What is mirror symmetry?

- It is a duality.
- It provides an isomorphism between complex and symplectic geometry.
- The complex and symplectic manifolds involved should be interesting.
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Try 1:

V a real finite dim'l vector space	$V^* = \mathrm{Hom}(V, \mathbb{R})$ the dual space.

Try 2:

$V \times V$ with complex structure $J(v_1, v_2) = (-v_2, v_1)$	$V \times V^*$ with symplectic structure $\omega((v_1, w_1), (v_2, w_2)) = \langle w_1, v_2 \rangle - \langle w_2, v_1 \rangle$

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A vector space is not a particularly interesting example.

We can make this more interesting by choosing V to have an integral structure, i.e.,

$$V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$$

where $\Lambda \cong \mathbb{Z}^n$. Set

$$\check{\Lambda} := \{w \in V^* \,|\, \langle w, \Lambda \rangle \subseteq \mathbb{Z}\} \subseteq V^*$$

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Alternatively view Λ as a family of lattices in the tangent bundle $\mathcal{T}_V = V \times V$ of V and $\check{\Lambda}$ as a family of lattices in $\mathcal{T}^*_V = V \times V^*$.

$X(V) := \mathcal{T}_V/\Lambda$ with complex structure J as before. Torus bundle $X(V) \to V$	$\check{X}(V) := \mathcal{T}^*_V/\check{\Lambda}$ with symplectic structure as before. Torus bundle $\check{X}(V) \to V$

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We need to replace the structure $\Lambda \subset V$ with something which looks like this locally.

Consider a manifold B with coordinate charts $\{\psi_i : U_i \to V\}$.

At $b \in U_i$, we obtain a lattice $(\psi_{i*})^{-1}(\Lambda) \subseteq \mathcal{T}_b$.

We would like this lattice to be independent of the chart.

This requires that the transition functions $\psi_i \circ \psi_j^{-1}$ are in fact affine linear transformations $v \mapsto Tv + v_0$ for some $T \in \mathrm{GL}(\Lambda)$ and $v_0 \in V$.

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$B = \mathbb{R}^n/\mathbb{Z}^n$. Then $X(B)$ is a complex torus of complex dimension n and $\check{X}(B)$ a symplectic torus of real dimension $2n$.

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What about trying to get complex and symplectic structure on *both* sides? This means we want a Kähler structure on $X(B)$ and $\check{X}(B)$.

It is convenient to do this on $X(B)$ using a Kähler potential which is pulled back from the base.

Definition

A multi-valued convex function K on B is a set of functions $K_i : U_i \to \mathbb{R}$ on an open cover $\{U_i\}$ of B with $K_i - K_j$ affine linear on $U_i \cap U_j$ and each K_i is convex.

Given $f : X(B) \to B$, $K \circ f$ provides a Kähler potential, i.e., $\omega = 2i\partial\bar{\partial}K$ is a Kähler form on $X(B)$.

Given $f : \check{X}(B) \to B$, locally on U_i if we have tropical affine coordinates $y_1, \ldots, y_n$, let $y_i, \check{x}_i$ be coordinates on the cotangent bundle with $\check{x}_i$ given by evaluation of $\partial/\partial y_i$. Then $\check{x}_i + \sqrt{-1}\partial K/\partial y_i$ provides complex coordinates on $\check{X}(B)$.

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Both these Kähler structures are Ricci-flat if K satisfies the *real* Monge-Ampère equation

$$\det(\partial^2 K/\partial y_i \partial y_j) = \text{constant}.$$

(This observation is due to Hitchin.)

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We say a tropical manifold B with a multi-valued convex function K is *affine Kähler*. It is *Monge-Ampère* if K satisfies the above Monge-Ampère equation.

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A *Calabi-Yau manifold* is an n-dimensional complex manifold with a Ricci-flat Kähler metric ω and a nowhere vanishing holomorphic n-form Ω.

Ricci-flatness is equivalent to

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Given B, K Monge-Ampère, $X(B)$ has a nowhere-vanishing holomorphic n-form provided that B is orientable.

In this case, the fibres of $f : X(B) \to B$ have a special property:

Definition

(Harvey-Lawson) Let X be a Calabi-Yau manifold with $\dim_{\mathbb{C}} X = n$. A real submanifold $M \subseteq X$ is *special Lagrangian* if

1. $\dim_{\mathbb{R}} M = n$.
2. $\omega|_M = 0$ (M is Lagrangian).
3. $\operatorname{Im} \Omega|_M = 0$ (M is special).

