virt}}\right),$$

where $w(E)$ is the index (degree of divisibility) of $\mathbf{u}(E)$.

Proof. Using Theorem 6.1, the given diagram is a part of the diagram of Theorem 5.1. The result will follow from Theorem 5.9. Thus we first verify the tropical transversality condition, and calculate $\mu(\boldsymbol{\tau})$.

Note that $\overline{N}_{\boldsymbol{\sigma}(v)} = 0$, while each $\overline{N}_{\boldsymbol{\sigma}(E)}$ can be identified with $\mathbb{Z}\cong\mathbb{Z}(1,-1)\subseteq\mathbb{Z}^2$ for any vertex v, edge E. Thus the morphism Ψ of Definition 5.6 takes the form

$$\Psi:\bigoplus_{E\in E(G)}\mathbb{Z}\to\bigoplus_{E\in E(G)}\mathbb{Z}$$

given by $\Psi\big((\ell_E)_{E\in E(G)}\big) = \big(\ell_E\mathbf{u}(E)\big)$. In particular, the image has finite index, and this index is $\prod_E w(E)$.

By Theorem 5.9, it is thus sufficient to show that $\underline{\mathrm{ev}}$ is flat. However, this follows immediately from Theorem 5.3, as $\boldsymbol{\sigma}'(L) = \boldsymbol{\sigma}(L)$ always in this situation. ♠

8. Applications to wall structures for type III degenerations of K3 surfaces

In this section, we will work with a specific kind of degeneration $g : X \to S$. Here, we will use S for the base log scheme rather than B as is done in the previous sections and in [ACGS2], as B will notationally play a different role. We make the following assumptions:

Assumptions 8.1. (1) *$\underline{S}$ is a one-dimensional non-singular scheme with closed point $0 \in \underline{S}$ inducing the divisorial log structure on S.*
(2) *$\underline{X} \to \underline{S}$ is a simple normal crossings degeneration of K3 surfaces.*
(3) *The fibre over 0, X_0, is the only singular fibre, necessarily a simple normal crossings divisor, and the log structure on X is the divisorial log structure induced by X_0.*
(4) *With $D := (X_0)_{\mathrm{red}}$ the reduction, we have $K_{X/S} + D = 0$.*
(5) *Any intersection of irreducible components of D is connected.*
(6) *D has a zero-dimensional stratum.*

Here we will give a useful inductive description for the so-called *canonical wall structure* defined in [GS8]. We make use of an observation of Ranganathan in [Ra, §6.5.2], following Parker [Pa], that gluing remains fairly easy in the case the normal crossings divisor D has at worst triple points and the domain curve is genus 0. Rather than give the general description of this aproach in our language, we just carry out the procedure in our particular application.

8.1. **Review of wall types and balancing.** We begin by recalling certain concepts from [GS8]. By the assumptions on $X \to S$ made above, all irreducible components of D are good in the sense of [GS8, §1.1]. Further, Assumptions 1.1 and 1.2 of [GS8] hold. Thus, in the notation of that paper, we may take $B = |\Sigma(X)|$ and $\mathscr{P}$ the set of cones of $\Sigma(X)$, so that $(B, \mathscr{P})$ is a pseudo-manifold as explained in [GS8, Prop. 1.3]. We set

$$\Delta := \bigcup_{\substack{\sigma\in\Sigma(X)\\ \operatorname{codim}\sigma\geq 2}} \sigma,$$

and $B_0 := B \smallsetminus \Delta$. Then [GS8, §1.3] gives the structure of integral affine manifold to B_0. Further, by [GS8, Prop. 1.15], the tropicalization of g induces a map $g_{\mathrm{trop}} : B \to \Sigma(S) = \mathbb{R}_{\geq 0}$ which is an affine submersion. We set $B' = g_{\mathrm{trop}}^{-1}(1)$, and

$$\mathscr{P}' = \{\sigma \cap g_{\mathrm{trop}}^{-1}(1) \,|\, \sigma \in \mathscr{P}\},$$

a polyhedral decomposition of B'. In the notation of the previous sections, $\mathscr{P}'$ is the set of polyhedra in $\Delta(X)$, but we wish to avoid this notation now to avoid conflict with Δ as the discriminant locus. Here we also set

$$\Delta' := B' \cap \Delta, \qquad B'_0 := B' \smallsetminus \Delta.$$

We use the convention that for $\sigma \in \mathscr{P}$ and $\sigma' = \sigma \cap B' \in \mathscr{P}'$, we write either X_σ or $X_{\sigma'}$ for the stratum of X corresponding to σ. Under this convention, irreducible components of D correspond to vertices of $\mathscr{P}'$, and a vertex v has integral coordinates if and only if the corresponding irreducible component X_v of X_0 appears with multiplicity 1 in X_0. Thus the cells of $\mathscr{P}'$ are not in general lattice polytopes. In case X_0 is reduced, then [GS8, Prop. 1.16] applies, but we do not wish to make this restriction. In any event, B' is still an affine manifold with singularities.

We recall from [GS8, Lem. 2.1]:

Lemma 8.2. *Let $f : C^\circ/W \to X$ be a stable punctured map to X, with $W = \operatorname{Spec}(Q \to \kappa)$ a geometric log point. For $s \in \operatorname{Int}(Q^\vee_{\mathbb{R}})$, let $h_s : G \to B$ be the corresponding tropical map. If $v \in V(G)$ satisfies $h_s(v) \in B_0$, then h_s satisfies the balancing condition at v. More precisely, if $E_1, \ldots, E_n$ are the legs and edges adjacent to v, oriented away from v, then the contact orders $\mathbf{u}(E_i)$ may be interpreted as elements of $\Lambda_{h_s(v)}$, the stalk of the local system Λ of integral tangent vectors $h_s(v)$. In this group, the balancing condition*

$$\sum_{i=1}^{m} \mathbf{u}(E_i) = 0 \tag{8.1}$$

is satisfied.

As a consequence, we say a realizable tropical type τ of tropical map to $\Sigma(X)/\Sigma(S)$ is *balanced* if, for any $s \in \operatorname{Int}(\tau)$, the corresponding tropical map $h_s : G \to B$ satisfies the balancing condition of this lemma.

We then define:

Definition 8.3. A *wall type* is a type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$ of tropical map to $\Sigma(X)$ defined over $\Sigma(S)$ such that:

1. G is a genus zero graph with $L(G) = \{L_{\mathrm{out}}\}$ and $u_\tau := \mathbf{u}(L_{\mathrm{out}}) \neq 0$.
2. τ is realizable and balanced.
3. Let $h : \Gamma(G, \ell) \to \Sigma(X)$ be the corresponding universal family of tropical maps, and $\tau_{\mathrm{out}} \in \Gamma(G, \ell)$ the cone corresponding to L_{out}. Then $\dim \tau = 1$ and $\dim h(\tau_{\mathrm{out}}) = 2$.

A *decorated wall type* is a decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with τ a wall type and

$$\mathbf{A} : V(G) \to \coprod_{\sigma \in \Sigma(X)} H_2(X_\sigma)$$

is a *refined decoration*, i.e., $\mathbf{A}(v) \in H_2(X_{\boldsymbol{\sigma}(v)})$ for $v \in V(G)$.

Remarks 8.4. (1) The notion of refined decoration, in which curve classes lie in the group of curve classes of the relevant stratum rather than of X, gives more control over moduli spaces. In particular, the moduli space $\mathscr{M}(X/S, \boldsymbol{\tau})$ for $\boldsymbol{\tau}$ carrying a refined decoration is simply a union of connected components of the moduli space where the decoration is obtained by pushing forward all curve classes $\mathbf{A}(v)$ to $H_2(X)$.

(2) We note that, other than the issue of refined decorations, this definition is slightly simpler than the one given in [GS8, Def. 3.6] in that (1) we have specialized to the case that $\dim X = 3$ and all irreducible components of D are good; (2) B does not have a boundary in our case; and (3) here we insist that the type be defined over $\Sigma(S)$, but by

[GS8, Prop. 3.7], this is implied by [GS8, Def. 3.6]. As a consequence, the two notions of wall type coincide in this particular case.

(3) We note that because of our restriction to $\dim X = 3$, given a wall type τ, there is a unique tropical map $h'_\tau : G \to B$ of type τ factoring through B'. We will write this as $h'_\tau : G \to B'$, or h' when clear from context. Such a tropical map is rigid.

It is shown in [GS8, Const. 3.13] that if $\boldsymbol{\tau}$ is a decorated wall type, then $\mathscr{M}(X/\Bbbk, \boldsymbol{\tau}) \cong \mathscr{M}(X/S, \boldsymbol{\tau})$, and this stack is proper over $\operatorname{Spec} \Bbbk$ and carries a zero-dimensional virtual fundamental class. As a consequence, from now on we will work over $\operatorname{Spec} \Bbbk$ rather than S, writing $\mathscr{M}(X, \boldsymbol{\tau})$, $\mathfrak{M}(\mathcal{X}, \tau)$ for the relevant moduli spaces.

We then define

$$W_{\boldsymbol{\tau}} := \frac{\deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}}}{|\operatorname{Aut}(\boldsymbol{\tau})|}.$$

We also define the number k_τ as follows. The map $h : \Gamma(G, \ell) \to \Sigma(X)$ induces a homomorphism

$$h_* : N_{\tau_{\mathrm{out}}} \to N_{\boldsymbol{\sigma}(L_{\mathrm{out}})}.$$

We then define

$$k_{\boldsymbol{\tau}} := |\operatorname{coker}(h_*)_{\mathrm{tors}}|.$$

Example 8.5. Let $v \in \mathscr{P}'$ be a vertex with corresponding irreducible component X_v. Let $D_v \subseteq X_v$ be the union of lower-dimensional strata of D contained in X_v. Suppose X_v contains a rational curve E with self-intersection -1. Suppose further that E meets D_v transversally at one point. This point is contained in a one-dimensional stratum X_ρ for $\rho \in \mathscr{P}'$ an edge. Necessarily, v is one endpoint of ρ. Let v' be the other endpoint of ρ, and μ, μ' be the multiplicities of the irreducible components $X_v, X_{v'}$ in the fibre X_0. Let k be such that $\mu | k\mu'$.

Consider a type τ where G has one vertex w, one leg L, and no edges. We take $\boldsymbol{\sigma}(w)$ to be the ray of $\mathscr{P}$ corresponding to v and $\boldsymbol{\sigma}(L)$ to be the cone of $\mathscr{P}$ corresponding to ρ. Before specifying $\mathbf{u}(L)$, we decorate τ by taking $\mathbf{A}(w) = k[E] \in H_2(X_v)$. Then $\mathbf{A}(w) \cdot X_{v'} = k$. Since $E \subseteq X_v \cup X_{v'}$ and $E \cdot X_0 = 0$, necessarily $\mathbf{A}(w) \cdot (\mu X_v + \mu' X_{v'}) = 0$. From this we conclude that $\mathbf{A}(w) \cdot X_v = -k\mu'/\mu$, which is an integer by assumption. Thus there is a unique choice of $\mathbf{u}(L)$ compatible with these intersections, by Corollary 2.2.

In this case, we may apply Theorem 6.1 to see that $\mathscr{M}(X/S, \boldsymbol{\tau}) \cong \mathscr{M}(\overline{X}_v, \bar{\boldsymbol{\tau}})$ in the notation of §6. Here $\overline{X}_v$ will be the log scheme structure on $\underline{X}_v$ induced by the divisor $D_v \subseteq \underline{X}_v$. Note that any stable log map of curve class kE is a multiple cover of E. Via the comparison of logarithmic and relative invariants of [AMW], we may use the calculation of [GPS, Prop. 5.2] to obtain that $\deg[\mathscr{M}(X/S, \boldsymbol{\tau})]^{\mathrm{virt}} = (-1)^{k+1}/k^2$.

On the other hand, a simple calculation shows that $k_\tau = k$.

Our goal here is to give an inductive method for computing $k_\tau W_\tau$, the quantity which plays a key role in the construction of the canonical wall structure of [GS8]. The methods here are particular to relative dimension two, and the structure of these invariants in higher dimensions is more subtle.

8.2. **An invariant of Looijenga pairs.** Recall a *Looijenga pair* is a pair (X, D) where X is a non-singular projective rational surface and D is a reduced nodal anti-canonical divisor with at least one node. In what follows, assume that D has at least three irreducible components. In this case, D will be a cycle of rational curves, and we write $D = D_1 + \cdots + D_n$. We are again in the situation of [GS8], and may again take $B = \Sigma(X)$ and $\mathscr{P}$ the set of cones of $\Sigma(X)$. Then $B_0 := B \setminus \{0\}$ carries the structure of an integral affine manifold, see [GS8, §1.3] again, but the original construction was given in [GHK, §1.2]. Then Lemma 8.2 still holds in this case.

