Cluster algebras and mirror symmetry

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Cluster algebras

Cluster algebras were invented by Fomin and Zelevinsky in 2001 motivated by the combinatorics of dual canonical bases of Lusztig.

Fix a lattice $N \cong \mathbb{Z}^n$ along with a skew-symmetric bilinear form

$$\{\cdot,\cdot\} : N \times N \to \mathbb{Z}.$$

Let $M = \mathrm{Hom}(N, \mathbb{Z})$.
A *seed* is a choice of ordered basis $\mathbf{i} = (e_1, \ldots, e_n)$ for N.
We write the dual basis as $f_1, \ldots, f_n$.
We also associate to the seed a torus

$$A_{\mathbf{i}} := \mathrm{Spec}\, k[M] = \mathrm{Spec}\, k[A_1^{\pm 1}, \ldots, A_n^{\pm 1}],$$

where $A_i z^{f_i}$. For a seed, we obtain a corresponding skew-symmetric matrix B with

$$B_{ij} = \{e_i, e_j\}.$$

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We can define a *mutation* μ_k of a seed for $1 \le k \le n$:
$\mu_k(\mathbf{i}) = (e_1', \ldots, e_n')$, where

$$e_i' = \begin{cases} e_i + [B_{ik}]_+ e_k & i \ne k \\ -e_k & i = k \end{cases}$$

where $[a]_+ = \max(0, a)$.
The *exchange relation* defines a birational map between $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{A}_{\mu_k(\mathbf{i})}$ via the equations

$$A_k A_k' = \prod_{j:B_{kj}>0} A_j^{B_{kj}} + \prod_{j:B_{kj}<0} A_j^{-B_{kj}},$$

$$A_i' = A_i \qquad \text{for } i \ne k.$$

(Remember that B depends on the seed, so after mutating, B changes. Here we use the B coming from the basis $\mathbf{i}$.)

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We can compose these birational transformations, so if $\mathbf{i}$ and $\mathbf{i}'$ are two seeds related by a sequence of mutations, we obtain a birational transformation between $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{A}_{\mathbf{i}'}$.

Gluing all these tori together via these birational transformations gives the *$\mathcal{A}$-cluster variety*, and the ring of functions on this variety is the *upper cluster algebra* associated to the initial seed.

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Cluster algebras

Example

The classic example is to take $N = \mathbb{Z}^2$, and

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can start with cluster variables A_1, A_2 and mutate μ_1. With $A_3 = A_1'$, we get

$$A_1 A_3 = A_2 + 1, \text{ or } A_3 = \frac{A_2+1}{A_1}.$$

New set of cluster variables is $\{A_2, A_3\}$.

μ_2:

$$A_2 A_4 = A_3 + 1, \text{ or } A_4 = \frac{1+A_1+A_2}{A_1 A_2}$$

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$$A_4A_6 = A_5 + 1, \text{ or } A_6 = A_1$$

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$$A_5A_7 = A_6 + 1, \text{ or } A_7 = A_2$$

so we get a cycle returning to the begininng.

Note the equations $A_{i-1}A_{i+1} = A_i + 1$ for i mod 5 define an affine del Pezzo surface of degree 5.
The cluster tori come from the five different ways of describing a del Pezzo surface of degree 5 as a blowup of a toric surface.

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Goal: Give an alternate description of cluster algebras motivated by mirror symmetry.

The ideas which follow are joint work with Paul Hacking, Sean Keel, and Maxim Kontsevich, making use of notions developed in the pursuit of understanding mirror symmetry, namely *scattering diagrams* and *theta functions*:

- M. Kontsevich and Y. Soibelman, "Affine structures and non-archimedean analytic spaces," 2004.
  Scattering diagrams in two dimensions.
- M. Gross and B. Siebert, "From real affine geometry to complex geometry," 2007.
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- M. Gross, P. Hacking and S. Keel, "Mirror symmetry for log Calabi-Yau surfaces I," 2011.
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Continuing with the previous notation, we have

Definition

A *wall* in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where:

1. $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is a convex rational polyhedral cone of codimension one (not necessarily strictly convex), with an element $m_0 \in M \setminus \{0\}$ tangent to $\mathfrak{d}$.
2. $f_{\mathfrak{d}} \in k[M][[x_1, \ldots, x_n]]$ such that

$$f_{\mathfrak{d}} = 1 + \sum_{k \geq 1} c_k z^{km_0}$$

where c_k is a polynomial in the ideal $(x_1, \ldots, x_n)$.