Special Lagrangian submanifolds are volume minimizing within their homology class.

The fibres of $f : X(B) \to B$ are special Lagrangian.

Thus the version of mirror symmetry we have seen so far involves dual special Lagrangian torus fibrations $f : X(B) \to B$ and $\check{f} : \check{X}(B) \to B$.

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Conjecture

(Strominger-Yau-Zaslow, 1996) Let X and $\check{X}$ be a mirror pair of Calabi-Yau manifolds. Then there exists special Lagrangian torus fibrations $f : X \to B$ and $\check{f} : \check{X} \to B$ which are dual.

- This conjecture remains unproven other than some very straightforward cases (abelian varieties, K3 surfaces). However, it has led to a clear philosophy for the geometry underlying mirror symmetry, as illustrated by the "toy" version of mirror symmetry given here.
- We still haven't fulfilled the third requirement of mirror symmetry: torus bundles always have Euler characteristic zero, and most interesting examples of mirror symmetry have non-zero Euler characteristic.

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Example

Let $X \subseteq \mathbb{CP}^4$ be defined by a quintic equation

$$x_0^5 + \cdots + x_4^5 = 0.$$

Then

$$\chi(X) = -200.$$

Let G be the subgroup of $\mathbb{Z}_5^5$ given by

$$G = \{(a_0, \ldots, a_4) \mid \sum_i a_i = 0\}.$$

Then $(a_0, \ldots, a_4) \in G$ acts on X by

$$(x_0, \ldots, x_4) \mapsto (\mu^{a_0} x_0, \ldots, \mu^{a_4} x_4)$$

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How will we get an interesting Calabi-Yau manifold such as the quintic?

Definition

A *tropical affine manifold with singularities* is a real (C^0) manifold B along with an open subset $B_0 \subseteq B$ such that $\Delta := B \setminus B_0$ is of codimension ≥ 2 and such that B_0 has the structure of a tropical affine manifold.

Example

Let $\Xi \subseteq \mathbb{R}^4$ be the convex hull of the points

$$(-1,-1,-1,-1)$$
$$(4,-1,-1,-1)$$
$$(-1,4,-1,-1)$$
$$(-1,-1,4,-1)$$
$$(-1,-1,-1,4)$$

Let $B = \partial\Xi$; this is homeomorphic to S^3.

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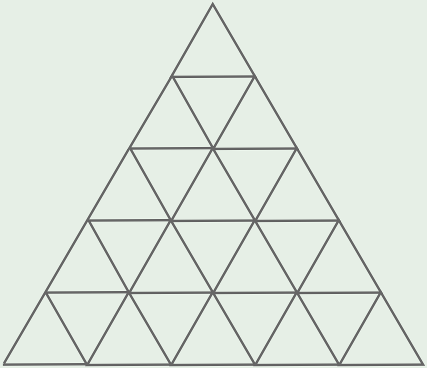
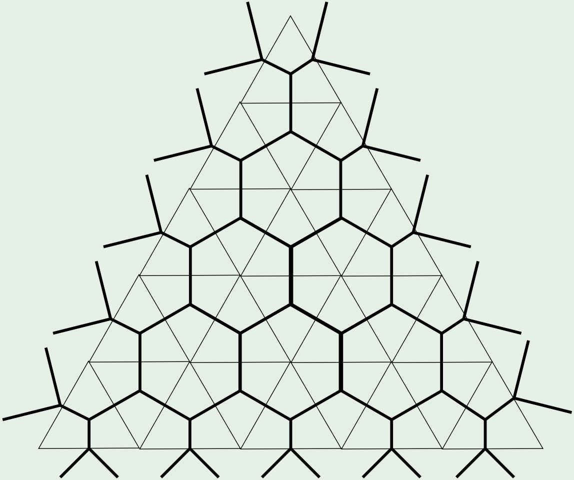
Example

Triangulate each two-face of Ξ using only integral points as vertices as follows:



Example

We will take the discriminant locus $\Delta \subseteq B$ to be contained in the union of two-faces, looking on each two-face like:



Example

We define an integral affine structure on $B_0 := B \setminus \Delta$ using coordinate charts as follows.

