Mathematical topics in General Relativity

Mihalis Dafermos

E2.08

M.Dafermos@dpmms.cam.ac.uk

Outline

1. The initial value problem in general relativity. Stability of Minkowski space. Strong and weak cosmic censorship conjectures.
2. Symmetry and the IVP. Spherical symmetry. Penrose diagrams.
3. Examples: self-gravitating Higgs fields, Yang-Mills fields, sigma models (wave maps), collisionless matter (Vlasov).
4. Weak cosmic censorship and the role of trapped surfaces.
5. Price's power-law-tails and the stability of black holes.
6. The internal structure of black holes and "mass inflation". Strong cosmic censorship.

1. The IVP in general relativity

References

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[2] Demetrios Christodoulou *The global initial value problem in general relativity* Proceedings of the Marcel Grossman Meeting, Rome

[3] Demetrios Christodoulou *On the global initial value problem and the issue of singularities* Classical Quantum Gravity **16** (1999), no. 12A, A23–A35

[4] Demetrios Christodoulou and Sergiu Klainerman *The global nonlinear stability of the Minkowski space* Princeton Univ. Press, 1993

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Causal structure

An n-dimensional spacetime $(\mathcal{M}, g)$ is a C^2 Hausdorff manifold with a time-oriented C^1 Lorentzian metric.

We call $V \in T\mathcal{M}$ *timelike* if $g(V,V) < 0$, *null* if $V \neq 0$ and $g(V,V) = 0$, and *spacelike* if $g(V,V) > 0$. If $V \neq 0$ and $g(V,V) \leq 0$, we call V *causal*.

A *time-orientation* on $\mathcal{M}$ is defined by a continuous timelike vector field T defined on all of $\mathcal{M}$.

A causal vector V is called *future directed* if $g(V,T) < 0$, and *past directed* if $g(V,T) > 0$. A parametrized C^1 curve $\gamma : I \to \mathcal{M}$ is said to be *timelike*, *null* or *spacelike* according to whether its tangent vector $\dot{\gamma}$ is timelike, null or spacelike, respectively. If $\dot{\gamma}$ is everwhere causal, γ is called *causal*, and if $\dot{\gamma}$ is future directed, then γ is called *future directed*.

A C^2 paremetrized curve $\gamma : I \to \mathcal{M}$ is said to be a *parametrized geodesic* if $\nabla_{\dot\gamma}\dot\gamma = 0$. The parameter $s \in I$ with respect to such a map is called an *affine parameter*.

Let $S \subset \mathcal{M}$. The *causal future* of S, denoted $J^+(S)$ is defined as the set of all points which can be reached from S by a future pointing causal curve. The *causal past* of S, denoted $J^-(S)$, is defined to be the set of all points which can be reached from S by a past-pointing causal curve.

The chronological future $I^+(S)$ is defined similarly where “causal” above is replaced by “timelike”.

A subset $S \subset \mathcal{M}$ is said to be *achronal* if $S \cap I^+(S) = \emptyset$.

A submanifold of $\mathcal{M}$ is called *spacelike* if its induced metric is Riemannian, *null* if its induced metric is degenerate, and *timelike* if its induced metric is Lorentzian. (For embedded curves, this definition coincides with the previous.) Also, one can easily see that a hypersurface is spacelike, timelike, or null if and only if its normal is everywhere timelike, spacelike, or null, respectively.

A curve is called *inextendible* if it is not a proper subset of another curve.

Definition 1 *A spacetime* $(\mathcal{M}, g)$ *is said to be* future causally geodesically complete *if all inextendible future-directed causal geodesics take on arbitrary large values of any affine parameter.*

A spacetime which is not future causally geodesically complete is called *future causally geodesically incomplete.*

Definition 2 *A spacetime* $(\mathcal{M}, g)$ *is said to be* globally hyperbolic *if there exists a spacelike hypersurface* Σ *such that all inextendible causal curves in* $\mathcal{M}$ *intersect* Σ *precisely once. Such a* Σ *is then called a* Cauchy surface.

The initial value problem for the vacuum

Let us assume in what follows that all manifolds, functions are C^∞ unless otherwise noted.