Example 8.6. Suppose we are in the situation of Assumptions 8.1, with $v \in \mathscr{P}'$ a vertex and X_v the corresponding irreducible component of D. With D_v the union of lower-dimensional strata of D contained in X_v, it follows from the assumption that $K_{X/S} + D = 0$ and adjunction that $K_{X_v} + D_v = 0$. Note that D_v is nodal, and then it is easy to see, e.g., by the classification of surfaces, that (X_v, D_v) is a Looijenga pair.

Fix a class β of logarithmic map of genus 0 to a Looijenga pair (X, D), with $q+1$ marked points, and contact orders $u_1, \ldots, u_q, u_{\mathrm{out}}$. Here, we assume that $u_k = w_k \nu_{i_k}$, $u_{\mathrm{out}} = w_{\mathrm{out}} \nu_{\mathrm{out}}$, where ν_{i_k} is a primitive generator of a ray ρ_{i_k} of $\mathscr{P}$ corresponding to the irreducible component D_{i_k} of D, and w_k is a positive integer, and similarly for w_{out} and ν_{out}. We do not assume the irreducible components D_{i_k} are distinct.

We then have a schematic evaluation map

$$\underline{\mathrm{ev}} : \mathscr{M}(X, \beta) \to \prod_{k=1}^{q} D_{i_k} \tag{8.2}$$

given by evaluating at the marked points with contact orders $u_1, \ldots, u_q$. It follows from a standard virtual dimension calculation via Riemann-Roch that the virtual dimension of $\mathscr{M}(X, \beta)$ is q. We define N_β by the identity

$$\underline{\mathrm{ev}}_*[\mathscr{M}(X, \beta)]^{\mathrm{virt}} = N_\beta \prod_{k=1}^{q} [D_{i_k}]. \tag{8.3}$$

We recall that subdivisions of $(B, \mathscr{P})$ correspond to log étale birational morphisms $(\widetilde{X}, \widetilde{D}) \to (X, D)$, see e.g., [GHK, Lem. 1.6] for this case. As the contact orders in β for X are integral points of B, they also determine a set of contact orders for $\widetilde{X}$. This set of contact orders along with a curve class $\widetilde{A} \in H_2(\widetilde{X})$ determines a class $\tilde{\beta}$ of logarithmic map to $\widetilde{X}$. We have:

Lemma 8.7. *Let $\pi : \widetilde{X} \to X$ be a log étale birational morphism with $\widetilde{X}$ non-singular. If $\mathscr{M}(X, \beta)$ is non-empty, then there exists a unique curve class $\widetilde{A} \in H_2(\widetilde{X})$ with $\pi_* \widetilde{A} = A$ such that $\mathscr{M}(\widetilde{X}, \tilde{\beta})$ is non-empty. Further, with this choice of curve class, $N_\beta = N_{\tilde{\beta}}$.*

Proof. We recall from [AW, Eq. (1)] that there is a Cartesian diagram (in all categories) with strict vertical arrows

$$\begin{tikzcd} \mathscr{M}(\widetilde{X}) \arrow[r, "\mathscr{M}(\pi)"] \arrow[d] & \mathscr{M}(X) \arrow[d] \\ \mathfrak{M}(\widetilde{\mathcal{X}} \to \mathcal{X}) \arrow[r, "\mathfrak{M}(\pi)"'] & \mathfrak{M}(\mathcal{X}) \end{tikzcd}$$

(where the moduli spaces have no restriction on type). Further, [AW, Prop. 5.2.1] shows that $\mathfrak{M}(\pi)$ is of pure degree 1, while [HW, Lem. 4.2] shows that $\mathfrak{M}(\pi)$ is proper. Taken together, this implies $\mathfrak{M}(\pi)$ is surjective, and hence the union of connected components of $\mathscr{M}(\widetilde{X})$ lying over $\mathscr{M}(X,\beta)$ is non-empty. Given a class $\tilde{\beta}$ of such a stable log map lying over $\mathscr{M}(X,\beta)$, the contact orders of $\tilde{\beta}$ are determined by the contact orders of β, and hence the only unknown is the curve class $\widetilde{A} \in H_2(\widetilde{X})$, which necessarily satisfies $\pi_*\widetilde{A} = A$ by construction of the map $\mathscr{M}(\pi)$. On the other hand, for any irreducible component $\widetilde{D}_\rho$ of $\widetilde{D}$, the intersection number $\widetilde{D}_\rho \cdot \widetilde{A}$ is completely determined by the contact orders of $\tilde{\beta}$, by Corollary 2.2. Since the intersection matrix of the set of exceptional curves of π is negative definite, and any two choices of lifts of A differ by a linear combination of exceptional curves, it follows that $\widetilde{A}$ is uniquely determined.

Identifying $\widetilde{D}_i$ with D_i using π, we now have a commutative diagram

$$\begin{tikzcd} \mathscr{M}(\widetilde{X},\tilde{\beta}) \arrow[d, "\mathscr{M}(\pi)"'] \arrow[dr, "\widetilde{\underline{\mathrm{ev}}'"] & \\ \mathscr{M}(X,\beta) \arrow[r, "\underline{\mathrm{ev}}"'] & \prod_{k=1}^q D_{i_k} \end{tikzcd}$$

By [AW, Thm. 1.1.1], $\mathscr{M}(\pi)_*[\mathscr{M}(\widetilde{X},\tilde{\beta})]^{\mathrm{virt}} = [\mathscr{M}(X,\beta)]^{\mathrm{virt}}$. The result follows. ♠

Alternatively, we may define N_β as follows. Choose points $x_k \in D_{i_k}$, $1 \le k \le q$, and write $\mathbf{x} := (x_1,\ldots,x_q)$. Define

$$\mathscr{M}(X,\beta,\mathbf{x}) := \mathscr{M}(X,\beta) \times_{\prod_k D_{i_k}} \mathbf{x}. \tag{8.4}$$

Proposition 8.8. *$\mathscr{M}(X,\beta,\mathbf{x})$ carries a virtual fundamental class of virtual dimension zero, and* $\deg[\mathscr{M}(X,\beta,\mathbf{x})]^{\mathrm{virt}} = N_\beta$.

Proof. Let $D_i^\circ \subseteq D_i$ be the open subset obtained by deleting the two double points of D contained in D_i. We then have a diagram:

$$\begin{tikzcd} \mathscr{M}(X,\beta,\mathbf{x}) \arrow[r] \arrow[d, "\varepsilon_{\mathbf{x}}"'] & \mathscr{M}(X,\beta)^\circ \arrow[r, "j''"] \arrow[d, "\varepsilon^\circ"'] & \mathscr{M}(X,\beta) \arrow[d, "\varepsilon"] \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta,\mathbf{x}) \arrow[r, "\iota'"] \arrow[d, "\underline{\mathrm{ev}}_{\mathbf{x}}"'] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)^\circ \arrow[r, "j'"'] \arrow[d, "\underline{\mathrm{ev}}^\circ"'] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta) \arrow[d, "\underline{\mathrm{ev}}"] \\ \mathbf{x} \arrow[r, "\iota"'] & \prod_{k=1}^q D_{i_k}^\circ \arrow[r, "j"'] & \prod_{k=1}^q D_{i_k} \end{tikzcd}$$

Here the evaluation space $\mathfrak{M}^{\mathrm{ev}}(\mathcal{X},\beta)$ is given by evaluation at the marked points with contact orders $u_1,\ldots,u_q$, and all other stacks are defined by the requirement that all squares are cartesian in the category of ordinary stacks. The relative obstruction theory for ε then pulls back to give relative obstruction theories for ε° and $\varepsilon_{\mathbf{x}}$.

By compatibility of flat pullback with proper pushfoward, and compatibility of virtual pullback with flat pullback, we have

$$\begin{aligned} N_\beta \prod_k [D^\circ_{i_k}] = j^*\left(N_\beta\prod_k [D_{i_k}]\right) &= j^*(\underline{\mathrm{ev}}\circ\varepsilon)_*[\mathscr{M}(X,\beta)]^{\mathrm{virt}}\\ &= (\underline{\mathrm{ev}}^\circ\circ\varepsilon^\circ)_*(j'')^*[\mathscr{M}(X,\beta)]^{\mathrm{virt}}\\ &= (\underline{\mathrm{ev}}^\circ\circ\varepsilon^\circ)_*[\mathscr{M}(X,\beta)^\circ]^{\mathrm{virt}}. \end{aligned}$$

Further, $\underline{\mathrm{ev}}^\circ$ is flat, being a base-change of an evaluation map $\mathfrak{M}(\mathcal{X},\beta)^\circ\rightarrow\prod_k [D^\circ_{i_k}/\mathbb{G}_m^2]$. Hence, the Gysin maps $\iota^!$ and $(\iota')^!$ agree, and by compatibility of Gysin maps with virtual pullback, we have

$$\begin{aligned} \deg[\mathscr{M}(X,\beta,\mathbf{x})]^{\mathrm{virt}} = \deg(\underline{\mathrm{ev}}_{\mathbf{x}}\circ\varepsilon_{\mathbf{x}})_*[\mathscr{M}(X,\beta,\mathbf{x})]^{\mathrm{virt}} &= \deg(\underline{\mathrm{ev}}_{\mathbf{x}}\circ\varepsilon_{\mathbf{x}})_*\iota^![\mathscr{M}(X,\beta)^\circ]^{\mathrm{virt}}\\ &= \deg\iota^!(\underline{\mathrm{ev}}^\circ\circ\varepsilon^\circ)_*[\mathscr{M}(X,\beta)^\circ]^{\mathrm{virt}}\\ &= N_\beta, \end{aligned}$$

as desired. ♠

While we will only need the following result in [GHKS], its method of proof will be used in §8.4.

Lemma 8.9. *Let (X,D), β be as above. Suppose given a stable log map $f:C\rightarrow X$ in $\mathscr{M}(X,\beta)$ lying in $\underline{\mathrm{ev}}^{-1}(x_1,\ldots,x_q)$ where none of the x_i are double points of D. Then $f(C)\cap D$ is a finite set.*

Proof. Denote by $p_1,\ldots,p_q,p_{\mathrm{out}}\in C$ the marked points with contact orders $u_1,\ldots,u_q,u_{\mathrm{out}}$ respectively, so that $f(p_k)=x_k$. However, we do not know the value of $f(p_{\mathrm{out}})$.

Let f be of tropical type $\tau=(G,\boldsymbol{\sigma},\mathbf{u})$, and fix $s\in\mathrm{Int}(\tau)$, yielding a tropical map $h_s:G\rightarrow B$. We write the legs of G corresponding to the marked points as $L_1,\ldots,L_q,L_{\mathrm{out}}$. Of course G may have many vertices, but h_s is balanced at those vertices which don't map to the origin of B. Also, as by assumption the marked points $p_1,\ldots,p_q$ of C do not map to a double point of D, the image under h_s of each leg L_k of G lies in a ray of $\mathscr{P}$. Now note that if $f(C)\cap D$ is not finite, then $f(C)$ necessarily contains a double point of D, and then the image of h_s must intersect the interior of a two-dimensional cone of $\mathscr{P}$. Thus it is now enough to show that the image of h_s is contained in the one-skeleton of $\mathscr{P}$.

Suppose this is not the case. First suppose there is a $v\in V(G)$ such that $h_s(v)\in\mathrm{Int}(\sigma)$ for a two-dimensional cone $\sigma\in\mathscr{P}$. As such a vertex corresponds to a contracted irreducible component of C, by stability, there must be at least three edges or legs adjacent to v. Further, the only possible leg adjacent to v is L_{out}. However, necessarily $h_s(L_{\mathrm{out}})$ is parallel to the ray $\boldsymbol{\sigma}(L_{\mathrm{out}})$ of $\mathscr{P}$, and in particular, is not contained in the ray $\mathbb{R}_{\geq 0}h_s(v)$. Then it follows easily from the balancing condition that one of the two possibilities occur: (1) There are at least two edges or legs E_1,E_2 adjcaent to v

with $h_s(E_i)$ not contained in the ray $\mathbb{R}_{\geq 0}h_s(v)$, with $h_s(E_1)$ mapping into one side of this ray and $h_s(E_2)$ mapping into the other. (2) No leg is adjacent to v and all edges E_i adjacent to v satisfy $h_s(E_i) \subseteq \mathbb{R}_{\geq 0}h_s(v)$. In the latter case, again by balancing, there must be another vertex v' with $h_s(v') \in \mathbb{R}_{>0}h_s(v)$ with $h_s(v)$ lying between 0 and $h_s(v')$. Repeating this argument, eventually we will come to a vertex where case (1) holds.