- For each three-face σ of Ξ, we have a natural affine chart ψ_σ on $\mathrm{Int}(\sigma)$ given by the inclusion of σ in the affine hyperplane in $\mathbb{R}^4$ containing σ.
- For each integral point v of a two-face, we can choose an open neighbourhood $U_v \subseteq B \setminus \Delta$ of v such that

$$\{\mathrm{Int}(\sigma) \,|\, \sigma \text{ a 3-face}\} \cup \{U_v \,|\, v \text{ an integral point}\}$$

  forms an open cover of B_0 and so that $U_v \cap U_{v'} = \emptyset$ if $v \neq v'$. Then define charts

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We now have a compactification problem, needing diagrams

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This compactification problem can be thought about at three levels:

1. Can $X(B_0)$ and $\check{X}(B_0)$ be compactified topologically in a useful way?
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Answers: Yes, yes, no.

Theorem

(G., 1999) $X(B_0)$ and $\check{X}(B_0)$ can be compactified topologically to get six-manifolds homeomorphic to the mirror quintic and the quintic respectively.

Let's look at the additional fibres added over the discriminant locus.

There are three types of singular fibres, fibres over a smooth point of Δ, and fibres over two types of vertices, which we call *positive* and *negative* vertices.

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There are three types of singular fibres, fibres over a smooth point of Δ, and fibres over two types of vertices, which we call *positive* and *negative* vertices.

1. Over smooth points of Δ, the fibre takes the form $I_1 \times S^1$, where I_1 is a pinched two-torus.
2. Over a positive vertex, the fibre is a pinched 3-torus, i.e., of the form $S^1 \times S^1 \times S^1 / \sim$, where $(a_1, b_1, c_1) \sim (a_2, b_2, c_2)$ if $c_1 = c_2 = 1 \in S^1$ or $(a_1, b_1, c_1) = (a_2, b_2, c_2)$.
3. Over a negative vertex, the fibre has a figure eight singular locus, given by of the form $S^1 \times S^1 \times S^1 / \sim$, where $(a_1, b_1, c_1) \sim (a_2, b_2, c_2)$ if $a_1 = a_2 = 1, b_1 = b_2$; $a_1 = a_1, b_1 = b_2 = 1$; or $(a_1, b_1, c_1) = (a_2, b_2, c_2)$.

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The Euler characteristic of a positive fibre is $+1$, of a negative fibre is -1, and these are interchanged between $X(B)$ and $\check{X}(B)$.

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What about symplectic or complex compactifications?

- Results of Castaño-Bernard and Matessi give a symplectic compactification of $\check{X}(B_0)$. In addition, Wei-Dong Ruan constructed a Lagrangian torus fibration on the quintic.
- There is no holomorphic compactification of $X(B)$ because the complex structure on $X(B_0)$ is not precisely correct.
- *Crucial point:* The complex structure on $X(B_0)$ has to be perturbed before it can be compactified. This perturbation is what makes mirror symmetry truly interesting, and is responsible for the relationship between complex deformation theory on one side and curve counting on the other. We usually describe this pertubation as given by "instanton corrections."

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We now need to travel to the tropics...

Suppose $L \subseteq B$ is a rationally defined affine linear subspace (i.e., $\mathcal{T}_{L,b}$ is a rationally defined subspace of $\mathcal{T}_{B,b}$ for $b \in L$.)

$\mathcal{T}_L/(\mathcal{T}_L \cap \Lambda) \subseteq X(B)$ holomorphic submanifold.	$\mathcal{T}_L^{\perp}/(\mathcal{T}_L^{\perp} \cap \check{\Lambda}) \subseteq \check{X}(B)$ Lagrangian submanifold.

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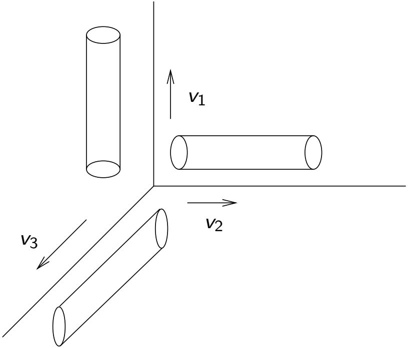
Again, these are not topologically very interesting. For example, if $\dim L = 1$, we obtain holomorphic curves which are either cylinders or tori.

Let's try to get a more interesting "approximate" holomorphic curve by gluing together cylinders, taking three rays meeting at $b \in B$:



We can try to glue the three cylinders by gluing in a surface contained in the fibre $f^{-1}(b)$.