In the case of the vacuum, the theory is completely described by a 4-dimensional spacetime $(\mathcal{M}, g)$ satisfying the equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$, or equivalently $R_{\mu\nu} = 0$.

Definition 3 *Let Σ be a 3-manifold, $\bar{g}_{ij}$ a Riemannian metric, and k_{ij} a symmetric 2-tensor on Σ. We call $(\Sigma, \bar{g}_{ij}, k_{ij})$ a* vacuum initial data set *if*

$$\nabla^j k_{ij} - \nabla_i trk = 0,$$

$$\bar{R} - |k|^2 + (trk)^2 = 0.$$

Theorem 1 (Choquet-Bruhat) *Let $(\Sigma, \bar{g}_{ij}, k_{ij})$ be an initial data set. Then there exists a unique spacetime $(\mathcal{M}, g)$ satisfying the following*

1.
$$(\Sigma, \bar{g}_{ij}, k_{ij}) \subset (\mathcal{M}, g),$$

2.
$$R_{\mu\nu} = 0,$$

3. *$(\mathcal{M}, g)$ is* globally hyperbolic *with Σ as a Cauchy surface,*

4. *If $(\tilde{\mathcal{M}}, \tilde{g})$ satisfies 1–3, then $(\tilde{\mathcal{M}}, \tilde{g}) \subset (\mathcal{M}, g)$ isometrically.*

We call $(\mathcal{M}, g)$ the maximal Cauchy development of $(\Sigma, \bar{g}_{ij}, k_{ij})$.

From the point of view of this course, to "study" general relativity means to relate properties of initial data sets with properties of their maximal Cauchy development.

All conjectures/results to be stated/proven in this course will be of that form.

Penrose's incompleteness theorem

Let $(\mathcal{M}, g)$ be a 4-dimensional spacetime, and let $E \subset \mathcal{M}$ be a spacelike surface. Given $p \in E$, there exists a neighborhood $p \in U$ such that $U \cap J^+_U(E) \setminus I^+_U(E)$ is a null hypersurface with two connected components, and $U \cap E$ can be thought of as a boundary of either.

Definition 4 *We say that a spacelike surface $E \subset \mathcal{M}$ is* trapped *if for all $p \in E$, the mean curvature of E at p as a subset of both of the above null hypersurfaces is negative.*

Theorem 2 (Penrose) *Let $(\Sigma, \bar{g}_{ij}, k_{ij})$ be an initial data set and let $(\mathcal{M}, g)$ be its maximal Cauchy development. Suppose that Σ is non-compact, and $(\mathcal{M}, g)$ contains a closed (compact without boundary) trapped surface $E \subset \Sigma$. Then $\mathcal{M}$ is future-causally geodesically incomplete.*

Notes:

1. The conditions are satisfied in particular if $E \subset \Sigma$. Such data can be explicitly constructed.
2. No examples are known for the vacuum for which the assumptions of the theorem are satisfied and Σ is "regular".
3. The theorem does not say what "goes wrong" and forces $(\mathcal{M}, g)$ to be incomplete. In particular, it is not a "singularity theorem".

Asymptotic flatness

The primary object of study in this course will correspond physically to what is known as an isolated system. This motivates

Definition 5 *A data set* $(\Sigma, \bar{g}_{ij}, k_{ij})$ *is said to be strongly asymptotically flat if there exists a compact* $C \subset \Sigma$ *such that* $\Sigma \setminus C$ *is diffeomorphic to* $\mathbf{R}^3 \setminus \{r \leq 1\}$*, and such that, with respect to the Euclidean coordinates* x^i *of* R^3*, we have*

$$g_{ij} = (1 + 2M/r)\delta_{ij} + o_4(r^{-3/2}),$$

$$k_{ij} = o_3(r^{-5/2}).$$

Positive energy theorem

Theorem 3 (Schoen-Yau) *Under the assumptions of the above definition,* $M \geq 0$*, and* $M = 0$ *iff* $(\Sigma, \bar{g}_{ij}, k_{ij}) \subset (\mathbf{R}^{3+1}, \delta_{ij})$.