Thus we may assume case (1) holds at the vertex v. Choose one of these two edges which is not a leg, say E_1. It has another vertex v'. Note that $h_s(v')$ is not the origin in B, so the balancing condition holds for h at v'. Thus, again there must be another edge or leg adjacent to v' which maps to the other side of the ray $\mathbb{R}_{\geq 0}h_s(v')$. Continuing in this fashion, there are three possibilities: (1) We arrive at L_{out}, in which case we made the wrong initial choice of edge E_i. Choose the other edge instead. (2) We get an infinite sequence of edges, of course impossible. (3) We obtain a loop, in which case the genus of C is positive. But C is assumed to be genus zero. Thus we arrive at a contradiction.

If there are no vertices of G mapping into the interior of a two-dimensional cone, but there is an edge E of G with $h_s(E)$ intersecting the interior of a two-dimensional cone, then the endpoints of E must map to different rays of $\mathscr{P}$. We can then repeat the same argument as above, starting at either endpoint of E.

So in either case we obtain a contradiction, showing the result. ♠

We observe that in the case (X, D) is a toric pair, the invariant N_β has already been encountered in [GPS]. In [GPS, §3], a number $N^{\mathrm{hol}}_{\mathbf{m}}(\mathbf{w})$ is defined. Here, we start with a toric surface X with fan Σ in a two-dimensional vector space $N_{\mathbb{R}}$; of course, the tropicalization of the toric pair (X, D) is $(N_{\mathbb{R}}, \Sigma)$. Let $m_1, \ldots, m_n$ be primitive generators of distinct rays $\rho_1, \ldots, \rho_n$ in Σ. Write $\mathbf{m} = (m_1, \ldots, m_n)$. Further, write $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ with $\mathbf{w}_i = (w_{i1}, \ldots, w_{in_i})$. Suppose we may write

$$w_{\mathrm{out}}m_{\mathrm{out}} = -\sum_{i,j} w_{ij}m_i$$

with m_{out} a primitive generator of a ray ρ_{out} of Σ. Let $D_1, \ldots, D_n, D_{\mathrm{out}}$ be the toric divisors of X corresponding to $\rho_1, \ldots, \rho_n, \rho_{\mathrm{out}}$. Then it is standard (see e.g., [GHK, Lem. 1.13]) that this data determines a unique curve class A such that for each prime toric divisor D, we have

$$A \cdot D = \begin{cases} \sum_j w_{ij} & D = D_i \text{ and } D \neq D_{\mathrm{out}}; \\ w_{\mathrm{out}} + \sum_j w_{ij} & D = D_i = D_{\mathrm{out}}; \\ w_{\mathrm{out}} & D = D_{\mathrm{out}} \text{ and } D \neq D_i \text{ for any } i; \\ 0 & \text{otherwise.} \end{cases} \tag{8.5}$$

Thus the above data determines a class of log curve β with underlying curve class A and marked points with set of contact orders $\{w_{ij}m_{ij} \,|\, 1 \leq i \leq n, 1 \leq j \leq n_i\} \cup \{w_{\mathrm{out}}m_{\mathrm{out}}\}$.

Lemma 8.10. *In the above situation, $N_\beta = N^{\mathrm{hol}}_{\mathbf{m}}(\mathbf{w})$.*

Proof. This follows from the description of N_β from Proposition 8.8, the definition of $N^{\mathrm{rel}}_{\mathbf{m}}(\mathbf{w})$ of [GPS, (4.7)] and [GPS, Thm. 4.4], as well as the comparison between logarithmic and relative moduli spaces of [AMW]. In more modern language, one proceeds as follows. By adapting the argument of [GPS, Prop. 4.3] to stable log maps, one shows that in fact for general choice of $\mathbf{x} = (x_{ij})_{1\le i\le n, 1\le j\le n_i}$, the moduli space $\mathscr{M}(X,\beta,\mathbf{x})$ is particularly simple. It consists of a finite number of reduced points, each corresponding to a log map from an irreducible domain curve C with $f^{-1}(D)$ consisting only of the marked points on C. The number $N^{\mathrm{hol}}_{\mathbf{m}}(\mathbf{w})$ is defined precisely to be the count of such maps. ♠

We give one further result required for [GHKS]. Suppose given an inclusion $\widetilde{N}\subseteq N$ of rank two lattices and a fan Σ in N determining a toric surface X. This also determines a fan $\widetilde{\Sigma}$ for $\widetilde{N}$, with corresponding toric surface $\widetilde{X}$. The inclusion $\widetilde{N}\subseteq N$ determines a covering $p:\widetilde{X}\rightarrow X$ of degree $\mu=[\widetilde{N}:N]$. Suppose given a class $\tilde\beta$ of log curve with underlying curve class $\widetilde{A}$ for the toric pair $(\widetilde{X},\widetilde{D})$ with contact orders $\tilde u_1,\ldots,\tilde u_n,\tilde u_{\mathrm{out}}\in\widetilde{N}$ contained in rays of $\widetilde{\Sigma}$, with $\tilde u_i=\tilde w_i\tilde\nu_i$ for $\tilde\nu_i$ primitive. Suppose $\tilde u_{\mathrm{out}}+\sum_i\tilde u_i=0$, and $\widetilde{A}$ is the unique curve class determined by this data as in (8.5). Then via the inclusion $\widetilde{N}\subseteq N$, the contact orders $\tilde u_i$, $\tilde u_{\mathrm{out}}$ may be viewed as contact orders u_i for the toric pair (X,D). (Note that under the inclusion $\widetilde{N}\subseteq N$, we have $\tilde u_i=u_i$.) This determines similarly a class β of log curve for the pair (X,D). Write $u_i=w_i\nu_i$ as bofore, with ν_i primitive in N. Of course, $\tilde w_i|w_i$.

Lemma 8.11. *In the above situation,*

$$\mu N_\beta = N_{\tilde\beta}\prod_{i=1}^n\frac{\tilde\mu\tilde w_i}{w_i}.$$

Proof. One first notes via elementary toric geometry that $p_i=p|_{\widetilde{D}_i}:\widetilde{D}_i\rightarrow D_i$ is the lattice index $[\widetilde{N}/\tilde\nu_i\mathbb{Z}:N/\nu_i\mathbb{Z}]$, which is $\tilde\mu w_i/w_i$. Further, in a neighbourhood of a point of $\widetilde{D}_i$, the map p has ramification of degree $w_i/\tilde w_i$ along $\widetilde{D}_i$.

Composing stable log maps with p gives a map $q:\mathscr{M}(\widetilde{X},\tilde\beta)\rightarrow\mathscr{M}(X,\beta)$. Note that since the targets are toric pairs, and the log tangent bundle of a toric variety is trivial, these moduli spaces are unobstructed. Hence their virtual fundamental class and fundamental class coincide. Further, the degree of the map q is μ. Indeed, choose a general closed point in $\mathscr{M}(X,\beta)$, corresponding to a stable log map $f:C\rightarrow X$. Because it is general, then as in the proof of Lemma 8.10, C is an irreducible curve with $f^{-1}(D)$ consisting only of the marked points of C. Then $C':=C\times_X\widetilde{X}$ is a (in general) non-normal curve. In a local coordinate t near the marked point p_i with contact order u_i and local coordinates near $f(p_i)$, $\widetilde{X}\rightarrow X$ takes the form $(x,y)\mapsto(x^{w_i/\tilde w_i},y)$ (with $\widetilde{D}_i$ given by $x=0$) and f is given by $t\mapsto(t^{w_i}\varphi(t),\psi(t))$, where $\varphi(t)$ is invertible and $\psi(t)$ is arbitrary. Then the local equation for C' in $\mathbb{A}^2$ with coordinates x,t is

$$x^{w_i/\tilde w_i}-t^{w_i}\varphi(t)=0.$$

Note that étale locally we may find a function $\bar\varphi(t)$ with with $\bar\varphi^{w_i/\tilde w_i}=\varphi$, so that the left-hand side of the above equation factors as $\prod_{i=1}^{w_i/\tilde w_i}(x-\zeta^i t^{\tilde w_i}\bar\varphi(t))$, with ζ a primitive root of unity. Thus, after normalizing C' to obtain a curve $\widetilde{C}$, we see that $\widetilde{C}\rightarrow C$ is an étale map, necessarily of degree μ. Since $C\cong\mathbb{P}^1$, $\widetilde{C}$ must split into μ connected components, each isomorphic to $\mathbb{P}^1$. In addition, the composition $\widetilde{C}\rightarrow C'\rightarrow\widetilde{X}$ of the normalization with the projection to $\widetilde{X}$ then induces a stable map on each irreducible component. From the equations above, one sees that each such stable map is tangency order $\tilde{w}_i$ with $\widetilde{D}_i$ and hence gives an element of the fibre of q. Conversely, by the universal property of the fibre product, any map in the fibre of q must arise in this way. Thus the degree of q is μ.

We now consider the commutative diagram

$$\begin{tikzcd} \mathscr{M}(\widetilde{X},\tilde{\beta}) \arrow[r, "q"] \arrow[d, "\underline{\widetilde{\mathrm{ev}}}"'] & \mathscr{M}(X,\beta) \arrow[d, "\underline{\mathrm{ev}}"] \\ \prod_{i=1}^n \widetilde{D}_i \arrow[r, "\prod p_i"'] & \prod_{i=1}^n D_i \end{tikzcd}$$

We then have

$$\left(\prod p_i\right)_* \underline{\widetilde{\mathrm{ev}}}_*[\mathscr{M}(\widetilde{X},\tilde{\beta})] = N_{\tilde{\beta}} \deg\left(\prod p_i\right)\prod[D_i]$$

while

$$\underline{\mathrm{ev}}_* q_*[\mathscr{M}(\widetilde{X},\tilde{\beta})] = \mu N_\beta \prod[D_i].$$

Since $\deg\left(\prod p_i\right) = \prod(\mu\tilde{w}_i/w_i)$, the result follows. ♠

8.3. **Gluing via Parker and Ranganathan's triple point method.** We now return to the situation of Assumptions 8.1. To state the main result of this section, fix a decorated wall type $\boldsymbol{\tau} = (G, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{A})$, and let $v_{\mathrm{out}} \in V(G)$ be the vertex adjacent to L_{out}. Let $h : \Gamma(G,\ell) \to \Sigma(X)$ be the universal family of tropical maps of type $\boldsymbol{\tau}$. Let $E_1, \ldots, E_q \in E(G)$ be the edges adjacent to v_{out} (with $q \geq 0$). If we split G at the edges $E_1, \ldots, E_q$, we obtain connected components $G_1, \ldots, G_q$ and G_{out}, where v_{out} is the unique vertex of G_{out} and E_i becomes the unique leg of G_i, which we will denote as $L_{i,\mathrm{out}}$. This in turn gives rise to decorated types $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q, \boldsymbol{\tau}_{v_{\mathrm{out}}}$.

Lemma 8.12. *$\boldsymbol{\tau}_i$ is a decorated wall type.*

Proof. Recall we write $h' : G \to B'$ for the unique tropical map to B' of type τ. If $u_{i,\mathrm{out}} := \mathbf{u}(L_{i,\mathrm{out}}) = 0$, then as this coincides with $\mathbf{u}(E_i)$, h' contracts the edge E_i. However, the length of the edge E_i is a parameter in the moduli space of tropical maps of type τ, and hence this edge length may vary freely. This contradicts the rigidity of h'. Thus Definition 8.3,(1) holds. Item (2) of that definition holds because it holds for $\boldsymbol{\tau}$.

For item (3) of Definition 8.3, to show $\dim \tau_i = 1$, it is sufficient to show that $h'_i := h'|_{G_i}$ is rigid. If it deformed as a tropical map to B' in a way so that the image of $L_{i,\mathrm{out}}$ also deforms to a line segment not passing through $h'(v_{\mathrm{out}})$, then this violates [GS8, Lem. 2.5,(1)]. Otherwise, if h'_i deforms so that the image of the leg $L_{i,\mathrm{out}}$ continues to pass through $h'(v_{\mathrm{out}})$, then this deformation can be glued to produce a deformation of h', again a contradiction. Thus $\dim \tau_i = 1$ (remembering that τ_i parameterizes maps to B rather than B') and, as $u_{i,\mathrm{out}} \neq 0$, one sees that $\dim h_i(\tau_{i,\mathrm{out}}) = 2$.

Thus $\boldsymbol{\tau}_i$ is a decorated wall type. ♠

Construction 8.13. Continuing with a fixed decorated wall type $\boldsymbol{\tau}$, our next goal is to associate an enumerative invariant of the form defined in the previous subsection to the vertex $v_{\rm out}$ of G. Let $x := h'(v_{\rm out}) \in B'$. We will first construct: (1) a pair $(\overline{B}_x, \widetilde{\Sigma}_x)$ of integral affine surface with singularity $\overline{B}_x$ and a polyhedral cone decomposition $\widetilde{\Sigma}_x$; (2) a Looijenga pair $(\widetilde{X}_x, \widetilde{D}_x)$ which tropicalizes to $(\overline{B}_x, \widetilde{\Sigma}_x)$; (3) a class $\boldsymbol{\tau}_{\rm out}$ of logarithmic map to $(\widetilde{X}_x, \widetilde{D}_x)$.