If $m_0 \in \mathfrak{d}$, we say $\mathfrak{d}$ is *incoming*, otherwise we say $\mathfrak{d}$ is *outgoing*.

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Definition

A *scattering diagram* $\mathfrak{D}$ is a collection of walls such that for each $k \geq 0$, the set

$$\{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \mid f_{\mathfrak{d}} \not\equiv 1 \mod (x_1, \ldots, x_n)^k\}$$

is finite.

Scattering diagrams

Given a scattering diagram $\mathfrak{D}$, set

- $\operatorname{Supp}\mathfrak{D} = \bigcup_{\mathfrak{d}\in\mathfrak{D}} \mathfrak{d}$.
- $\operatorname{Sing}(\mathfrak{D})$ = locus where $\operatorname{Supp}\mathfrak{D}$ is not a manifold.

For a path $\gamma : [0,1] \to M_{\mathbb{R}} \setminus \operatorname{Sing}(\mathfrak{D})$, with endpoints not in $\operatorname{Supp}(\mathfrak{D})$, we can define

$$\theta_{\gamma,\mathfrak{D}} \in \operatorname{Aut}_{k[[x_1,\ldots,x_n]]}(k[M][[x_1,\ldots,x_n]])$$

called the *path ordered product*.

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called the *path ordered product*.

First, if γ crosses a wall $(\mathfrak{d}, f_{\mathfrak{d}})$, we associate an automorphism to this wall crossing

$$\theta_{\gamma,\mathfrak{d}}(z^m) = z^m f_{\mathfrak{d}}^{\langle n_0, m\rangle}$$

where $n_0 \in N$ is defined by

- n_0 annihilates $\mathfrak{d}$;
- n_0 is primitive;
- $\langle \gamma'(t_0), n_0\rangle < 0$ at the time t_0 when γ crosses $\mathfrak{d}$.

$\theta_{\gamma,\mathfrak{D}}$ is then defined to be the (possibly infinite) composition of such automorphisms in the order traversed.

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- n_0 annihilates $\mathfrak{d}$;
- n_0 is primitive;
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Scattering diagrams

The fundamental construction: Start with a seed $\mathbf{i}$.

Let $v_i = \{e_i, \cdot\} \in M$.

Define

$$\mathfrak{D}_{in} := \{(e_i^{\perp}, 1 + x_i z^{v_i}) \,|\, 1 \leq i \leq n\}.$$

Theorem

There exists a scattering diagram $\mathfrak{D} \supseteq \mathfrak{D}_{in}$ such that $\mathfrak{D} \setminus \mathfrak{D}_{in}$ contains no incoming walls and $\theta_{\gamma,\mathfrak{D}} = \mathrm{id}$ for every loop γ for which this is defined.

This is a special case of a result of G.–Siebert generalizing a two-dimensional result of Kontsevich and Soibelman.

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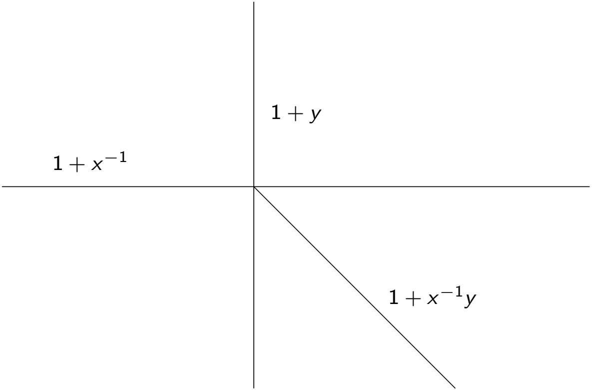
Examples. Take $N = \mathbb{Z}^2$,