Noting that $H_1(f^{-1}(b), \mathbb{Z}) = \Lambda_b$, the tangent vectors v_1, v_2 and v_3 represent the boundaries of the three cylinders in $H_1(f^{-1}(b), \mathbb{Z})$. Thus the three circles bound a surface if

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This leads us to the notion of a *tropical curve* in a tropical affine manifold:

Definition

A *parameterized tropical curve* in a tropical manifold B is a graph Γ (possibly with non-compact edges with zero or one adjacent vertices) along with

- a weight function w associating a non-negative integer to each edge;
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Definition

(*cont'd.*)

1. If E is an edge of Γ and $w(E) = 0$, then $h|_E$ is constant; otherwise $h|_E$ is a proper embedding of E into B as a line segment, ray or line of rational slope.
2. *The balancing condition.* For every vertex of Γ with adjacent edges $E_1, \ldots, E_n$, let $v_1, \ldots, v_n \in \Lambda_{h(V)}$ be primitive tangent vectors to $h(E_1), \ldots, h(E_n)$ pointing away from $h(V)$. Then

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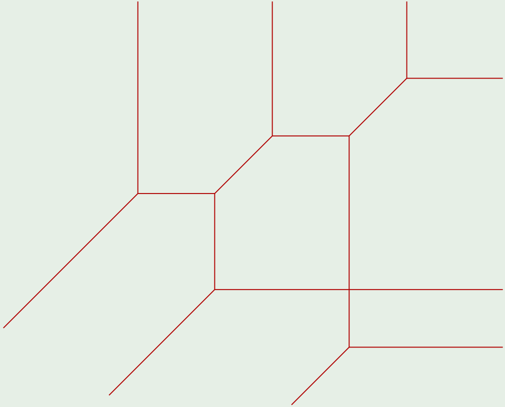
Example

Take $B = \mathbb{R}^2$. Here is a tropical curve:



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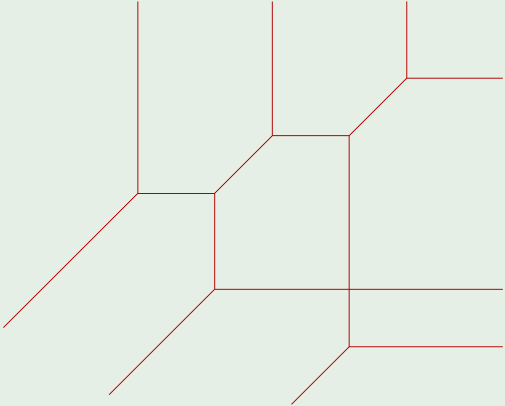

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This curve can be viewed as an approximation to a curve of degree 3 in $\mathbb{CP}^2$.

Mikhalkin showed that curves in $\mathbb{CP}^2$ through a given number of points can in fact be counted by counting tropical curves of this nature.

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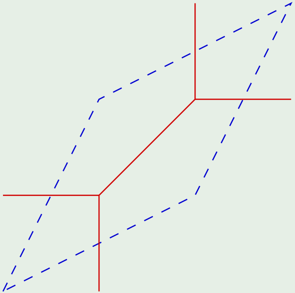
Take $B = \mathbb{R}^2/((1,2)\mathbb{Z} + (2,1)\mathbb{Z})$. We have a genus two tropical curve



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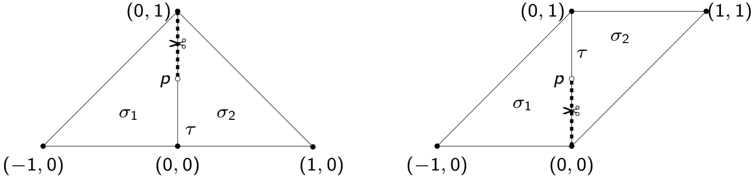
Locally, the singularity can be described via two charts, as depicted:



The diagram shows the affine embedings of two charts, obtained by cutting the union of two triangles as indicated in the two figures. Note that the vertical line segment is an invariant direction, being a straight line in both charts.