In what follows, let $\sigma'_x \in \mathscr{P}'$ be the minimal cell containing x, $\sigma_x \in \mathscr{P}$ the minimal cone of $\mathscr{P}$ containing x. We write X_x for the stratum X_{σ_x} of X.

Construction of a pair $(\overline{B}_x, \Sigma_x)$. First, we define $\overline{B}_x$ an affine manifold with singularities along with a decomposition Σ_x into not-necessarily strictly convex polyhedral cones. If $x \in B'_0$, let $v_x \in \Lambda_{B,x}$ be a primitive integral tangent vector to the ray $\mathbb{R}_{\geq 0} x \subseteq B_0$. Then $\overline{B}_x$ is identified with $\Lambda_{B,\mathbb{R},x}/\mathbb{R}v_x$, with lattice structure coming from $\Lambda_{B,x}/\mathbb{Z}v_x$. In this case, we define

$$\Sigma_x := \{(\sigma + \mathbb{R}v_x)/\mathbb{R}v_x \,|\, \sigma_x \subseteq \sigma \in \mathscr{P}\}. \tag{8.6}$$

Specifically, if $\dim \sigma'_x = 2$, then Σ_x just consists of one cone, namely all of $\overline{B}_x$. If $\dim \sigma'_x = 1$, then Σ_x consists of two half-spaces and their common face, the image of the tangent space to σ'_x.

If $x \in \Delta'$, i.e., x is a vertex of $\mathscr{P}'$, we obtain $(\overline{B}_x, \Sigma_x)$ as the tropicalization of the corresponding irreducible component X_x of D. Here X_x carries the divisorial log structure coming from the divisor $D_x := \partial X_x$, as in §6. Note that by the discussion in §6, in this case, (8.6) is still the correct description of Σ_x. However, further, as in §8.2, $\overline{B}_x$ also carries an integral affine structure with a singularity at the origin.

Refinement $\widetilde{\Sigma}_x$ **of** Σ_x**.** Note that by construction and the assumption that $\boldsymbol{\tau}$ is a wall-type, for any edge or leg E adjacent to $v_{\rm out} \in V(G)$, $h(\tau_E)$ is two-dimensional and $\rho_E := (h(\tau_E) + \mathbb{R}v_x)/\mathbb{R}v_x$ is a ray in $\overline{B}_x$.

We now let $\widetilde{\Sigma}_x$ be a refinement of Σ_x chosen so that (1) every cone in $\widetilde{\Sigma}_x$ is a strictly convex rational polyhedral cone integral affine isomorphic to the standard cone $\mathbb{R}^2_{\geq 0}$, and (2) the rays ρ_{E_i}, $1 \leq i \leq q$ and $\rho_{L_{\rm out}}$ are one-dimensional cones of $\widetilde{\Sigma}_x$. In what follows, the precise choice of $\widetilde{\Sigma}_x$ will be unimportant, see Lemma 8.7.

The type $\tau_{\rm out}$**.** Note that the adjacent edges to $v_{\rm out} \in V(G)$ determine a type $\tau_{\rm out} = (G_{\rm out}, \boldsymbol{\sigma}_{\rm out}, \mathbf{u}_{\rm out})$ of tropical map to $(\overline{B}_x, \widetilde{\Sigma}_x)$, as follows. First, $G_{\rm out}$ is the graph underlying $\boldsymbol{\tau}_{v_{\rm out}}$ as previously described, with only one vertex $v_{\rm out}$ and legs $E_1, \ldots, E_q, L_{\rm out}$. We set $\boldsymbol{\sigma}_{\rm out}(v_{\rm out}) = \{0\} \in \widetilde{\Sigma}_x$, $\boldsymbol{\sigma}_{\rm out}(E_i) = \rho_{E_i}$ and $\boldsymbol{\sigma}_{\rm out}(L_{\rm out}) = \rho_{L_{\rm out}}$. Finally, the images of $\mathbf{u}(E_i)$ (with the edges E_i oriented away from $v_{\rm out}$) and $\mathbf{u}(L_{\rm out})$ under the quotient map given by dividing out by $\mathbb{R}v_x$ yield $\mathbf{u}_{\rm out}(E_i), \mathbf{u}_{\rm out}(L_{\rm out})$.

The pair $(\widetilde{X}_x, \widetilde{D}_x)$**.** If $\dim \sigma'_x = 2$ or 1, then we may interpret the pair $(\overline{B}_x, \widetilde{\Sigma}_x)$ as a fan, hence defining a toric variety $\widetilde{X}_x$ with toric boundary $\widetilde{D}_x$. In particular, $(\overline{B}_x, \widetilde{\Sigma}_x)$ is the tropicalization of the pair $(\widetilde{X}_x, \widetilde{D}_x)$. In both these cases, there is a morphism $\pi : \widetilde{X}_x \to X_x$. Indeed, if $\dim \sigma'_x = 2$, this is just a constant map to a point. If $\dim \sigma'_x = 1$, consider the quotient map $\overline{B}_x \to \mathbb{R}$ given by dividing out by the tangent space to σ'_x. In this case $\mathbb{R}$ carries the fan $\Sigma_{\mathbb{P}^1}$ defining $\mathbb{P}^1$, and the quotient map induces a map of fans $\widetilde{\Sigma}_x \to \Sigma_{\mathbb{P}^1}$, hence defining a morphism $\pi : \widetilde{X}_x \to \mathbb{P}^1$. This $\mathbb{P}^1$ can be identified with X_x.

If, on the other hand, $\dim \sigma'_x = 0$, then the refinement $\widetilde{\Sigma}_x$ of Σ_x defines a toric blow-up $\pi : \widetilde{X}_x \to X_x$. We take $\widetilde{D}_x$ to be the strict transform of D_x.

Decorating τ_{out}. Finally, we define a curve class $\widetilde{A}_{\mathrm{out}}$ to turn τ_{out} into a decorated type $\boldsymbol{\tau}_{\mathrm{out}}$ for log maps to the pair $(\widetilde{X}_x, \widetilde{D}_x)$. Note we already have a curve class $A_{\mathrm{out}} = \mathbf{A}(v_{\mathrm{out}}) \in H_2(X_x)$. Of course, if $\dim \sigma'_x = 2$, this curve class is 0, and if $\dim \sigma'_x = 1$, this curve class is some multiple of the class of X_x.

We first introduce some additional notation. For each ray $\rho \in \widetilde{\Sigma}_x$, denote by $\widetilde{D}_\rho$ the corresponding irreducible component of $\widetilde{D}_x$. For each leg $L \in L(G_{\mathrm{out}})$, denote by w_L for the index of $\mathbf{u}_{\mathrm{out}}(L)$, i.e., $\mathbf{u}_{\mathrm{out}}(L)$ is w_L times a primitive tangent vector. This represents the order of tangency imposed by the contact order $\mathbf{u}_{\mathrm{out}}(L)$ with the divisor $\widetilde{D}_{\boldsymbol{\sigma}_{\mathrm{out}}(L)}$.

Lemma 8.14. *There is at most one curve class $\widetilde{A}_{\mathrm{out}} \in H_2(\widetilde{X}_x)$ with $\pi_* \widetilde{A}_{\mathrm{out}} = A_{\mathrm{out}}$ and for ρ any ray in $\widetilde{\Sigma}_x$,*

$$\widetilde{A}_{\mathrm{out}} \cdot \widetilde{D}_\rho = \sum_{\substack{L \in L(G_{\mathrm{out}}) \\ \boldsymbol{\sigma}_{\mathrm{out}}(L) = \rho}} w_L. \tag{8.7}$$

If $W_{\boldsymbol{\tau}} \neq 0$, then such an $\widetilde{A}_{\mathrm{out}}$ exists.

Proof. If $\dim \sigma'_x = 2$ or 1, the unique existence of a class $\widetilde{A}_{\mathrm{out}}$ satisfying (8.7) is easy. Indeed, there is a unique tropical map of type τ_{out}, and by the balancing condition at v_{out}, this defines a balanced tropical map to $\overline{B}_x$ with all legs mapping to rays of $\widetilde{\Sigma}_x$. As is standard, this defines a curve class $\widetilde{A}_{\mathrm{out}} \in H_2(\widetilde{X}_x)$, see for example [GHK, Lem. 1.13]. Furthermore, it is characterized precisely by the intersection numbers with the boundary divisors as in (8.7).

If $\dim \sigma'_x = 2$, then $\pi_* \widetilde{A}_{\mathrm{out}} = A_{\mathrm{out}}$ is trivial as $A_{\mathrm{out}} = 0$, and thus the last statement on the existence of $\widetilde{A}_{\mathrm{out}}$ is vacuous.

If $\dim \sigma'_x = 1$, then we need to verify that if $W_{\boldsymbol{\tau}} \neq 0$, then $\pi_* \widetilde{A}_{\mathrm{out}} = A_{\mathrm{out}}$. Let $\boldsymbol{\tau}_{v_{\mathrm{out}}}$ be the type of punctured map to X/S corresponding to the vertex v_{out} after splitting $\boldsymbol{\tau}$ at the edges $E_1, \ldots, E_q$. Necessarily, if $W_{\boldsymbol{\tau}} \neq 0$, then the moduli space $\mathscr{M}(X/S, \boldsymbol{\tau}_{v_{\mathrm{out}}})$ is non-empty. The requirement that this moduli space be non-empty then allows us to determine A_{out}, using Corollary 2.2. In particular, $A_{\mathrm{out}} = d[X_x]$ for some $d \geq 0$, and d can be determined by intersecting A_{out} with an irreducible component of D transverse to X_v. This is calculated as follows. Let $\sigma'_1, \sigma'_2 \in \mathscr{P}'$ be the two two-cells containing σ'_x, with additional vertices v_1, v_2 respectively not contained in σ'_x. Also write $\sigma_1, \sigma_2 \in \mathscr{P}$ for the corresponding cones in $\mathscr{P}$. Then the corresponding irreducible components X_{v_i} of D each meet X_x transversally in one point.

By Corollary 2.2, we may now calculate $d = X_{v_1} \cdot \mathbf{A}(v_{\mathrm{out}})$ as follows. Let $\delta : \Lambda_{B,x} \to \mathbb{Z}$ be the quotient map by the tangent space to $\sigma_x \in \mathscr{P}$, with sign chosen so that elements of $\Lambda_{B,x}$ pointing into σ_1 map to positive integers. Then

$$d = \sum_E \delta(\mathbf{u}(E)),$$

where E runs over edges and legs adjacent to v_{out} with $\boldsymbol{\sigma}(E) = \sigma_1$. On the other hand, a simple toric argument shows that given our definition of $\widetilde{A}_{\mathrm{out}}$, $\pi_* \widetilde{A}_{\mathrm{out}} = d[X_x]$ for the same choice of d. This completes the argument in the $\dim \sigma'_x = 1$ case.

Finally, consider the case $\dim \sigma'_x = 0$. For the uniqueness statement, we argue as in the proof of Lemma 8.7: the intersection matrix of the exceptional locus is negative definite. In particular, any two lifts $\widetilde{A}_{\mathrm{out}}$ of A_{out} differ by a linear combination of exceptional divisors. The uniqueness of the lift with the given intersection numbers with the boundary divisors then follows.

For the second statement, suppose $W_{\boldsymbol{\tau}} \neq 0$. Let $\boldsymbol{\tau}_{v_{\mathrm{out}}}$ be as in the $\dim \sigma'_x = 1$ case, so that if $W_{\boldsymbol{\tau}} \neq 0$, then the moduli space $\mathscr{M}(X/S, \boldsymbol{\tau}_{v_{\mathrm{out}}})$ is non-empty. By Theorem 6.1, this moduli space is isomorphic to $\mathscr{M}(X_x, \bar{\boldsymbol{\tau}}_{v_{\mathrm{out}}})$, where $\bar{\boldsymbol{\tau}}_{v_{\mathrm{out}}}$ is the type of tropical map to $(\overline{B}_x, \Sigma_x)$ constructed from $\boldsymbol{\tau}_{v_{\mathrm{out}}}$ as in §6. On the other hand, the type τ_{out} is a type of tropical map to $(\overline{B}_x, \widetilde{\Sigma}_x)$ with the same underlying graph and contact orders as $\bar{\boldsymbol{\tau}}_{v_{\mathrm{out}}}$. It then follows as in the proof of Lemma 8.7 that there must be at a curve class $\widetilde{A}_{\mathrm{out}}$ with $\pi_* \widetilde{A}_{\mathrm{out}} = A_{\mathrm{out}}$, determining a type $\boldsymbol{\tau}_{\mathrm{out}}$ with $\mathscr{M}(\widetilde{X}_x, \boldsymbol{\tau}_{\mathrm{out}})$ non-empty. However, necessarily $\widetilde{A}_{\mathrm{out}}$ satisfies (8.7) by Corollary 2.2. ♠

The invariant $N_{\boldsymbol{\tau}_{\mathrm{out}}}$. We have now constructed a pair $(\widetilde{X}_x, \widetilde{D}_x)$, a curve class $\widetilde{A}_{\mathrm{out}} \in H_2(\widetilde{X}_x)$, and hence a class $\boldsymbol{\tau}_{\mathrm{out}}$ of log map to $\widetilde{X}_x$. Thus we obtain an invariant $N_{\boldsymbol{\tau}_{\mathrm{out}}}$ as defined in (8.3), independent of the choice of $\widetilde{\Sigma}_x$ by Lemma 8.7.