Stability of Minkowski space

Theorem 4 (Christodoulou-Klainerman) *Let* $(\Sigma, \bar{g}_{ij}, k_{ij})$ *be an asymptotically flat initial data set "sufficiently close" to trivial data. Let* $(\mathcal{M}, g)$ *denote its maximal Cauchy development. Then* $(\mathcal{M}, g)$ *is future-causally geodesically complete. Moreover,* $(\mathcal{M}, g)$ *tends to Minkowski space along all past and future directed geodesics in a suitable sense. In particular, it is possible to associate to* g *an asymptotic structure known as* null infinity*.*

Applications. Rigorous formulation of the notion of gravitational radiation, Bondi mass law formula, "Christodoulou memory" effect.

Note. The existence of "null infinity" can be inferred for all asymptotically flat initial data sets. Weak cosmic censorship: null infinity is "long enough".

Conjecture 1 (Penrose) *Let* $(\Sigma, \bar{g}_{ij}, k_{ij})$ *be "generic" compact or asymptotically flat initial data, and let* $(\mathcal{M}, g)$ *denote their maximal Cauchy development. Then* $(\mathcal{M}, g)$ *is inextendible as a manifold with* C^0 *metric, i.e. there does not exist a* $(\tilde{\mathcal{M}}, \tilde{g})$ *such that* $\tilde{g}$ *is* C^0 *and* $(\mathcal{M}, g) \subset (\tilde{\mathcal{M}}, \tilde{g})$ *as a proper subset.*

The above conjecture is known as strong cosmic censorship.

Matter

In the case of matter, in addition to $(\mathcal{M}, g)$, the unknowns now include a number of matter fields $\Psi_1 \ldots \Psi_n$ defined on $\mathcal{M}$, the Einstein equations take the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu},$$

where $T_{\mu\nu}$ is a known function of g, Ψ_i, and to yield a closed system, the equations of motion of the Ψ_i must be appended.

For large classes of these systems, theorems analogous to Theorem 1 can be again proven. Moreover, the Penrose incompleteness theorem holds as long as $T(L, L) \geq 0$ for all null vectors L, and the positive energy theorem holds as long as $T(W, W) \geq 0$ and $T^{\alpha\beta} W_\beta$ is non-spacelike, for all timelike vectors W.

Symmetry and the I.V.P.

References

[1] Mihalis Dafermos *Spherically symmetric spacetimes with a trapped surface* gr-qc/0403032, preprint, 2004

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Symmetry: There is a Lie group G which acts on $(\mathcal{M}, g, \ldots)$, by isometry, and preserves the matter fields $(\ldots)$.

The assumption of symmetry is *evolutionary*, i.e.:

Assume that a Lie group G acts on $(\Sigma, \bar{g}_{ij}, K_{ij}, \ldots)$, preserving $\bar{g}$, K, and the initial matter fields.

Then G acts on the maximal development $(\mathcal{M}, g, \ldots)$, preserving g and the initial matter fields.

Spherical symmetry

$G = SO(3)$ (group of rotations in $\mathbf{R}^3$).

We will restrict to asymptotically flat initial data.

More specifically, we will require that initial 3-manifold $(\Sigma, \bar{g})$ is a warped product of $[0, \infty)$ with S^2 (one end) or $(-\infty, \infty)$ with S^2 (two ends), and has metric

$$ds^2 + r^2\gamma$$

where γ is the standard metric, and r is a function of s.

In other words, $\mathcal{S} = \Sigma/SO(3)$ is a one-dimensional manifold (in the case of 1 end, with boundary, and $r = 0$ on the boundary). As $s \to \pm\infty$, $r \to \infty$. If $\pi_1 : \Sigma \to \mathcal{S}$ denotes the natural projection, then $r(p) = \sqrt{Area(\pi_1^{-1}(p))/4\pi}$.

