# Logarithmic geometry

**Log schemes.** Kato: A log structure on a scheme is a magic powder which makes it non-singular.

**Def:** A *log scheme* is a triple

$X = (\underline{X}, \mathcal{M}_X, \alpha_X)$ where

- $\underline{X}$ is a scheme
- $\mathcal{M}_X$ is a sheaf of (commutative) monoids on $X$

  (monoid = group with inverses)
- $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ is a homomorphism of sheaves of monoids, with the monoid structure on $\mathcal{O}_X$ being given by multiplication.

This data satisfies one property:

$$\alpha_X^{-1}(\mathcal{O}_X^*) \xrightarrow[\cong]{\alpha} \mathcal{O}_X^*$$

A morphism $f : (\underline{X}, M_X, \alpha_X) \to (\underline{Y}, M_Y, \alpha_Y)$

of log schemes is data

- $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ an ordinary morphism of schemes.

- $f^\flat : f^{-1} M_Y \longrightarrow M_X$ a morphism of sheaves of monoids

such that the following diagram is commutative:

$$\begin{tikzcd}
f^{-1} M_Y \arrow[r, "f^\flat"] \arrow[d, "f^{-1}\alpha_Y"'] & M_X \arrow[d, "\alpha_X"] \\
f^{-1} \mathcal{O}_Y \arrow[r, "f^\#"'] & \mathcal{O}_X
\end{tikzcd}$$

Examples: 1. The divisorial log structure. Let $D \subseteq X$ be a Weil divisor, $U = X \setminus D$

(viewing $D$ as just giving a closed subset, i.e., if $D = \sum a_i Y_i$ with $a_i > 0$, think of $D$ as the closed subset $\bigcup Y_i$)

Define $\mathcal{M}_X$ by, for $V \subseteq X$,

$$\mathcal{M}_X(V) = \left\{ f \in \mathcal{O}_X(V) \;\middle|\; f|_{U \cap V} \in \mathcal{O}_X^*(U \cap V) \right\}.$$

Note that we have an obvious map

$\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ (an inclusion)

Also, $\mathcal{O}_X^* \subseteq \mathcal{M}_X$ is a subsheaf and $\mathcal{O}_X^* = \alpha^{-1}(\mathcal{O}_X^*)$.

Given $(\underline{X}, D)$, $(\underline{Y}, E)$, a morphism of schemes $\underline{f} : \underline{X} \to \underline{Y}$ induces a morphism of divisorial log schemes provided that $f^{-1}(E) \subseteq D$.

This is because given $\varphi \in M_Y(V)$, this is a function away from $E$, and hence $f^\# \varphi$ is invertible away from $f^{-1}(E)$, hence also invertible away from $D$.

$$\begin{tikzcd}
f^{-1} M_Y \arrow[r, dashed, "f^\flat"] \arrow[d] & M_X \arrow[d, hook] \\
f^{-1}\mathcal{O}_Y \arrow[r, "f^\#"'] & \mathcal{O}_X
\end{tikzcd}$$

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

## ② The standard log point.

$$\operatorname{Spec} k^{\dagger} = (\operatorname{Spec} k,\ k^{*} \oplus \mathbb{N},\ \alpha_X)$$

$$\alpha_X(r, n) = \begin{cases} r & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

$$\text{"} = r\, 0^{n} \text{"} \qquad (0^0 = 1)$$

$$\alpha_X^{-1}(k^{*}) = k^{*} \oplus 0$$

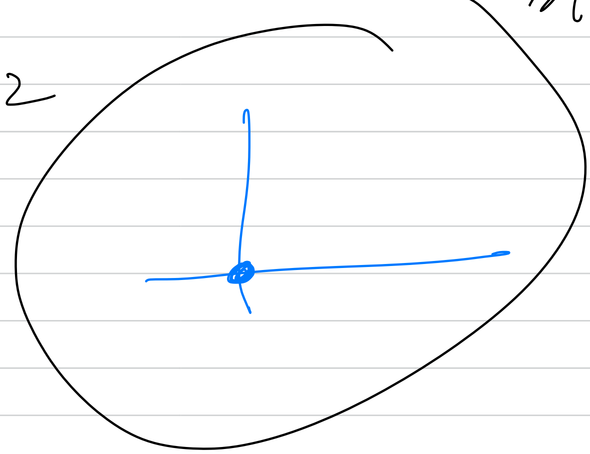
**Example of an "exotic" morphism:**

$$\operatorname{Spec} k^{\dagger} \xrightarrow{f} (\mathbb{A}^2, V(xy))$$

where $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$.