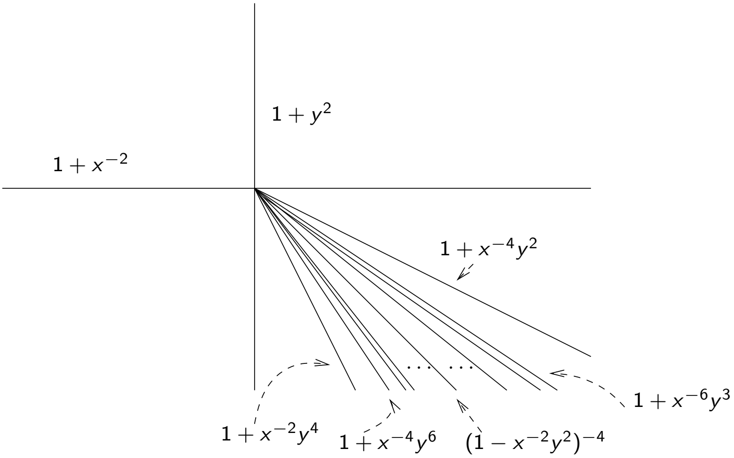
$$B = \begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}.$$

For $\ell = 1$ we obtain

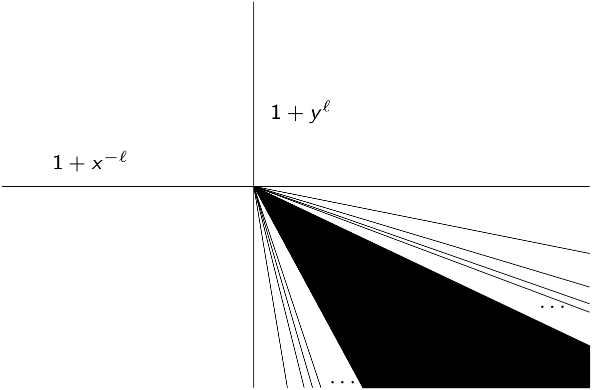


Scattering diagrams

$\ell = 2$



$\ell \geq 3$.



Scattering diagrams

$N = \mathbb{Z}^3$,

$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

The structure of $\mathfrak{D}$ in general. We will set all x_i's to be 1 for simplicity of discussion, but this may cause convergence issues, which we shall ignore.

(1) $\mathcal{C}_+ = \{m \in M_\mathbb{R} \mid \langle e_i, \cdot \rangle > 0\}$ does not intersect any walls of $\mathfrak{D}$.

(2) If we perform a mutation μ_k to get a seed $\mathbf{i}'$, we obtain new scattering diagrams $\mathfrak{D}'_{in}$ and $\mathfrak{D}'$.

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Scattering diagrams

Let $T_k : M_{\mathbb{R}} \to M_{\mathbb{R}}$ be defined by

$$T_k(m) = \begin{cases} m + \langle e_k, m\rangle v_k & \langle e_k, m\rangle \geq 0 \\ m & \langle e_k, m\rangle \leq 0 \end{cases}$$

Define $T_k(\mathfrak{D})$ to be the scattering diagram obtained from $\mathfrak{D}$ by

- Applying T_k to each wall (might break a wall into two).
- Replacing the initial wall

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Thus in particular the chamber $\mathcal{C}'_+$ defined by the seed $\mathbf{i}'$ gives a chamber $T_k^{-1}(\mathcal{C}'_+)$ of $\mathfrak{D}$. Every mutation then corresponds to a chamber of $\mathfrak{D}$.

Every such scattering diagram $\mathfrak{D}$ then has a region decomposed into a chamber structure.

If we associate the torus $\operatorname{Spec} k[M]$ to each such region and glue these tori together using the wall-crossing automorphisms, we obtain the $\mathcal{A}$-cluster variety.

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Broken lines

Goal: Find a "nice" basis for the cluster algebra indexed by points of M (a canonical basis).

Currently there are many constructions in special cases, many of which differ. I will give a construction which we believe will give a basis in all cases when a basis is known to exist.

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Broken lines

Fix a seed and the corresponding $\mathfrak{D}$.

Definition

A *broken line* for $m_0 \in M \setminus \{0\}$ with endpoint $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$ general is

- a proper piecewise linear path with a finite number of domains of linearity
$$\gamma : (-\infty, 0] \to M_{\mathbb{R}}$$
- a monomial $c_L z^{m_L} \in k[M]$ attached to each domain of linearity $L \subseteq (-\infty, 0]$ of γ;

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Definition

1. $\gamma'(t) = -m_L$ for $t \in L$.
2. The monomial attached to the first domain of linearity of γ is z^{m_0}.
3. $\gamma(0) = Q$.
4. γ bends only when it crosses a wall. When γ crosses $(\mathfrak{d}, f_{\mathfrak{d}})$ in passing between domains of linearity L and L', then $c_{L'} z^{m_{L'}}$ is a term in $\theta_{\gamma,\mathfrak{d}}(c_L z^{m_L})$.

