Project: Global existence for non-linear wave equations on $\mathbb{R}^{3+1}$

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The purpose of this project is to understand the question of *small data global existence* for non-linear wave equations on $\mathbb{R}^{3+1}$ of the form

$$\Box\psi = Q(\nabla\psi, \nabla\psi) \tag{1}$$

where Q is a quadratic non-linearity. Equation (1) can the thought of as a toy model for several more complicated equations of mathematical physics, like the compressible Euler equations for irrotational fluid flow or the Einstein equations of general relativity.

The behaviour of (1) for small data solutions can be considerably different according to whether Q is a null form, e.g.

$$Q = -(\partial_t\psi)^2 + (\partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2, \tag{2}$$

or otherwise, e.g.

$$Q = (\partial_t\psi)^2. \tag{3}$$

In the latter case of (3), Fritz John [3] proved that solutions to (1) **blow up in finite time** for *arbitrarily small*, smooth compactly supported initial data. On the other hand, in the case where Q satisfies the so-called "null condition", Klainerman [4] and Christodoulou [1] independently proved that solutions arising from such data exist for all time.

The purpose of this project is to understand the proof of the global existence of [1, 4]. The group can select which approach they prefer to follow.

The Einstein equations of general relativity, expressed in harmonic coordinates, can be written as a tensorial, quasilinear version of (1):

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu} = Q(g, \partial g, \partial g). \tag{4}$$

The reduced equations (4) do not, however, satisfy the classical null condition of Klainerman [4]. Nonetheless, stability of the trivial solution (Minkowski space) has been proven, originally in the monumental monograph of Christodoulou and Klainerman [2]. In the original proof of [2], the equations are rewritten in a more geometric gauge in which the essential "null structure" of the non-linearity is made evident. In a more recent proof due to Lindblad–Rodnianski [5], a weaker version of the null condition is identified which is sufficient for proving global existence. While a full treatment of the question of stability of Minkowski space is beyond the scope of this project, the project should end with an expository discussion of the problem at the level of the introduction of [5].

References

[1] D. Christodoulou *Global solutions of nonlinear hyperbolic equations for small initial data* Comm. Pure Appl. Math. **39** (1986), 267–282

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[3] F. John *Blow-up for quasi-linear wave equations in three space dimensions* Comm. Pure Appl. Math. **34** (1981), 29–51

[4] S. Klainerman *The null condition and global existence to nonlinear wave equations*, Lect. Appl. Math **23** (1986), 293–326

[5] H. Lindblad and I. Rodnianski *Global stability of Minkowski space-time in harmonic gauge*, Ann. of Math. **171** (2010), No. 3, 1401–1477