Consider only scheme maps with image $0 \in \mathbb{A}^2$

$\mathbb{A}^2$



$$\begin{tikzcd}
\mathcal{M}_{\mathbb{A}^2,0} = f^{-1}\mathcal{M}_{\mathbb{A}^2} \arrow[r, dashed, "f^\flat"'] \arrow[d, "\alpha"'] & k^* \oplus \mathbb{N} \arrow[d, "\alpha"] \\
\mathcal{O}_{\mathbb{A}^2,0} = f^{-1}\mathcal{O}_{\mathbb{A}^2} \arrow[r, "f^\#"'] & \mathcal{O}_{\operatorname{Spec} k} = k
\end{tikzcd}$$

$$\mathcal{O}_{\mathbb{A}^2,0} \ni \varphi \longmapsto \varphi(0) \in k$$

First exercise: Note

$$\mathcal{M}_{\mathbb{A}^2,0} = \{ \varphi x^a y^b \mid \varphi \in \mathcal{O}^*_{\mathbb{A}^2,0} \}$$

germ of an invertible function at $0$

Convince yourself the possibilities for $f$ are precisely maps of the form

$$\varphi x^a y^b \longmapsto (\varphi(0)\cdot\psi(a,b), \eta(a,b)) \in k^* \oplus \mathbb{N}$$

where $\psi : \mathbb{N}^2 \to k^*$ is arbitrary and $\eta : \mathbb{N}^2 \to \mathbb{N}$ is arbitrary of the form $\eta(a,b) = \alpha a + \beta b$

for some $\alpha, \beta > 0$.

Lots of choices.

**Why do we care?**

There are new morphisms which are not smooth as morphisms of schemes but are smooth as morphisms of log schemes.

$$\underline{f} : \underline{\mathbb{A}^2} \longrightarrow \underline{\mathbb{A}^1}$$
$$(x,y) \longmapsto x \cdot y$$

$$f^{-1}(0) = V(xy)$$

Singular!

Smooth "=" all fibres are non-singular.

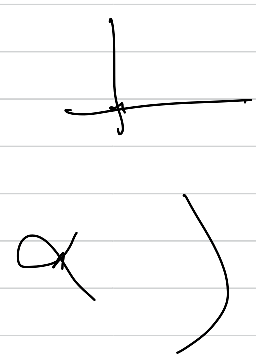
$$f : (\mathbb{A}^2, V(xy)) \longrightarrow (\mathbb{A}^1, 0)$$

This morphism is <u>log smooth</u>.

In particular, this allows us to put a log structure on $V(xy)$ making it behave as if it is non-singular.

<u>Gromov–Witten theory.</u>

<u>Classical version:</u> $X$ a non-singular variety. A <u>stable map</u> $f : C \longrightarrow X$ is a morphism from a curve $C$ with at worst nodal singularities (locally looks like $V(xy)$



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such that $f$ has a finite automorphism

group. An automorphism of $f$ is an automorphism $\varphi: C \to C$ such that $f \circ \varphi = f$.

We can also consider map

$$f : (C, p_1, \dots, p_n) \longrightarrow X$$

where $p_1, \dots, p_n \in C$ are distinct non-singular points of $C$. Now an automorphism $\varphi : (C, p_1, \dots, p_n) \circlearrowleft$ must satisfy $\varphi(p_i) = p_i$

<u>Example</u>: $X \subseteq \mathbb{P}^3$ a non-singular cubic surface. $C = \mathbb{P}^1$

$f : C \longrightarrow X$ a closed immersion with image a line.

There are 27 such maps.