The monodromy about the singularity in Λ is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

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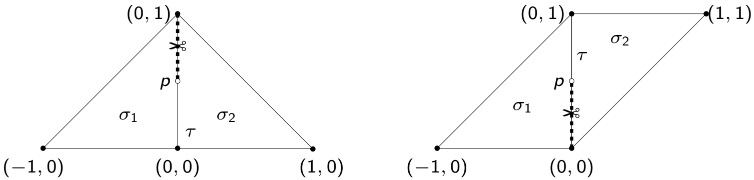

The diagram shows the affine embedings of two charts, obtained by cutting the union of two triangles as indicated in the two figures.
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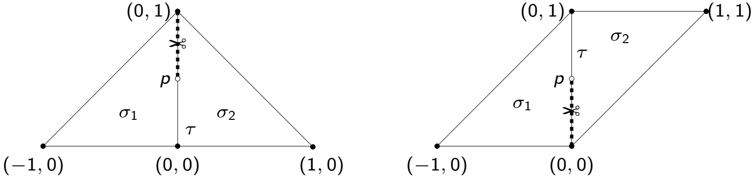

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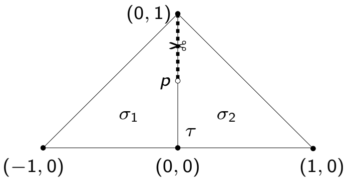

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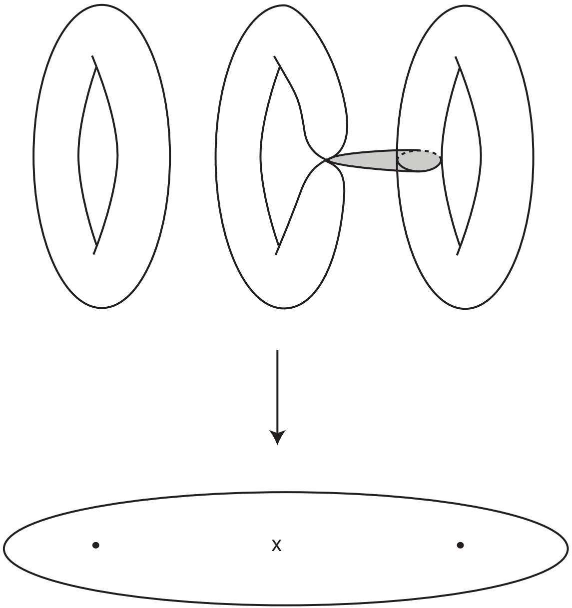
In this case, the SYZ fibration over B has a singular fibre over the singular point of B, a pinched torus, and the total space of the fibration over B has two holomorphic disks, one as depicted: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This disk corresponds to a tropical curve on B of the form:

The tropical curve must enter the singularity along the invariant direction.

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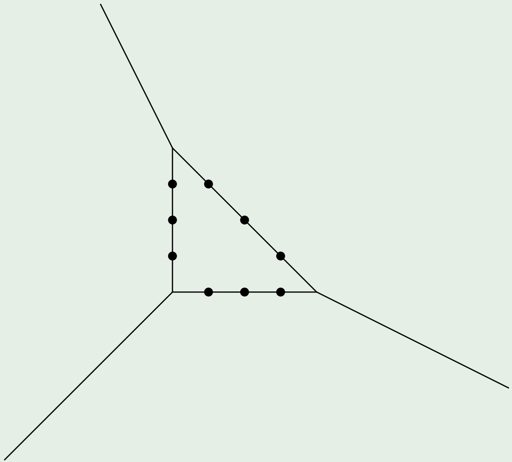
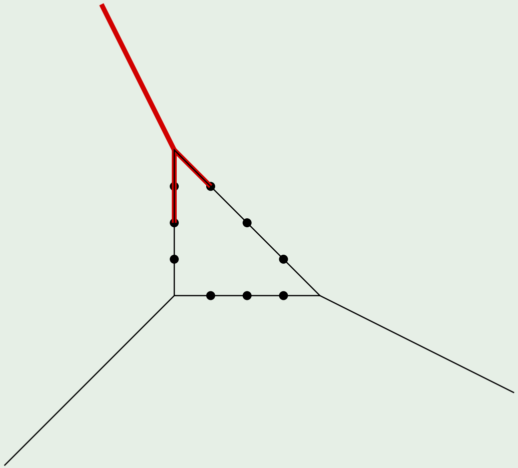
Example

The following is a depiction of a tropical affine manifold with singularities corresponding to a cubic surface in $\mathbb{CP}^3$:



Example

The following tropical curve corresponds to a line on the cubic surface, and there are 27 such!



How do we prove a correspondence theorem between tropical curves and holomorphic curves?

