PROJECT: NON-UNIQUENESS OF WEAK SOLUTIONS TO THE INCOMPRESSIBLE EULER EQUATIONS

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There is a plethora of important PDE whose initial value problem admits multiple weak solutions arising from the same initial configurations. A notable such example can be found in the context of *incompressible fluids* with constant density, satisfying the *incompressible Euler equations*:

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}, \tag{1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n, t \in \mathbb{R}$, is the fluid velocity vector and $p(x,t)$ the pressure.

Definition. A vector function $u(x,t) \in L^2_{\mathrm{loc}}$ is called a *weak solution* of the incompressible Euler equations, if it satisfies the following integral form of (1):

$$\begin{cases} \int_{\mathbb{R}^n \times \mathbb{R}} u \frac{\partial v}{\partial t} + u \cdot \nabla_u v \, dxdt = 0 \\ \int_{\mathbb{R}^n \times \mathbb{R}} u \cdot \nabla \varphi \, dxdt = 0 \end{cases}, \tag{2}$$

for every divergence free vector $v \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$, $\nabla \cdot v = 0$, and function $\varphi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R})$.

Scheffer in [12] first made the striking discovery of the existence of weak solutions of (2) which are *compactly supported in space-time* for $n = 2$. This implies in particular the **non-uniqueness** of the trivial solution $u \equiv 0$, considered as a solution arising from trivial initial data. A different and simpler construction of such solutions was later given by Shnirelman [13] again for $n = 2$.

In contrast to the non-uniqueness phenomenon of weak solutions, one can show that C^1 solutions u, p of (1) are determined uniquely by the value of u at a single time slice $u(x, t_0)$ and that the total *kinetic energy*, $\frac{1}{2} \int |u|^2 dx$, is a constant function of time, i.e., the energy of u is *conserved*.

The students undertaking this project should focus on understanding the nature of the aforementioned non-unique 'unphysical' solutions to the Euler equations following Shnirelman [13]. They should be able to convey the idea behind the simpler construction in [13] and present adequate details on the analytical part of the argument.

[We note in conclusion that this project could serve as an introduction to very recent research resolving a long standing problem in the subject, namely, to find the lowest regularity for which solutions to (2) conserve energy. We elaborate briefly here. Onsager conjectured in '49 [14] that for $n = 3$ a weak solution u in the Hölder space $C_t C_x^\alpha$, $\alpha > \frac{1}{3}$, must conserve energy and that there should exist weak solutions $u \in L_t^\infty C_x^\alpha$, for $\alpha < \frac{1}{3}$, that do not conserve energy. The first part of the conjecture was positively confirmed by Constantine, E and Titi [5], following a weaker result of Eyink [8]. On the other hand, the existence of low regularity solutions to (2) with nonconstant in-time kinetic energy, up to the threshold Hölder exponent $\frac{1}{3}$, had remained unsolved until recently. Major advances were achieved in the past few years by De Lellis and Székelyhidi [6, 7], where the authors managed to adapt a method originating in Nash's proof of his C^1 isometric embedding theorem [11], called 'convex integration', constructing weak solutions in $L_t^\infty C_x^\alpha$, $\alpha < \frac{1}{10}$, that do not conserve energy. Following the preceding developments, after a series of papers [9, 2, 1, 3], Onsager's

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conjecture was finally settled by Isett [10], who constructed weak solutions in $L_t^\infty C_x^\alpha$, for all $\alpha < \frac{1}{3}$, having compact support in time. A simplified, more general result was obtained by Buckmaster, De Lellis, Székelyhidi and Vicol [4] two days ago!]

References

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