Gromov-Witten theory tells us that there is a moduli space of maps

$M_{g,n}(X, \beta)$ where

- $g$ is the genus of the domain curve.
- $n$ is the number of marked points on $C$
- $\beta \in H_2(X, \mathbb{Z})$ is the homology class represented by $f$ ($f_*[C]$) (or think degrees of curves.)

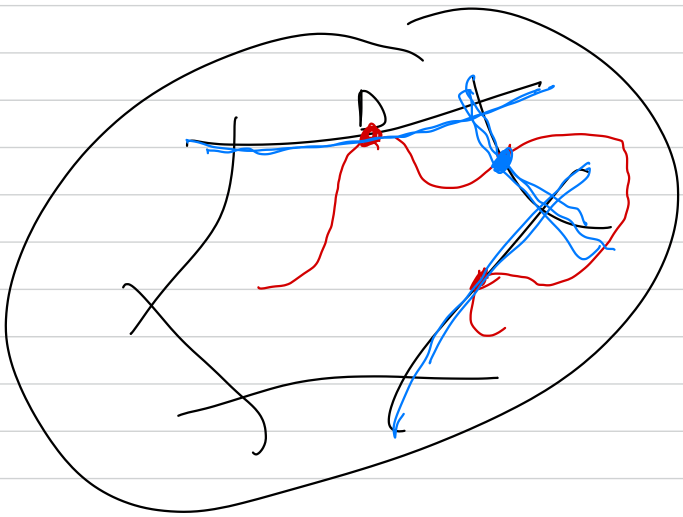
e.g. $X$ the cubic surface

$M_{0,0}(X, 1)$ consists of 27 points.
$\uparrow$ degree 1

Gromov-Witten theory gives us

a way of counting the # of points in this moduli space.

Log Gromov-Witten theory



$(X, D)$ $X$ non-singular

$D$ is nice

We wish to consider stable maps

$$f: (C, p_1, \dots, p_n) \longrightarrow (X, D)$$

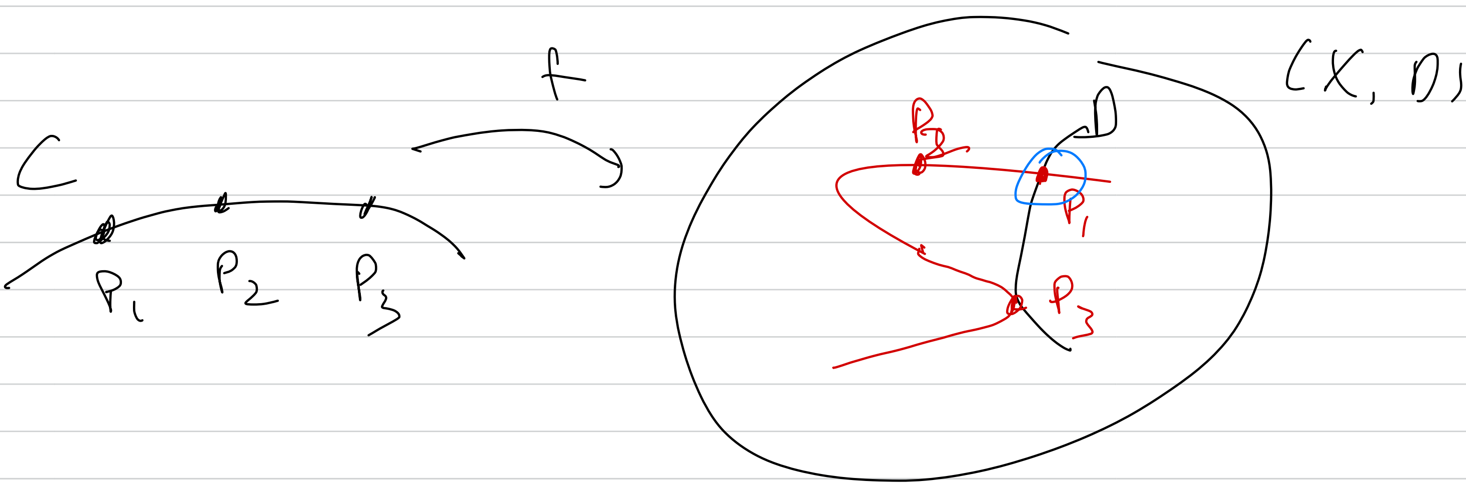
where we impose some orders of tangency with irreducible components of $D$ at each of the marked points.