The application of our gluing formalism is then:

Theorem 8.15. *We have*

$$k_\tau W_{\boldsymbol{\tau}} = \frac{w_{L_{\mathrm{out}}} N_{\boldsymbol{\tau}_{\mathrm{out}}} \prod_{i=1}^q k_{\tau_i} W_{\boldsymbol{\tau}_i}}{|\operatorname{Aut}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q)|}, \tag{8.8}$$

where the automorphism group of the denominator is the set of permutations σ of $\{1, \ldots, q\}$ such that $\boldsymbol{\tau}_i$ is isomorphic to $\boldsymbol{\tau}_{\sigma(i)}$ as decorated types.

Proof. We have a standard gluing situation obtained by splitting $\boldsymbol{\tau}$ at the edges $E_1, \ldots, E_q$. In general this gluing situation will not be tropically transverse. However, in this case a relatively mild birational étale modification of X takes care of this. Flatness of the map $\underline{\mathrm{ev}}$ in Theorem 5.1 also is an issue, but this is dealt with via Parker [Pa] and Ranganathan's approach [Ra] to this situation.

Step I. *Refining* $\Sigma(X)$. Note that a refinement of the polyhedral decomposition $\mathscr{P}'$ of B' into rational convex polyhedra gives a refinement of $(B, \mathscr{P})$, i.e., of $\Sigma(X)$, and hence a log étale modification $\pi : \widetilde{X} \to X$. In particular, we choose a refinement $\widetilde{\mathscr{P}'}$ of $\mathscr{P}'$ with the following properties: (1) x is a vertex of $\widetilde{\mathscr{P}'}$; (2) the integral affine manifold with singularity $(\overline{B}_x, \Sigma_x)$ coincides with $(\overline{B}_x, \widetilde{\Sigma}_x)$ in the notation of Construction 8.13. It is not difficult to see that this can be done as a series of toric blow-ups, and hence $\widetilde{X} \to X$ is projective. We omit the details.

Step II. *Lifting the type $\boldsymbol{\tau}$.* Having chosen the log étale modification $\widetilde{X} \to X$, we use [J22] to choose a *lift* of τ to $\Sigma(\widetilde{X})$. In general, there may be many choices of lift of a type, but in the rigid case, the description of lifts given in [J22, §4] reduces to a quite simple procedure. Given the map $h' : G \to B'$, we first take the minimal refinement

$\widetilde{G}$ of the graph G (i.e., subdivide edges or legs via the addition of vertices) with the property that for any edge or leg E of $\widetilde{G}$, $h'(E)$ is contained in an element of $\widetilde{\mathscr{P}'}$. There remains, however, some ambiguity as to the treatment of L_{out}, as it might have been subdivided into a number of edges and one leg. We choose to discard all but the edge (or leg) adjacent to v_{out}, so that it is now a leg, which we take to be the unique leg of $\widetilde{G}$. This provides a type $\tilde{\tau} = (\widetilde{G}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}})$, a lifting of τ in the terminology of [J22, §4].

Now it is shown in the proof of [J22, Cor. 9.4] that

$$k_\tau W_{\boldsymbol{\tau}} = k_{\tilde{\tau}} \sum_{\tilde{\boldsymbol{\tau}}} W_{\tilde{\boldsymbol{\tau}}}, \tag{8.9}$$

where the sum is over all decorations $\tilde{\boldsymbol{\tau}}$ of $\tilde{\tau}$ with $\pi_*\widetilde{\mathbf{A}} = \mathbf{A}$. What this means is that if $v \in V(\widetilde{G})$ is contained in the interior of an edge or leg E of G, then $\pi_* : H_2(\widetilde{X}_{\tilde{\boldsymbol{\sigma}}(v)}) \to H_2(X_{\boldsymbol{\sigma}(E)})$ satisfies $\pi_*\widetilde{\mathbf{A}}(v) = 0$. If on the other hand $v \in V(\widetilde{G})$ is a vertex which is also a vertex of G, then $\pi_* : H_2(\widetilde{X}_{\tilde{\boldsymbol{\sigma}}(v)}) \to H_2(X_{\boldsymbol{\sigma}(v)})$ satisfies $\pi_*\widetilde{\mathbf{A}}(v) = \mathbf{A}(v)$.

We now note that for a given type $\tilde{\boldsymbol{\tau}}$ appearing in (8.9), $W_{\tilde{\boldsymbol{\tau}}} = 0$ unless $\widetilde{\mathbf{A}}(v_{\text{out}})$ coincides with the curve class $\widetilde{A}_{\text{out}}$ constructed from $\boldsymbol{\tau}$ in Lemma 8.14. Thus we may assume this equality and the condition of Lemma 8.14 in the sequel.

The type $\tilde{\boldsymbol{\tau}}$ now gives rise to the invariant $N_{\tilde{\boldsymbol{\tau}}_{\text{out}}}$, and $N_{\boldsymbol{\tau}_{\text{out}}} = N_{\tilde{\boldsymbol{\tau}}_{\text{out}}}$ by Lemma 8.7. We also have formula (8.9) for $W_{\boldsymbol{\tau}_i}$ using the lift $\tilde{\tau}_i$ of τ_i induced by restricting the lift $\tilde{\tau}$ to G_i. Note here we use what [J22] calls a maximally extended lift, in that we don't remove any segments from the subdivided leg $L_{i,\text{out}}$. We now observe it is sufficient to prove (8.8) after replacing $\boldsymbol{\tau}$ with $\tilde{\boldsymbol{\tau}}$ and $\boldsymbol{\tau}_i$ with $\tilde{\boldsymbol{\tau}}_i$. Indeed, if (8.8) has been proved in this case, then the left-hand-side of (8.8) can be expanded using (8.9), giving

$$k_\tau W_{\boldsymbol{\tau}} = \sum_{\tilde{\boldsymbol{\tau}}} w_{L_{\text{out}}} N_{\boldsymbol{\tau}_{\text{out}}} \frac{\prod_{i=1}^q k_{\tilde{\tau}_i} W_{\tilde{\boldsymbol{\tau}}_i}}{|\operatorname{Aut}(\tilde{\boldsymbol{\tau}}_1, \ldots, \tilde{\boldsymbol{\tau}}_q)|}.$$

Here $\tilde{\boldsymbol{\tau}}_1, \ldots, \tilde{\boldsymbol{\tau}}_q$ are the decorated types induces by $\tilde{\boldsymbol{\tau}}$. If instead, one sums over all choices of decorations of the lifts $\tilde{\tau}_1, \ldots, \tilde{\tau}_q$, the same choice of $\tilde{\boldsymbol{\tau}}$ will occur

$$|\operatorname{Aut}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q)| / |\operatorname{Aut}(\tilde{\boldsymbol{\tau}}_1, \ldots, \tilde{\boldsymbol{\tau}}_q)|$$

times. Thus we obtain

$$\begin{aligned} k_\tau W_{\boldsymbol{\tau}} &= \sum_{\tilde{\boldsymbol{\tau}}_1, \ldots, \tilde{\boldsymbol{\tau}}_q} w_{L_{\text{out}}} N_{\boldsymbol{\tau}_{\text{out}}} \prod_{i=1}^q \frac{k_{\tilde{\tau}_i} W_{\tilde{\boldsymbol{\tau}}_i}}{|\operatorname{Aut}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q)|} \\ &= \frac{w_{L_{\text{out}}} N_{\boldsymbol{\tau}_{\text{out}}}}{|\operatorname{Aut}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q)|} \prod_{i=1}^q \sum_{\tilde{\boldsymbol{\tau}}_i} k_{\tilde{\tau}_i} W_{\tilde{\boldsymbol{\tau}}_i}, \end{aligned}$$

which gives (8.8) using (8.9) again.

Thus, replacing X with $\widetilde{X}$, we may assume that x is a vertex of $\mathscr{P}'$ and that every edge or leg of G adjacent to v_{out} maps to an edge of $\mathscr{P}'$ under h'.

Step III. *The gluing situation.* We now are assuming $\boldsymbol{\tau}$ satisfies $x = h'(v_{\text{out}})$ is a vertex of $\mathscr{P}'$ and every edge or leg adjacent to v_{out} is mapped to an edge of $\mathscr{P}'$ under h'. Let $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q, \boldsymbol{\tau}_{v_{\text{out}}}$ be as usual, and denote by $\tilde{\boldsymbol{\tau}}_i$ the decorated type obtained by gluing $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_{v_{\text{out}}}$. Alternatively, after splitting $\boldsymbol{\tau}$ at the edges $E_{i+1}, \ldots, E_q$,

$\tilde{\boldsymbol{\tau}}_i$ is the decorated type corresponding to the connected component containing v_{out}. In particular, $\tilde{\boldsymbol{\tau}}_0 = \boldsymbol{\tau}_{v_{\mathrm{out}}}$ and $\tilde{\boldsymbol{\tau}}_q = \boldsymbol{\tau}$. For each edge E_j, write $D_j := \underline{X}_{\boldsymbol{\sigma}(E_j)}$.

Let $\underline{\mathrm{ev}}_i : \mathscr{M}(X, \tilde{\boldsymbol{\tau}}_i) \to \prod_{j=i+1}^q D_j$ be the schematic evaluation map at the punctured points corresponding to the legs $E_{i+1}, \ldots, E_q$ of $\tilde{\boldsymbol{\tau}}_i$. Define N_i to be the rational number such that

$$\underline{\mathrm{ev}}_{i,*}[\mathscr{M}(X, \tilde{\boldsymbol{\tau}}_i)]^{\mathrm{virt}} = N_i \prod_{j=i+1}^{q} [D_j].$$

Thus (using Theorem 6.1) $N_0 = N_{\boldsymbol{\tau}_{\mathrm{out}}}$ and $N_q = \deg[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}} = W_{\boldsymbol{\tau}} |\operatorname{Aut}(\boldsymbol{\tau})|$. We note a simple calculation shows that the virtual dimension of $\mathscr{M}(X, \tilde{\boldsymbol{\tau}}_i)$ is $q - i$, but this also follows from the computations below.

We will now inductively determine N_i from N_{i-1}. We consider the gluing situation given by $\tilde{\boldsymbol{\tau}}_i$ with set of splitting edges $\mathbf{E} = \{E_i\}$. After splitting, we obtain the types $\tilde{\boldsymbol{\tau}}_{i-1}$ and $\boldsymbol{\tau}_i$.