This is a significant part of a program I began with Bernd Siebert in 2001. The goal is to understand mirror symmetry by studying degenerations of the complex manifolds involved.

For a very simple example, consider a degeneration of a cubic surface

$$\mathcal{X} := \{tf_3 + x_0x_1x_2 = 0\} \subseteq \mathbb{A}^1 \times \mathbb{CP}^3.$$

The total space $\mathcal{X}$ has nine singular points at

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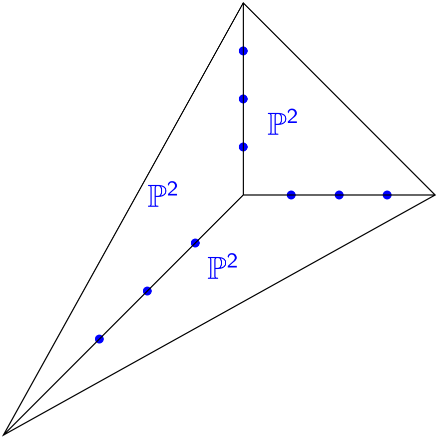
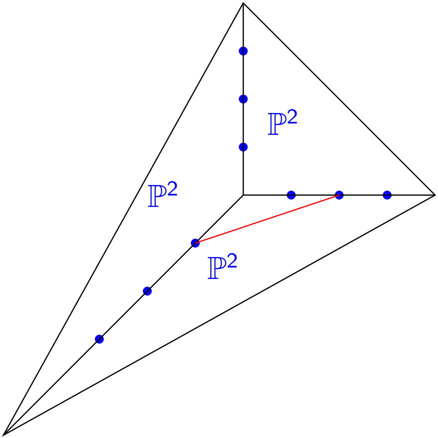
Schematically the central fibre looks like



Which lines on the central fibre deform to lines on the general fibre?

Answer:



To properly answer this question, one needs to introduce logarithmic geometry of Illusie, Fontaine and Kato.

This is a fundamental tool whenever we want to study degenerations.

Log structures on schemes or analytic spaces is a "magic powder" which allows one to treat certain singular schemes as being non-singular.

Theorem

(G.-Siebert, Abramovich-Chen 2011) There is a theory of logarithmic Gromov-Witten invariants which allows calculations of Gromov-Witten invariants on the general fibre of a degeneration by calculating logarithmic Gromov-Witten invariants of the singular fibre of a degeneration.

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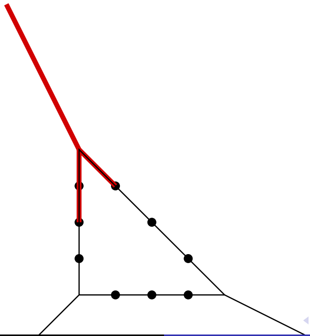
There is a direct connection between logarithmic and tropical geometry: in fact there is a functor from the category of log schemes to a "tropical" category, which we call the *tropicalization functor*.

In the example of the cubic surface, the tropicalization of the central fibre with its induced log structure is precisely B as drawn before, and the tropicalization of the stable log map as drawn before. Here B can be thought of as the "dual intersection graph" of the degeneration.



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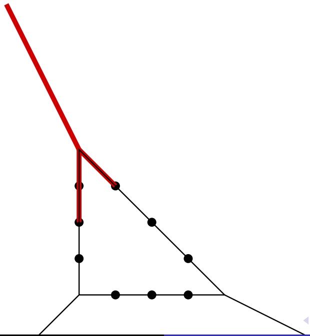

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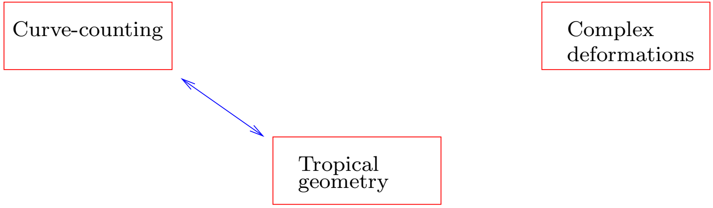

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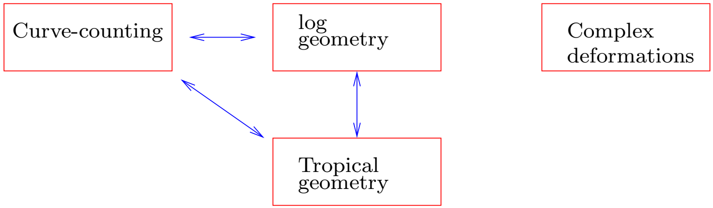

At the heart of my program with Siebert is an approach to connecting tropical geometry with complex deformations. This approach gives a very general mirror construction.