Basic problem: One wants a compact moduli space. We have trouble when limits of curves fall into $D$, as then it is hard to define tangency conditions.

Log geometry solves this!

## Simple example of how this works

$D \subseteq X$, $X$ non-singular, $D$ a non-singular prime divisor




$f^{-1}(D) \subseteq \{P_1, P_2, P_3\}$, so we do get a morphism of log schemes

$$(C, \{P_1, P_2, P_3\}) \longrightarrow (X, D)$$

Near $P_1$, we can write a section of

$M_X$ as $\varphi \cdot t^a$ where $t=0$

is the local defining equation for $D \subseteq X$

and $\varphi$ is invertible.

$$f^{\#}(\varphi \cdot t^a) = (\varphi \circ f) \cdot (t \circ f)^a$$

$t \circ f$ vanishes at $p_1$ to order $1$ because

$f$ is transversal to $D$ at $p_1$

So if $u$ is a local coordinate

at $p_1 \in C$, then $f^{\#}(\varphi \cdot t^a) = \psi \cdot u^a$

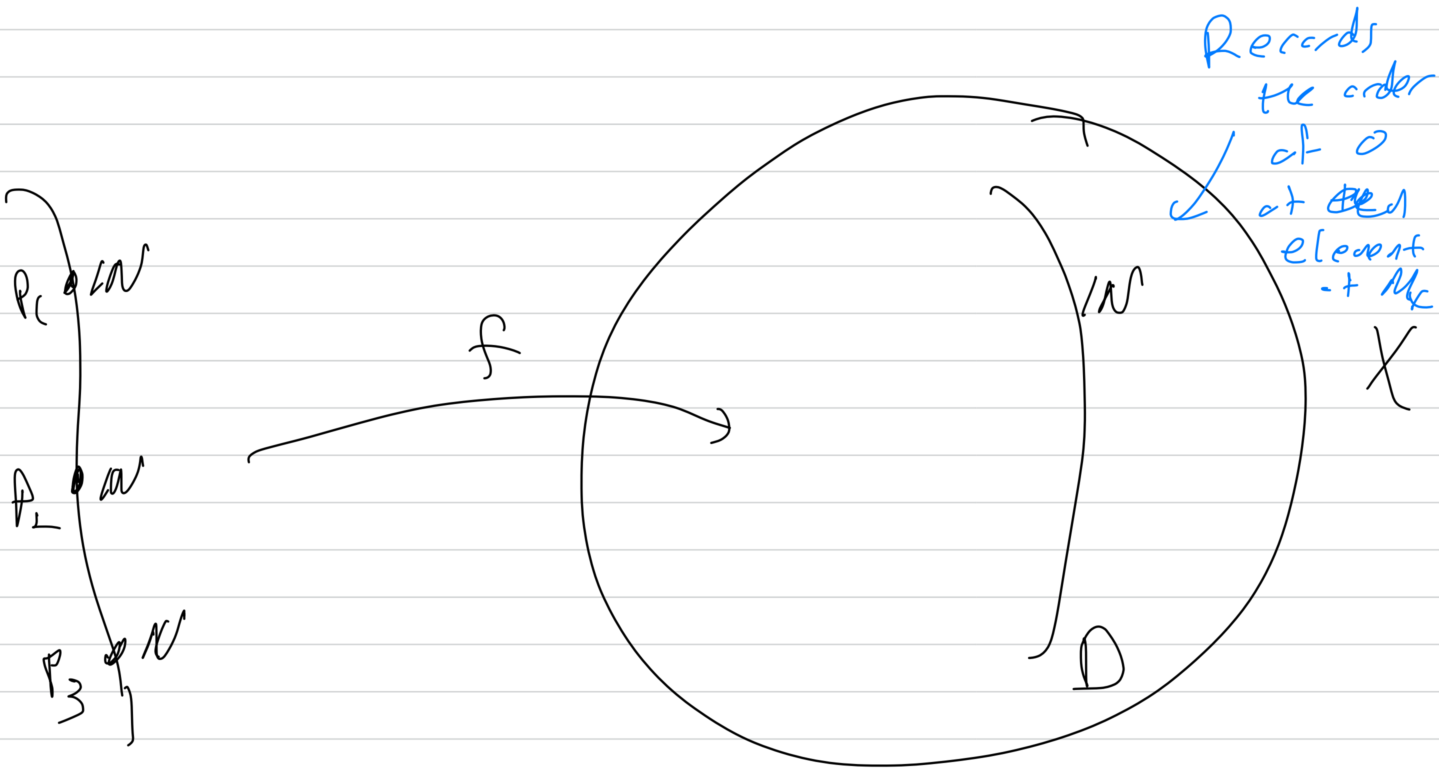
where $\psi$ is invertible.