Step IV. *Calculating the tropical multiplicity.* We calculate $\mu(\tilde{\tau}_i, \mathbf{E})$. Let Q_i be the basic monoid for the type τ_i and $\widetilde{Q}_i$ the basic monoid for the type $\tilde{\tau}_i$. Note that by rigidity of the types τ_i we have $Q_i^* = \mathbb{Z}$ for all i. Similarly, the types $\tilde{\tau}_i$ are rigid, so $\widetilde{Q}_i^* = \mathbb{Z}$. Further, $N_{\tau_{i,\mathrm{out}}}$ can be identified with $Q_i^* \oplus \mathbb{Z}$ in such a way so that the map $Q_i^* \oplus \mathbb{Z} \to P_{E_i}^*$ given by $(q_i, \ell_i) \mapsto \mathrm{ev}_{v_i}(q_i) + \ell_i \mathbf{u}(L_{i,\mathrm{out}})$ coincides with $h_{\tau_i,*} : N_{\tau_{i,\mathrm{out}}} \to N_{\boldsymbol{\sigma}_i(L_{i,\mathrm{out}})}$. We then obtain a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \widetilde{Q}_i^* \arrow[r] \arrow[d] & \widetilde{Q}_{i-1}^* \arrow[d, "="] & \\
0 \arrow[r] & N_{\tau_{i,\mathrm{out}}} \arrow[r] \arrow[d, "h_{\tau_i,*}"] & \widetilde{Q}_{i-1}^* \times N_{\tau_{i,\mathrm{out}}} \arrow[r] \arrow[d, "\Psi_i"] & \widetilde{Q}_{i-1}^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{E_i}^* \arrow[r, "="'] \arrow[d] & P_{E_i}^* \arrow[r] \arrow[d] & 0 & \\
 & \operatorname{coker} h_{\tau_i,*} \arrow[r] \arrow[d] & \operatorname{coker} \Psi_i \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

giving a long exact sequence

$$0 \longrightarrow \widetilde{Q}_i^* \longrightarrow \widetilde{Q}_{i-1}^* \longrightarrow \operatorname{coker} h_{\tau_i,*} \longrightarrow \operatorname{coker} \Psi_i \longrightarrow 0.$$

Thus we see that $\operatorname{coker} \Psi_i$ is finite, hence we are in a tropically transverse gluing situation, and

$$\mu(\tilde{\tau}_i, \mathbf{E}) = |\operatorname{coker} \Psi_i| = k_{\tau_i} |\widetilde{Q}_{i-1}^* / \widetilde{Q}_i^*|^{-1}.$$

Step V. *Gluing.* We have two evaluation maps

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tau_i) \to \underline{X}_{\boldsymbol{\sigma}(E_i)} \text{ and } \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \tilde{\tau}_i) \to \underline{X}_{\boldsymbol{\sigma}(E_i)}$$

whose product gives the morphism $\underline{\mathrm{ev}}$ in Theorem 5.1. Since we are only evaluating at one leg, the criterion for flatness of Theorem 5.3 holds automatically. For example, for the leg $L_{i,\mathrm{out}}$ of $\boldsymbol{\tau}_i$ and contraction $\phi:\tau_i'\to\tau_i$, we necessarily have $\dim\boldsymbol{\sigma}'(L)-\dim\boldsymbol{\sigma}(L)\le 1$. Indeed, $\dim\boldsymbol{\sigma}(L)=2$ and $\dim\boldsymbol{\sigma}'(L)\le 3$ as B is three-dimensional.[6]

Thus we may apply Theorems 5.1 and 5.5 to see that

$$\begin{aligned}\phi'_*[\mathscr{M}(X,\tilde{\boldsymbol{\tau}}_i)]^{\mathrm{virt}} &= \mu(\tilde{\tau}_i,\mathbf{E})[\mathscr{M}^{\mathrm{sch}}(X,\tilde{\boldsymbol{\tau}}_i)]^{\mathrm{virt}}\\ &= \mu(\tilde{\tau}_i,\mathbf{E})\Delta^!([\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}}\times[\mathscr{M}(X,\tilde{\boldsymbol{\tau}}_{i-1})]^{\mathrm{virt}}).\end{aligned}$$

On the other hand, we have a diagram

$$\begin{tikzcd}
\mathscr{M}^{\mathrm{sch}}(X,\tilde{\boldsymbol{\tau}}_i) \arrow[r] \arrow[d, "\underline{\mathrm{ev}}''"'] & \mathscr{M}(X,\boldsymbol{\tau}_i)\times\mathscr{M}(X,\tilde{\boldsymbol{\tau}}_{i-1}) \arrow[d, "\underline{\mathrm{ev}}'"] \\
D_i\times\prod_{j=i+1}^q D_j \arrow[r] \arrow[d] & D_i\times\prod_{j=i}^q D_j \arrow[d] \\
D_i \arrow[r, "\Delta"'] & D_i\times D_i
\end{tikzcd}$$

where $\underline{\mathrm{ev}}'$ is the product of the evaluation map $\mathscr{M}(X,\boldsymbol{\tau}_i)\to D_i$ at $L_{i,\mathrm{out}}$ and the evaluation map $\underline{\mathrm{ev}}_{i-1}$. The bottom two vertical arrows are the obvious projections, and the composition of the right-hand vertical arrows agrees with the morphism $\underline{\mathrm{ev}}\circ\hat{\varepsilon}$ of Theorem 5.1. Next, by definition

$$\underline{\mathrm{ev}}'_*([\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}}\times[\mathscr{M}(X,\tilde{\boldsymbol{\tau}}_{i-1})]^{\mathrm{virt}}) = (\deg[\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}})N_{i-1}\left([p]\times\prod_{j=i}^q[D_j]\right),$$

where $p\in D_i$ is any closed point. By compatibility of push-forward and Gysin pull-back, we see that

$$\begin{aligned}\underline{\mathrm{ev}}''_*[\mathscr{M}^{\mathrm{sch}}(X,\tilde{\boldsymbol{\tau}}_i)]^{\mathrm{virt}} &= \Delta^!\underline{\mathrm{ev}}'_*([\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}}\times[\mathscr{M}(X,\tilde{\boldsymbol{\tau}}_{i-1})]^{\mathrm{virt}})\\ &= (\deg[\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}})N_{i-1}\left([p]\times\prod_{j=i+1}^q[D_j]\right).\end{aligned}$$

Putting this all together, we conclude that

$$N_i = k_{\tau_i}|\widetilde{Q}^*_{i-1}/\widetilde{Q}^*_i|^{-1}\deg[\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}}N_{i-1}.$$

Hence, inductively we obtain

$$\deg[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}} = N_q = N_{\boldsymbol{\tau}_{\mathrm{out}}}|Q^*_0/Q^*_q|^{-1}\prod_{i=1}^q k_{\tau_i}\deg[\mathscr{M}(X,\boldsymbol{\tau}_i)]^{\mathrm{virt}}. \tag{8.10}$$

Now note that Q^*_0 can be identified with $N_{\boldsymbol{\sigma}(v_{\mathrm{out}})}$, as the vertex v_{out} of $G_{\tau_{\mathrm{out}}}$ can be placed at any integral point of $\boldsymbol{\sigma}(v_{\mathrm{out}})$. Further, $N_{\boldsymbol{\sigma}(L_{\mathrm{out}})}/N_{\boldsymbol{\sigma}(v_{\mathrm{out}})}$ can be identified with the integral tangent vectors of the ray $\boldsymbol{\sigma}_{\mathrm{out}}(L_{\mathrm{out}})$ in $\widetilde{\Sigma}_x$, while $w_{L_{\mathrm{out}}}$ is the index of the image $\mathbf{u}_{\mathrm{out}}(L_{\mathrm{out}})$ of $\mathbf{u}(L_{\mathrm{out}})$ in $N_{\boldsymbol{\sigma}(L_{\mathrm{out}})}/N_{\boldsymbol{\sigma}(v_{\mathrm{out}})}$. Thus $w_{L_{\mathrm{out}}}|Q^*_0/Q^*_q|$ coincides with k_τ.

[6]From our point of view, this is the fundamental point of Parker's and Ranganathan's observation about gluing in the genus zero, triple point case: the fact that the boundary D only has triple points means this numerical criterion for flatness always holds when gluing along one edge.

Next note that $\mathrm{Aut}(\boldsymbol{\tau}) = \mathrm{Aut}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q) \times \prod_{i=1}^q \mathrm{Aut}(\boldsymbol{\tau}_i)$. Thus multiplying both sides of (8.10) by k_τ and dividing by $|\mathrm{Aut}(\boldsymbol{\tau})|$ gives the desired result. ♠

8.4. **Behaviour of N_β under blow-ups.** Here we will give another sample application of our gluing formalism. This material does not represent anything radically new; rather, it is a modern version of an argument appearing in [GPS, §5], albeit carried out there using older technology in a somewhat restrictive circumstance. However, the formula given here will be essential in [GHKS].

As in §8.2, we fix a Looijenga pair (X, D). For convenience, we will assume D has at least three nodes. Write $D = D_1 + \cdots + D_n$ the irreducible decomposition, in cyclic order, so that $D_i \cdot D_{i+1} = 1$, with indices taken modulo n. This gives rise as before to an integral affine manifold with singularities B along with its cone decomposition $\mathscr{P}$.

Consider a collection of distinct boundary divisors $D_{j_1}, \ldots, D_{j_s}$, which for convenience we assume are pairwise disjoint. Choose, for each k, distinct points $p_{k1}, \ldots, p_{kn_k} \in D_{j_k}$, also distinct from the nodes of D. Denote by $\pi : \widetilde{X} \to X$ the blow-up of X at all of these points. Denote by $E_{k\ell}$ the exceptional curve over $p_{k\ell}$. Let $\widetilde{D}$ be the strict transform of D, so that $(\widetilde{X}, \widetilde{D})$ is also a Looijenga pair. This gives rise to an integral affine manifold with singularities $\widetilde{B}$ along with its cone decomposition $\widetilde{\mathscr{P}}$. Note that π induces a bijection between the components of $\widetilde{D}$ and D, which in turn induces a natural piecewise linear identification between $(\widetilde{B}, \widetilde{\mathscr{P}})$ and $(B, \mathscr{P})$.

We fix a class $\tilde{\beta}$ of logarithmic map to $\widetilde{X}$ of genus zero and $q+1$ marked points with non-zero contact orders $u_1, \ldots, u_q, u_{\mathrm{out}} \in \widetilde{B}(\mathbb{Z})$, all non-zero and contained in rays of $\widetilde{\mathscr{P}}$. As in §8.2, we assume u_k is contained in a ray ρ_{i_k} corresponding to the divisor $\widetilde{D}_{i_k}$. The attached curve class is $\widetilde{A} \in H_2(\widetilde{X})$. We thus obtain the invariant $N_{\tilde{\beta}}$ of (8.3).

In this situation, set $w_{k\ell} := \widetilde{A} \cdot E_{k\ell}$. Assume from now on that:

$$w_{k\ell} \geq 0 \text{ for all } k, \ell. \tag{8.11}$$

We denote by $\mathbf{P}_{k\ell} = P_{k\ell 1} + \cdots + P_{k\ell\mu}$ an unordered partition of $w_{k\ell}$ into μ positive integers, for some $\mu \geq 0$ (with $\mu = 0$ only if $w_{k\ell} = 0$). Write $\mathbf{P} = (\mathbf{P}_{k\ell})$ a collection of partitions of all $w_{k\ell}$. We write $\mathrm{Aut}(\mathbf{P}_{k\ell})$ for the subgroup of permutations σ of $\{1, \ldots, \mu\}$ with $P_{k\ell\sigma(m)} = P_{k\ell m}$ for $1 \leq m \leq \mu$, and write $\mathrm{Aut}(\mathbf{P}) = \prod_{k\ell} \mathrm{Aut}(\mathbf{P}_{k\ell})$.

For a given $\mathbf{P}$, we write $\beta(\mathbf{P})$ for the class of logarithmic map to X defined as follows. The curve class is $A(\mathbf{P}) = \pi_* \widetilde{A}$. The class has $q+1$ marked points, still with contact orders $u_1, \ldots, u_q, u_{\mathrm{out}}$ using the piecewise linear identification of $(\widetilde{B}, \widetilde{\mathscr{P}})$ and $(B, \mathscr{P})$. It has an additional marked point for every $P_{k\ell m}$, with contact order $P_{k\ell m}\nu_{j_k}$, where ν_{j_k} is the primitive generator of the ray of $\mathscr{P}$ corresponding to D_{j_k}.

We then have:

Theorem 8.16.

$$N_{\tilde{\beta}} = \sum_{\mathbf{P}} \frac{N_{\beta(\mathbf{P})}}{|\mathrm{Aut}(\mathbf{P})|} \prod_{k,\ell,m} \frac{(-1)^{P_{k\ell m}}}{P_{k\ell m}}, \tag{8.12}$$

where the sum is over all collections of partitions $\mathbf{P}$.

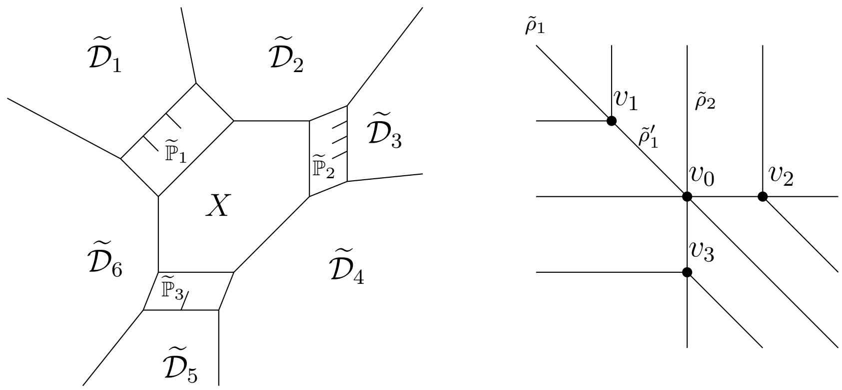

FIGURE 1. The boundary divisor $\widetilde{\mathcal{D}}$ of $\widetilde{\mathcal{X}}$ and the corresponding affine manifold $\widetilde{B}'$. The vertices are all singular points of the affine structure.