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Definition

Let $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$ be a general choice of point.

For a broken line γ, denote by $\mathrm{Mono}(\gamma)$ the monomial attached to the *last* domain of linearity of γ.

Define

$$\mathrm{Lift}_Q(m_0) = \sum_{\gamma} \mathrm{Mono}(\gamma)$$

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Theorem

If Q, Q' are two general points on $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$, and γ is a path joining Q and Q', then

$$\theta_{\gamma,\mathfrak{D}}(\mathrm{Lift}_Q(m_0)) = \mathrm{Lift}_{Q'}(m_0).$$

Corollary

If $Q \in \mathcal{C}^+$ and $\mathrm{Lift}_Q(m_0)$ is a finite sum, then

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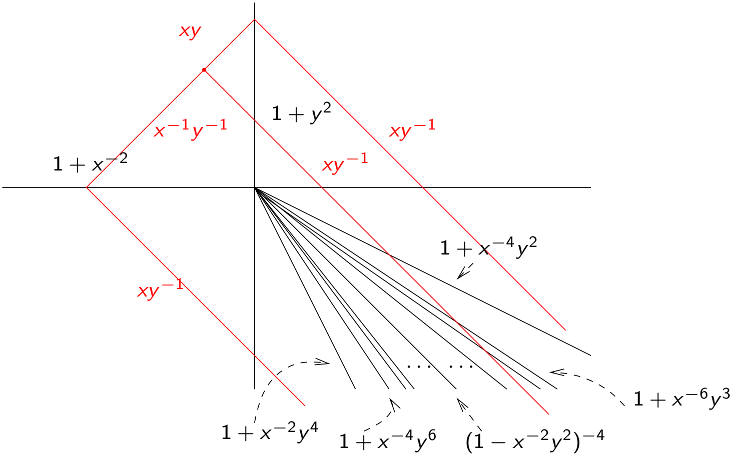
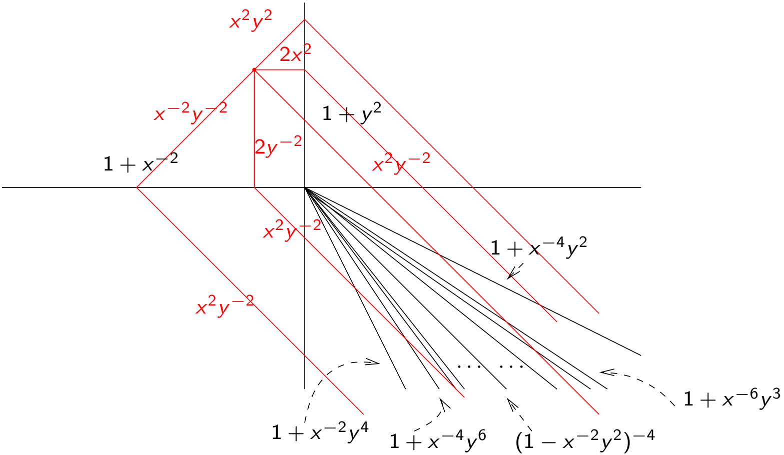


$$\vartheta_{(1,-1)} = xy^{-1}(1 + x^{-2} + y^2).$$

Broken lines



$$\vartheta_{(2,-2)} = x^2y^{-2}(1+2x^{-2}+2y^2+x^{-4}+y^{-4}) = \vartheta_{1,-1}^2 - 2.$$

Features of theta functions:

- If m_0 lies in a chamber corresponding to a seed, then ϑ_{m_0} is a cluster monomial on the corresponding torus.
- In general ϑ_{m_0} might involve an infinite number of terms, and canonical bases won't exist in general. However, we are extending the range in which we can prove theta functions give finite sums and canonical bases.
- This setup can be used to prove other conjectures in cluster algebras, such as positivity of the Laurent phenomenon, and give simple proofs of known results whose current proofs involve representation theory.

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