At $p_3$, where $f$ is simply tangent,

$t \circ f$ will vanish to order 2.

So $f^{\#}(\varphi \cdot t^a) = \psi \cdot u^{2a}$

Recalling that $\mathcal{O}_X^* \xhookrightarrow[\alpha^{-1}]{} M_X$, We define the ghost sheaf of $\mathcal{M}_X$ to be $\mathcal{M}_X / \mathcal{O}_X^*$




$$f^\flat : f^{-1}\mathcal{M}_X \longrightarrow \mathcal{M}_C$$

induces $\bar{f}^\flat : f^{-1}\overline{\mathcal{M}}_X \longrightarrow \overline{\mathcal{M}}_C$

At each $p_i$ with $f(p_i) \subseteq D$,

we then obtain a map

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\cdot c} & \mathbb{N} \\ \| & & \\ (f^{-1}\overline{\mathcal{M}}_X)_{p_i} & \longrightarrow & \overline{\mathcal{M}}_{C,p_i} \end{array}$$

where $c$ is the order of tangency of $f$ at the point $p_i \in C$.

# Algebraic stacks

Stacks are generalizations of schemes.

**Moduli spaces:** Fix a field $k$.

Consider a contravariant functor

$$F : Sch/k \longrightarrow Sets$$

$$F(S) = \left\{ \begin{array}{c} C \\ \downarrow \\ S \end{array} \quad \begin{array}{l} \text{a proper (flat) morphism,} \\ \text{each of whose fibres} \\ \text{is a non-singular genus } g \text{ curve} \end{array} \right\} \Big/ \cong$$

$$\begin{array}{c} C \\ \downarrow \\ S \end{array} \cong \begin{array}{c} C' \\ \downarrow \\ S \end{array} \quad \text{if } \exists \text{ a commutative diagram}$$

$$\begin{tikzcd} C \arrow[rr, "\cong"] \arrow[dr] & & C' \arrow[dl] \\ & S & \end{tikzcd}$$

e.g. if $S = \operatorname{Spec} k$, this is just the set of isomorphism classes of genus $g$ curves.

Want: A scheme $M_g$ which represents this functor, i.e., for each $S$, we have a bijection

$$F(S) \to \operatorname{Hom}_k(S \to M_g).$$

(i.e., an isomorphism of functors $F \cong h_{M_g}$.)

Such a scheme should have a universal curve $\begin{array}{c}\mathcal{C}\\ \downarrow\\ M_g\end{array}$ given by the identity $1_{M_g} \in \operatorname{Hom}(M_g, M_g) = h_{M_g}(M_g)$

with the property that for any

$\begin{array}{c} C \\ \downarrow \\ S \end{array} \in F(S)$, $\exists !$ morphism

$S \longrightarrow \mathcal{M}_g$ such that

the diagram
$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

gives an isomorphism $C \xrightarrow{\simeq} S \times_{\mathcal{M}_g} \mathcal{C}$.