Proof. **Step I.** *Building the degeneration.* We build a log smooth degeneration $\widetilde{\mathcal{X}} \to \mathbb{A}^1$ of $\widetilde{X}$, following [AG, §3.1], which in turn is inspired by [GPS], as follows.[7] We first construct a blow-up $\mathcal{X} \to X \times \mathbb{A}^1$, blowing up the closed subscheme $\bigcup_{k=1}^s D_{j_k} \times \{0\}$. Next, we form the blow-up $\widetilde{\mathcal{X}} \to \mathcal{X}$ with center the strict transform of $\bigcup_{k,\ell}\{p_{k\ell}\} \times \mathbb{A}^1$. Thus $\widetilde{\mathcal{X}}_t \cong \widetilde{X}$ for $t \neq 0$. Let $\widetilde{\mathcal{D}} \subseteq \widetilde{\mathcal{X}}$ be the union of $\widetilde{\mathcal{X}}_0$ and the strict transform of $D \times \mathbb{A}^1$. We also write $\widetilde{\mathcal{D}}_i$ for the strict transform of $D_i \times \mathbb{A}^1$. Then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ is a log Calabi-Yau pair, and the composition of

$$\tilde{\pi} : \widetilde{\mathcal{X}} \to \mathcal{X} \to X \times \mathbb{A}^1$$

with the projection to $\mathbb{A}^1$ gives a log smooth morphism $g : \widetilde{\mathcal{X}} \to \mathbb{A}^1$, with $\mathbb{A}^1$ carrying the divisorial log structure given by $0 \in \mathbb{A}^1$.

We have the map $\Sigma(g) : \Sigma(\widetilde{\mathcal{X}}) \to \Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}$, with $\Delta(\widetilde{\mathcal{X}})$ the inverse image of 1, a polyhedral complex. On the other hand, in the notation of [GS8], $\widetilde{B} := |\Sigma(\widetilde{\mathcal{X}})|$ carries an integral affine structure with singularities, and we may write $\mathscr{P}_{\widetilde{\mathcal{X}}}$ for the collection of cones of $\Sigma(\widetilde{\mathcal{X}})$. Similarly, we write $\widetilde{B}' := |\Delta(\widetilde{\mathcal{X}})|$ for the fibre of $g_{\mathrm{trop}} = \Sigma(g) : \widetilde{B} \to \mathbb{R}_{\geq 0}$ over 1, carrying an integral affine structure with singularities and the polyhedral decomposition $\mathscr{P}'_{\widetilde{\mathcal{X}}}$ consisting of the cells of $\Delta(\widetilde{\mathcal{X}})$. See [GS8, Prop. 1.16].

See Figure 1 for what $\widetilde{\mathcal{D}}$ and $\widetilde{B}'$ will look like. In particular, the central fibre is a union

$$\widetilde{\mathcal{X}}_0 = X \cup \bigcup_{k=1}^s \widetilde{\mathbb{P}}_k,$$

where $\mathbb{P}_k$, $1 \leq k \leq s$, is the exceptional divisor of the first blow-up lying over $D_{j_k} \times \{0\}$, and $\widetilde{\mathbb{P}}_k$ is the strict transform of $\mathbb{P}_k$ under the second blow-up. Note that $\mathbb{P}_k$ is a Hirzebruch surface and $\widetilde{\mathbb{P}}_k \to \mathbb{P}_k$ is a blow-up at n_k points. The exceptional curves of this blowup, which we write as $e_{k1}, \ldots, e_{kn_k}$, are by construction disjoint from X. We also write $f_k \in H_2(\widetilde{\mathbb{P}}_k)$ for the the class of a fibre of the ruling $\widetilde{\mathbb{P}}_k \to D_{j_k}$.

We write the vertices of $\mathscr{P}'_{\widetilde{\mathcal{X}}}$ as $v_0, \ldots, v_s$, corresponding to the irreducible components $X, \widetilde{\mathbb{P}}_1, \ldots, \widetilde{\mathbb{P}}_s$ respectively. We also have rays $\tilde{\rho}_i$, $1 \leq i \leq n$ corresponding to

[7]During this proof, we waive the typographic convention that $\mathcal{X}$ is the Artin fan of X.

the one-dimensional stratum $X \cap \widetilde{\mathcal{D}}_i$ if $i \notin \{j_1, \ldots, j_s\}$, and corresponding to the one-dimensional stratum $\widetilde{\mathbb{P}}_k \cap \widetilde{\mathcal{D}}_i$ if $i = j_k$. Further, we have a segment $\tilde{\rho}'_{j_k}$ connecting v_0 and v_k, corresponding to the stratum $\widetilde{\mathbb{P}}_k \cap X$.

The structure of $\widetilde{B}'$ near the vertices $v_1, \ldots, v_s$ has been analyzed in detail in [AG, §3.3.1]. We note that in that reference, X was a toric variety, but the analysis away from the vertex v_0 remains the same. In particular, consider the two one-cells $\tilde{\rho}_{j_k}$, $\tilde{\rho}'_{j_k}$. Then by [AG, Cor. 3.7], the tangent spaces to these two one-cells are left invariant under affine monodromy about v_k. More precisely, if x is a point near v_k, then we may view the group $\Lambda_{\tilde{\rho}_{j_k}}$ (resp. $\Lambda_{\tilde{\rho}'_{j_k}}$) of integral tangent vectors to $\tilde{\rho}_{j_k}$ (resp. $\tilde{\rho}'_{j_k}$) as a well-defined sublattice of Λ_x via parallel transport. Furthermore, under these identifications, $\Lambda_{\tilde{\rho}_{j_k}}$ and $\Lambda_{\tilde{\rho}'_{j_k}}$ agree. Thus $\tilde{\rho}_{j_k} \cup \tilde{\rho}'_{j_k}$ may be viewed as a straight line through v_k, despite the singularity in the affine structure at v_k.

Given a punctured map $f : C^\circ/W \to \widetilde{\mathcal{X}}$ defined over $\mathbb{A}^1$ with W a log point and type τ, let $t \in \tau$ be such that the tropicalization $h_t : G \to \widetilde{B}$ factors through $\widetilde{B}'$. Let $v \in V(G)$ be a vertex with $h_t(v) = v_k$ for some $k > 0$. If $E_1, \ldots, E_\ell$ are the edges or legs adjacent to v, oriented away from v, let u_i be the image of $\mathbf{u}(E_i)$ in $\Lambda_x/\Lambda_{\tilde{\rho}_{j_k}}$, well-defined under parallel transport. Then h_t satisfies the weaker balancing condition

$$\textstyle\sum_{i=1}^{\ell} u_i = 0 \text{ in } \Lambda_x/\Lambda_{\tilde{\rho}_{j_k}} \tag{8.13}$$

by [AG, Prop. 3.10].

As $\widetilde{\mathcal{X}}$ is non-compact, we should be more precise about what group our curve classes live in. Here we will take, for any stratum $Y \subseteq \widetilde{\mathcal{X}}$, $H_2(Y) := \mathrm{Pic}(Y)^*$. Note that as X is a rational surface, when Y is a compact stratum of $\widetilde{\mathcal{X}}$, $H_2(Y)$ coincides with the usual integral singular homology group $H_2(Y, \mathbb{Z})$.

Step II. *Degenerating the enumerative problem.* We begin by choosing points $x_k \in \widetilde{D}_{i_k}$, $1 \leq k \leq q$, not coinciding with any double point of $\widetilde{D}$ and not coinciding with any of the points $p_{k\ell}$. With $\mathbf{x} = (x_1, \ldots, x_q)$, consider the moduli space $\mathscr{M}(\widetilde{X}, \tilde{\beta}, \mathbf{x})$ of (8.4). Then $N_{\tilde{\beta}} = \deg[\mathscr{M}(\widetilde{X}, \tilde{\beta}, \mathbf{x})]^{\mathrm{virt}}$ by Proposition 8.8. Similarly, we may consider sections $\tilde{x}_k \subseteq \widetilde{\mathcal{X}}$ which are the strict transforms of $x_k \times \mathbb{A}^1$, and obtain

$$\mathscr{M}(\widetilde{\mathcal{X}}/\mathbb{A}^1, \tilde{\beta}, \tilde{\mathbf{x}}) := \mathscr{M}(\widetilde{\mathcal{X}}/\mathbb{A}^1, \tilde{\beta}) \times_{\prod_k \widetilde{\mathcal{X}}} \prod_{k=1}^{q} \tilde{x}_k.$$

Of course, for $0 \neq t \in \mathbb{A}^1$, we have

$$\mathscr{M}(\widetilde{X}, \tilde{\beta}, \mathbf{x}) = \mathscr{M}(\widetilde{\mathcal{X}}/\mathbb{A}^1, \tilde{\beta}, \tilde{\mathbf{x}}) \times_{\mathbb{A}^1} t,$$

so as in [ACGS1, Thm. 1.1], we have

$$\deg[\mathscr{M}(\widetilde{X}, \tilde{\beta}, \mathbf{x})]^{\mathrm{virt}} = \deg[\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \tilde{\beta}, \tilde{\mathbf{x}}_0)]^{\mathrm{virt}}.$$

Here $\tilde{\mathbf{x}}_0 = ((\tilde{x}_k)_0)$ denotes the tuple of points with $(\tilde{x}_k)_0 = \tilde{x}_k \cap g^{-1}(0)$. As in [ACGS1, Thm. 5.4], we then have a decomposition

$$N_{\tilde{\beta}} = \deg[\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \tilde{\beta}, \tilde{\mathbf{x}}_0)]^{\mathrm{virt}} = \sum_{\boldsymbol{\tau}} \frac{m_\tau}{|\mathrm{Aut}(\tau)|} j_{\boldsymbol{\tau}*} \deg[\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}, \tilde{\mathbf{x}}_0)]^{\mathrm{virt}}, \tag{8.14}$$

where the sum is over all rigid decorated tropical maps to $\widetilde{B}'$ which can arise as a degeneration of the class $\tilde{\beta}$. Here, $j_{\boldsymbol{\tau}} : \mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}, \tilde{\mathbf{x}}_0) \to \mathscr{M}(\widetilde{\mathcal{X}}_0/0, \tilde{\beta}, \tilde{\mathbf{x}}_0)$ is the canonical map forgetting the marking by $\boldsymbol{\tau}$. As in Definition 8.3, we assume the decoration functions $\mathbf{A}$ are refined, i.e., have $\mathbf{A}(v) \in H_2(\widetilde{\mathcal{X}}_{\boldsymbol{\sigma}(v)})$ rather than in $H_2(\widetilde{\mathcal{X}})$.

Step III. *Classifying rigid tropical maps: first steps.* Thus let $h : G \to \widetilde{B}'$ be a rigid tropical map of some type τ and decoration $\mathbf{A}$ contributing to the decomposition (8.14). Let $L_1, \ldots, L_q \in L(G)$ be the legs corresponding to the marked points mapping to the tuple of points of $\tilde{\mathbf{x}}_0$. Note G has one additional leg, denoted L_{out}. Because of the assumption that the x_k are not double points of $\widetilde{D}$, necessarily $h(L_k)$ is contained in the ray $\tilde{\rho}_{i_k}$. We may now apply the proof of Lemma 8.9 in this setting. The target affine manifold with singularities $\widetilde{B}'$ is slightly more complicated than the B of the proof of the lemma, but the same argument works as the weaker balancing condition (8.13) at the vertices $v_1, \ldots, v_s$ is still sufficient. This allows us to conclude that in any event, whether or not h is rigid, its image lies in $\bigcup_{i=1}^n \tilde{\rho}_i \cup \bigcup_{k=1}^s \tilde{\rho}'_{j_k}$. However, rigidity also then implies that vertices of G must map to vertices of $\widetilde{\mathscr{P}'}$, as otherwise the location of these vertices may be freely moved along the edge containing their image.

We now note that it is immediate that each such rigid type τ is tropically transverse in the sense of Definition 5.8. As in Step V of the proof of Theorem 8.15, since again $\widetilde{\mathcal{D}}$ only has triple points, we do not need to worry about the flatness hypothesis of Theorem 5.9, and will use this gluing result in what follows without reference to these hypotheses.

Step IV. *Balancing at each vertex mapping to v_k.* Continue with the notation of the previous step, and let $v \in V(G)$ be a vertex such that $h(v) = v_k$ for some $k > 0$. Let $E_1, \ldots, E_m$ be the edges or legs adjacent to v mapping to $\tilde{\rho}_{j_k}$. Note these in fact must necessarily be legs, as if E_i were an edge, the other vertex of E_i would map to the interior of $\tilde{\rho}_{j_k}$. Similarly, let $E'_1, \ldots, E'_{m'}$ be the edges or legs adjacent to v mapping to $\tilde{\rho}'_{j_k}$. Note these are necessarily edges with opposite endpoint mapping to v_0. Let w_i (resp. w'_i) be the index of $\mathbf{u}(E_i)$ (resp. $\mathbf{u}(E'_i)$), and let $w = \sum w_i$, $w' = \sum w'_i$. We now make a sequence of observations.