Proposition 1 *Let* $(\Sigma, g, K, \ldots)$ *be spherically symmetric initial data, in the sense described above, and let* $(\mathcal{M}, g, \ldots)$ *denote its maximal Cauchy development. Then* $SO(3)$ *acts by isometry on* $(\mathcal{M}, g)$ *so as for* $\mathcal{Q} = \mathcal{M}/SO(3)$ *to inherits the structure of a 2-dimensional manifold (possibly with boundary* Γ *which we call the* centre*) on which a Lorentzian metric* $g_{ij}dy^i dy^j$ *is defined. Let* $\pi_1 : \mathcal{M} \to \mathcal{Q}$ *denote the natural projection. The group orbits in* $\mathcal{M}$ *are either spacelike spheres or points, so this induces a function*

$$r : \mathcal{Q} \to \mathbf{R}$$

defined by $r(p) = \sqrt{Area(\pi_1^{-1}(p))/4\pi}$*. This function is called the* area-radius*. Then* $\Gamma = \{p \in \mathcal{Q} : r(p) = 0\}$, $\Gamma \neq \emptyset$ *iff* $\Gamma \cap \mathcal{S} \neq \emptyset$*, in which case* Γ *is a connected timelike curve through* $\mathcal{S}$*. The metric* g_{ij} *and* r *together yield* $g_{\alpha\beta}$ *by*

$$g_{\alpha\beta}dx^\alpha dx^\beta = g_{ij}dy^i dy^j + r^2\gamma.$$

$\mathcal{Q}$ is foliated by "ingoing" null rays emanating from $\mathcal{S}$.

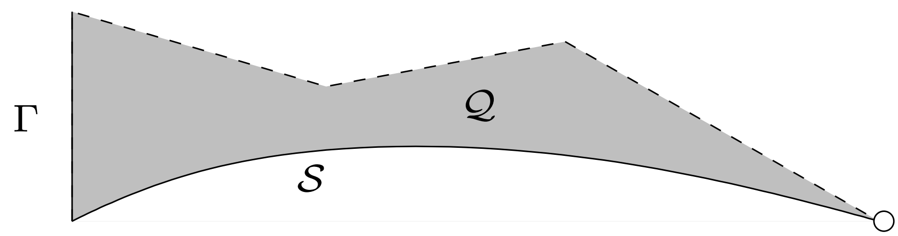
It is also foliated by the collection of "outgoing" null rays emanating from $\Gamma \cup \mathcal{S}$.

Parametrizing these rays by functions v, u, respectively, we define global null coordinates on $\mathcal{Q}$, which we can select to be bounded. Moreover, we will select v, u to be *oriented towards the future*.

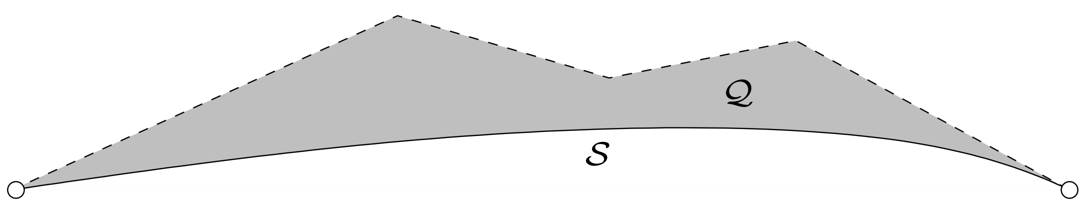
The coordinate-range of $\mathcal{Q}$ with respect to such coordinates u and v, considered as a subset of $\mathbf{R}^2$ is known as a *Penrose diagram*.

We will always restrict consideration to the future $J^+(\mathcal{S}) \cup \mathcal{S}$, i.e. we shall denote $\mathcal{Q} \cap J^+(\mathcal{S}) \cup \mathcal{S}$ in what follows, again by $\mathcal{Q}$.

One end:



Two ends:



The boundary $\partial\mathcal{Q} = \overline{\mathcal{Q}} \setminus \mathcal{Q}$ is necessarily achronal.

(Here $\overline{\mathcal{Q}}$ refers to the topology of the plane. The point of these depictions is that we can apply causal and topological relations to the boundary $\partial\mathcal{Q}$ as well!)

The Einstein equations under spherical symmetry in null coordinates

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -\Omega^2 du dv + r^2\gamma_{AB}dx^A dx^B \\ T_{\mu\nu}dx^\mu dx^\nu &= 2T_{uv}du dv + T_{AB}dx^A dx^B \end{aligned}$$

Why is there no T_{uA} component?