[Forgot to mention:

$$F(S_1 \longrightarrow S_2) : F(S_2) \to F(S_1)$$

is given by $\begin{array}{c} C_2 \\ \downarrow \\ S_2 \end{array} \longmapsto \begin{array}{c} C_2 \times_{S_2} S_1 \\ \downarrow \\ S_1 \end{array}$

This is a way of going from a family of curves over $S_2$ to a family of curves over $S_1$.]

Problem: $F$ is not representable!

Consider a curve $C$ with an automorphism $\varphi: C \longrightarrow C$ with $\varphi^2 = id$

Let $S = \mathbb{A}^1 \setminus \{0\} = k \setminus \{0\} = k^*$
$\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad \operatorname{Spec} k[x, x^{-1}]$

Family ①: $C \times_{\operatorname{Spec} k} S$
$$\downarrow$$
$$S$$

(every fibre is isomorphic to $C$.)

Family ②: $(C \times S) / \mathbb{Z}_2$
$$\downarrow$$
$$S/\mathbb{Z}_2 \cong S$$

where $1 \in \mathbb{Z}_2$ acts on $C \times S$

by $(c, s) \longmapsto (\varphi(c), s^{-1})$

Here $s \longmapsto s^{-1}$ is the morphism $S \to S$ induced by $k[x, x^{-1}] \to k[x, x^{-1}]$

$$x \longmapsto x^{-1}$$
$$x^{-1} \longmapsto x$$

$1 \in \mathbb{Z}_2$ acts on $S$ by $s \mapsto s^{-1}$.

All fibres are isomorphic $C$, but this is a different family!

Suppose $M_g$ was representable. We should get two maps

$$S \longrightarrow M_g$$ corresponding

to these two different families.

But the images of these two maps is the <u>same</u> point of $\mathcal{M}_g$, corresponding to the curve $C$.

But only exists one such map, so we can't get two different families!

But: $\underset{\sim}{\mathcal{M}_g}$ is represented by a stack.

Deligne-Mumford.

Basic idea: replace the set $F(S)$ with category, in particular, a groupoid.

A <u>groupoid</u> is a category all of whose morphisms are isomorphisms.

A stack $\mathcal{F}$ is a category fibred in groupoids over $Sch/k$ with a long list of extra properties.

This means $\mathcal{F}$ is a category with a covariant functor $\varphi : \mathcal{F} \to Sch/k$ such that if $x, y \in Ob\,\mathcal{F}$ with $\varphi(x) = \varphi(y)$, then every element of $Hom(x, y)$ is an isomorphism.

Example: $\mathcal{M}_g$ : category whose objects are families of genus $g$ curves

$$\begin{array}{c} C \\ \downarrow \\ S \end{array}$$

as before and whose morphisms are commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_2 \arrow[d] \\ S_1 \arrow[r] & S_2 \end{tikzcd}$$

inducing an isomorphism $C_1 \xrightarrow{\cong} S_1 \times_{S_2} C_2$

Algebraic stacks are such categories which have a smooth cover by schemes.

Example: Let $G$ be a ~~finite~~ algebraic group. We define $BG$ to be the category whose objects are $G$-torsors

$$\begin{matrix} E \\ \downarrow \\ S \end{matrix}$$

for $S$ a scheme.

Here being a $G$-torsor means that $G$ acts on $E$ via commutative diagrams (for $g \in G$)

\begin{tikzcd}
\mathcal{E} \arrow[rr, "g"] \arrow[dr] & & \mathcal{E} \arrow[dl] \\
& S &
\end{tikzcd}

and locally on $S$, $\mathcal{E}$ is isomorphic to $S \times G$

$$BG \to Sch_{\mathcal{U}}$$
$$(\mathcal{E} \to S) \mapsto S$$

If $G$ acts on a scheme $X$, we have a stack $[X/G]$ whose objects are diagrams

\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi"] \arrow[d, "G\text{-torsor}"'] & X \\
S &
\end{tikzcd}

with $g \cdot \varphi(e) = \varphi(g \cdot e)$ for $g \in G$, $e \in \mathcal{E}$.

Canonical application is the construction of the moduli space $\mathcal{M}_g$.

<u>Most accessible reference:</u>

Olsson: Algebraic spaces and stacks.