Observation A. No edge or leg of G maps to the one-dimensional cells of $\mathscr{P}'_{\widetilde{\mathcal{X}}}$ adjacent to v_k which are not $\tilde{\rho}_{j_k}$ or $\tilde{\rho}'_{j_k}$. Indeed, as these cells are unbounded, only legs may map to such cells. As the images of the legs L_i all map to cells of the form $\tilde{\rho}_j$, the only possibility is that L_{out} maps to such a cell. However, that would violate the balancing condition (8.13).

Observation B. Suppose $f : C \to \widetilde{\mathcal{X}}_0$ is a punctured map defined over a log point in the moduli space $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}, \tilde{\mathbf{x}}_0)$. Let $\underline{C}_v \subseteq \underline{C}$ be the union of irreducible components of $\underline{C}$ correponding to v in the marking by $\boldsymbol{\tau}$. Then $f(\underline{C}_v)$ is of curve class $A := \mathbf{A}(v) \in H_2(\widetilde{\mathbb{P}}_k)$. By Corollary 2.2, necessarily the intersection number of A with $\widetilde{\mathcal{D}}_{j_k} \cap \widetilde{\mathbb{P}}_k$ is w and the intersection number of A with $\widetilde{\mathbb{P}}_k \cap X$ is w'. Meanwhile, by Observation A, the intersection number of A with the other two strata of $\widetilde{\mathbb{P}}_k$ must be zero. As a consequence, A is a linear combination of f_k and the $e_{k\ell}$. Since $\underline{C}_v$ is connected, there is no choice but for $A = w(f_k - e_{k\ell}) + w'e_{k\ell}$ for some ℓ. In particular, the image $f(\underline{C}_v)$ is contained in a fibre of $\pi|_{\widetilde{\mathbb{P}}_k} : \widetilde{\mathbb{P}}_k \to D_{j_k}$, reducible if $w \neq w'$.

Observation C. We will now show that the contribution to $N_{\tilde{\beta}}$ is zero unless either one of the following hold:

(IV.1) $m = 0$, i.e., there are no legs adjacent to v;
(IV.2) $m = 1$ and $E_1 = L_{\mathrm{out}}$; or
(IV.3) $w = w'$.

Case (IV.3) is really balancing at v, which can now be viewed as a well-defined equality in $\Lambda_{\tilde{\rho}_j}$ by Step I.

Assume we are not in case (IV.1) or (IV.2), so that there is at least one leg L_i adjacent to v. Assume that $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}, \tilde{\mathbf{x}}_0)$ is non-empty so that there exists a punctured map as in Observation B. By Observation B, we have $f(\underline{C}_v)$ is a fibre of $\widetilde{\mathbb{P}}_k \to D_{j_k}$, reducible if $w \neq w'$. On the other hand, by the assumption that L_i is adjacent to v, there is a marked point of C on C_v map to $(\tilde{x}_i)_0$ under f. However, since the points x_i were chosen to be distinct from any $p_{k\ell}$, $f(\underline{C}_v)$ cannot be reducible. Hence $w = w'$.

Observation D. We have $m' > 0$. Indeed, if $m' = 0$, then since G is connected, the only possibility is G has one vertex v, which maps to v_k, and a number of legs, mapping to $\tilde{\rho}_{j_k}$. By Observation B, we then have $\mathbf{A}(v) = we_{k\ell}$ for some ℓ, and then necessarily the total curve class of the type is also $ce_{k\ell}$. But this curve class deforms to $wE_{k\ell}$ on a general fibre of $\widetilde{\mathcal{X}} \to \mathbb{A}^1$, and then $w_{k\ell} = -w < 0$, contradicting (8.11).

Step V. *Univalency or bivalency of each vertex mapping to v_j.*

Continuing with the notation of Step IV, assume that $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}, \tilde{\mathbf{x}}_0)$ is non-empty. We first observe that we can't have two distinct legs $L_i, L_{i'}$ adjacent to v. Indeed, by Observation B above, $f(\underline{C}_v)$ lies in a fibre of $\widetilde{\mathbb{P}}_j \to D_{j_k}$. Thus the corresponding marked points of C contained in $\underline{C}_v$ must map to the same point of $\widetilde{\mathcal{D}}_{j_k} \cap \widetilde{\mathbb{P}}_k$. However, since the x_i are chosen to be distinct, this is not possible.

Now suppose that one of the following two cases hold.

(V.1) Either (a) L_{out} is adjacent to v and another leg L_i is adjacent to v or (b) L_{out} is adjacent to v and balancing fails at v, i.e., $c \neq c'$ or (c) $m' > 1$ and balancing fails at v.
(V.2) $m' > 1$.

We will show the contribution to (8.14) from the type $\boldsymbol{\tau}$ is zero in these cases.

We may split $\boldsymbol{\tau}$ along the edges $E'_1, \ldots, E'_{m'}$, giving decorated types $\boldsymbol{\tau}_v$ and $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{m'}$ with underlying graphs $G_v, G_1, \ldots, G_{m'}$. We note that here we use the fact the genus of the underlying graph is zero, so that splitting at any edge produces an additional connected component. In either Case (V.1) or (V.2), since $m' > 0$ in any event by Observation D, there is a j such that $L_{\mathrm{out}} \notin L(G_j)$. Thus all legs in G_j other than the leg corresponding to E'_j correspond to marked points which are constrained to map to some subset of the points $\{(\tilde{x}_i)_0\}$. From [ACGS2, Prop. 3.28 and (4.17)], taking into account the just-mentioned point-constraints, the virtual dimension of $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_j, \tilde{\mathbf{x}}_0)$ is zero.

If we are in Case (V.1), $\underline{C}_v$ maps to a fixed fibre of $\widetilde{\mathbb{P}}_k \to D_{j_k}$ (a reducible fibre in the unbalanced case). Thus the image of the evaluation map $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_v, \tilde{\mathbf{x}}_0) \to \widetilde{\mathbb{P}}_k \cap X = D_{j_k}$ at the punctured point corresponding to an edge E'_i is a point, being either the image of $(\tilde{x}_i)_0$ under the projection map $\widetilde{\mathbb{P}}_k \to D_{j_k}$ in the case (a) or a point $p_{k\ell}$ in the case (b) and (c). It then follows immediately in this case that $\Delta^!([\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_v, \tilde{\mathbf{x}}_0)]^{\mathrm{virt}} \times [\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_j, \tilde{\mathbf{x}}_0)]^{\mathrm{virt}}) = 0$, with Δ as in Theorem 5.1 for the gluing of $\boldsymbol{\tau}_v$ and $\boldsymbol{\tau}_j$. Hence by Theorem 5.9, the contribution from the type $\boldsymbol{\tau}$ is zero.

Next suppose we are in Case (V.2), but not Case (V.1). The only remaining possibility is that $m' > 1$, there is only one leg adjacent to v, and the type $\boldsymbol{\tau}$ is balanced at v. Suppose L_{out} is the only leg adjacent to v In this case, the evaluation map $\underline{\mathrm{ev}}_v : \mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_v, \tilde{\mathbf{x}}_0) \to D_{j_k}$ will be surjective. On the other hand, in this case $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_i, \tilde{\mathbf{x}}_0)$ is virtual dimension 0 for all i, and another application of Theorem 5.9 shows the contribution from the type $\boldsymbol{\tau}$ vanishes. A similar argument applies if L_i is the leg adjacent to v, this time because the evaluation map $\underline{\mathrm{ev}}_v$ has image a point and the virtual dimension of $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_i, \tilde{\mathbf{x}}_0)$ is zero for at least one i.

In summary, if $\boldsymbol{\tau}$ does contribute non-trivially to (8.14), either v is univalent, necessarily with $\mathbf{A}(v)$ a positive multiple of $f_k - e_{k\ell}$ for some k, or v is bivalent and balanced, with $\mathbf{A}(v)$ a positive multiple of f_k.

Step VI. *Calculating the non-trivial contributions from rigid tropical maps.* The only remaining possibility for a rigid tropical map contributing non-trivially to the right-hand side of (8.14) is now as follows. There is one vertex w with $h(w) = v_0$. Attached to w are those legs L amongst $L_1, \ldots, L_q, L_{\mathrm{out}}$ for which $\boldsymbol{\sigma}(L) = \tilde{\rho}_k$ with $k \notin \{j_1, \ldots, j_s\}$. For those legs L with $\boldsymbol{\sigma}(L) = \tilde{\rho}_{j_k}$, we instead have an edge adjacent to w mapping surjectively to $\tilde{\rho}'_{j_k}$, with opposite vertex v_L bivalent and adjacent to L, which maps surjectively to $\tilde{\rho}_{j_k}$. Finally, there may be an additional set of edges adjacent to w mapping to the various $\tilde{\rho}'_{j_k}$, with opposite vertex being univalent. For each k, ℓ, we will have some number $m_{k\ell}$ of such univalent vertices mapping to v_j, with attached curve classes $P_{k\ell 1}(f_k - e_{k\ell}), \ldots, P_{k\ell m_{k\ell}}(f_k - e_{k\ell})$, for some positive integers $P_{k\ell 1}, \ldots, P_{k\ell m_{k\ell}}$. Note these curve classes also determine the contact order for the edge adjacent to such a univalent vertex, again by Corollary 2.2.

Recall from Step I that curve classes take values in the duals of the Picard groups of strata. In particular, $H_2(X \times \mathbb{A}^1) \cong H_2(X)$. If $\boldsymbol{\tau}$ is a decorated type contributing to (8.14) and $A(\boldsymbol{\tau})$ is the total curve class of the decorated type $\boldsymbol{\tau}$, then under the blow-down $\tilde{\pi} : \widetilde{\mathcal{X}} \to X \times \mathbb{A}^1$, $\pi_* A(\boldsymbol{\tau})$ must agree with $A := \pi_* \widetilde{A}$. Since the curve class $\mathbf{A}(v)$ is contracted by $\tilde{\pi}$ for any $v \neq w$, we see that $\mathbf{A}(w) = A$. Further, since $\widetilde{A} \cdot E_{k\ell} = w_{k\ell}$, it follows by intersecting $A(\boldsymbol{\tau})$ with the exceptional divisor over the strict transform of $p_{k\ell} \times \mathbb{A}^1$ that $\mathbf{P}_{k\ell} = P_{k\ell 1} + \cdots + P_{k\ell m_{k\ell}}$ is a partition of $w_{k\ell}$. In conclusion, the type $\boldsymbol{\tau}_w$ associated with the vertex w may now be viewed as a class of map to X, and is precisely the type $\beta(\mathbf{P})$, $\mathbf{P} = (\mathbf{P}_{k\ell})$.

We are now ready to compute the contribution of this type to the right-hand-side of (8.14). First, we note that for each univalent vertex v with associated curve class $P_{k\ell m}(f_k - e_{k\ell})$, we may apply Theorem 6.1 and [GPS, Prop. 5.2] to see that $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_w, \tilde{\mathbf{x}}_0)$ is virtual dimension zero with virtual fundamental class of degree

$(-1)^{P_{k\ell m}}/P_{k\ell m}^2$. On the other hand, if v is a bivalent vertex mapping to v_k with associated curve class df_k, then $\mathscr{M}(\widetilde{\mathcal{X}}_0/0, \boldsymbol{\tau}_w, \tilde{\mathbf{x}}_0)$ is easily seen to consist of one curve mapping $d:1$ to a non-singular fibre of $\widetilde{\mathbb{P}}_k \to D_{j_k}$, totally branched over $\widetilde{\mathbb{P}}_k \cap \widetilde{\mathcal{D}}_{j_k}$ and $\widetilde{\mathbb{P}}_k \cap X$. This map is unobstructed, and the fundamental class of the moduli space is degree $1/d$.

A simple calculation now shows that the multiplicity $\mu(\tau)$ is $\prod_{E\in E(G)} w_E$, where w_E is the index of the contact order $\mathbf{u}(E)$. If E is an edge adjacent to a bivalent vertex with associated curve class df_k, then $w_E = d$, while if E is an edge adjacent to a univalent vertex with associated curve class $P_{k\ell m}(f_k - e_{k\ell})$, we have $w_E = P_{k\ell m}$. The desired contribution to $N_{\tilde{\beta}}$ from $\boldsymbol{\tau}$ in (8.14) now agrees with the summand in (8.12) corresponding to the collection of partitions $\mathbf{P}$. Indeed, this follows from the description of the virtual degrees of the moduli spaces above, the calculation of the multiplicity $\mu(\tau)$, the definition of $N_{\beta(\mathbf{P})}$, and Theorem 5.9. ♠

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