We wish to construct a complex manifold from B. What we will in fact construct is, with the choice of some extra data, a degenerating flat family

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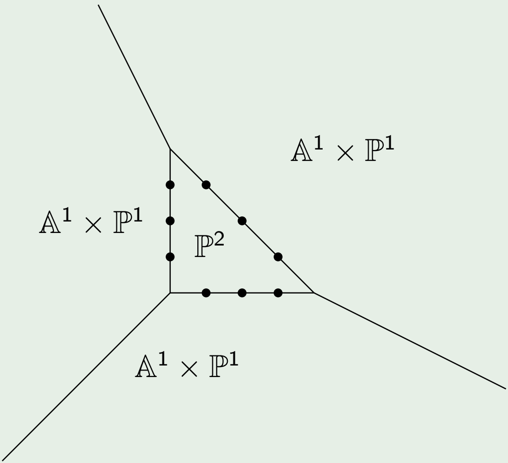
Example

Our old friend B interpreted as an intersection complex:



Call the central fibre $\check{X}_0$.

Theorem

(G.-Siebert, 2011) There is a construction of a smoothing $\mathcal{X} \to \mathrm{Spec}\, k[[t]]$ of X_0 controlled by tropical disks on B.

The tropical disks, counted in the right way, instruct us how to glue various pieces together to construct the smoothing.

This gives an algorithm for constructing mirrors:

1. Start with a "nice" degenerating family $\mathcal{X} \to S$ and let B be the dual integersection complex of this family.
2. Reinterpret B as an intersection complex and using the above theorem, build a mirror family $\check{\mathcal{X}} \to \mathrm{Spec}\, k[[t]]$.

Morally, the tropical disks governing the smoothing $\check{\mathcal{X}} \to \mathrm{Spec}\, k[[[t]]$ correspond to holomorphic disks in a fibre $\mathcal{X}_s$ of $\mathcal{X} \to S$ whose boundary lies in a fibre of an SYZ fibration $\mathcal{X}_s \to B$.

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One modifies the gluing of two charts by wall-crossing automorphisms determined by whether we cross the upper disk or the lower disk. The difference between the two gluing automorphisms removes the ambiguity produced by the monodromy.

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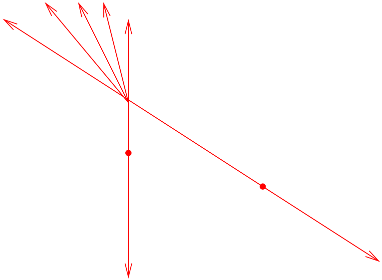

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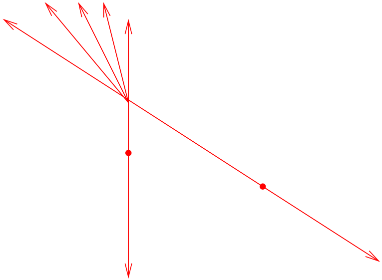

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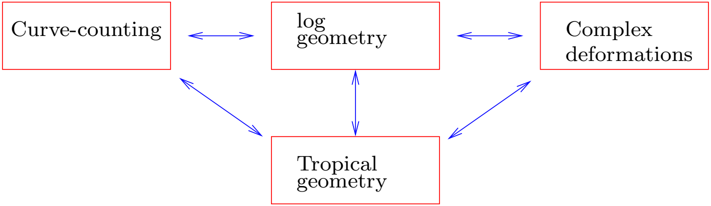
We have now obtained a symmetric picture:



Remarks.

- These 2007 techniques work when B has "nice" singularities, where we have a good local model for the deformation we are trying to construct, and we just need to glue together these local models.
- More recently, with P. Hacking and S. Keel, we introduced a notion of "theta function" (generalizing the notion of theta function on abelian varieties) which allow us to construct mirrors without local models. So far, this was done in a 2011 paper for the case of a mirror of a rational surface Y equipped with a cycle of rational curves in the anti-canonical linear system. However, we hope the use of theta functions will allow a generalization of Gross-Siebert to give the strongest possibly mirror construction.

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