Note: $g_{uv} = -\frac{1}{2}\Omega^2$. $g^{uv} = -2\Omega^2$.

Non-vanishing Christoffel symbols.

$$\Gamma^u_{AB} = -g^{uv} r\partial_v r\gamma_{AB}$$

$$\Gamma^v_{AB} = -g^{uv} r\partial_u r\gamma_{AB}$$

$$\Gamma^A_{Bv} = (\partial_v r)r^{-1}\delta^A_B$$

$$\Gamma^A_{Bu} = (\partial_u r)r^{-1}\delta^A_B$$

$$\Gamma^u_{uu} = \partial_u \log\Omega^2$$

$$\Gamma^v_{vv} = \partial_v \log\Omega^2$$

$$\Gamma^A_{BC} =$$

Along a constant-v ray, we can choose coordinates such that $\Gamma^u_{uu} = 0$.

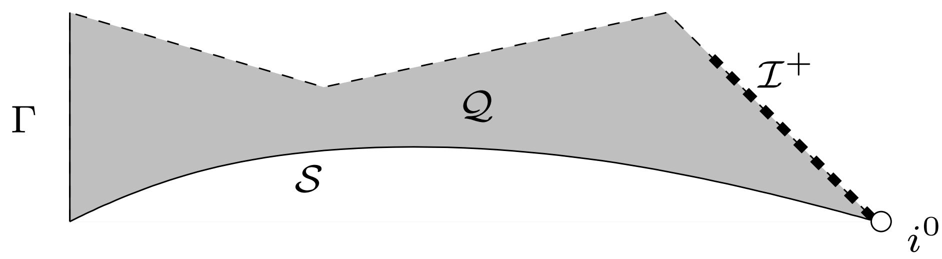
$$
\begin{aligned}
R_{uu} &= \partial_\alpha \Gamma^\alpha_{uu} - \partial_u \Gamma^\alpha_{u\alpha} + \Gamma^\alpha_{uu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{u\beta}\Gamma^\beta_{u\alpha} \\
&= \partial_u \Gamma^u_{uu} - (\partial_u \Gamma^u_{uu} + \partial_u \Gamma^A_{uA}) \\
&\quad + \Gamma^u_{uu}(\Gamma^u_{uu} + \Gamma^A_{uA}) \\
&\quad - \sum_A (\Gamma^A_{uA}\Gamma^A_{uA}) - \Gamma^u_{uu}\Gamma^u_{uu} \\
&= -2\partial_u \partial_u \log r - 2(\partial_u \log r)^2 \\
&= -\frac{2}{r}\partial_u \partial_u (r)
\end{aligned}
$$

Suppose $T_{uu} \geq 0$, $T_{vv} \geq 0$.

Since, by the Einstein equations, $R_{uu} = 2T_{uu}$ it follows that in such coordinates we have along the constant-u ray that $\partial_u \partial_u r \leq 0$. Thus, if $\partial_u r < 0$ at $v = V$, then $\partial_u r < 0$ at $v > V$.

Similarly for R_{vv} on a constant-v ray, and thus for $\partial_v r$.

This allow us to talk unambiguously about future null infinity $\mathcal{I}^+$.



(We will restrict to the 1-end case for now. Let ∂_v be "outgoing".)

Definition 6 *The set $\mathcal{I}^+$ is defined by*

$$\begin{aligned}\mathcal{I}^+ = \quad & \{(u,v) \in \partial\mathcal{Q} : \forall R > 0, \\ & \exists (u,v'), v' < v : r(u,v') \geq R\}.\end{aligned}$$

Let us assume that $\partial_u r < 0$ along Σ. (No *antitrapped* surfaces.)

Proposition 2 $\partial_u r < 0$ *everywhere, i.e. no antitrapped surfaces form.*

Proposition 3 *If $\mathcal{I}^+ \neq \emptyset$, then $\mathcal{I}^+$ is a connected subset of the ingoing null curve in* $\mathbf{R}^{1+1}$ *emanating from i^0.*

Definition 7 *We define the* regular region $\mathcal{R}$ *by*

$$\mathcal{R} = \{p \in \mathcal{Q} : \partial_u r < 0, \partial_v r > 0,$$

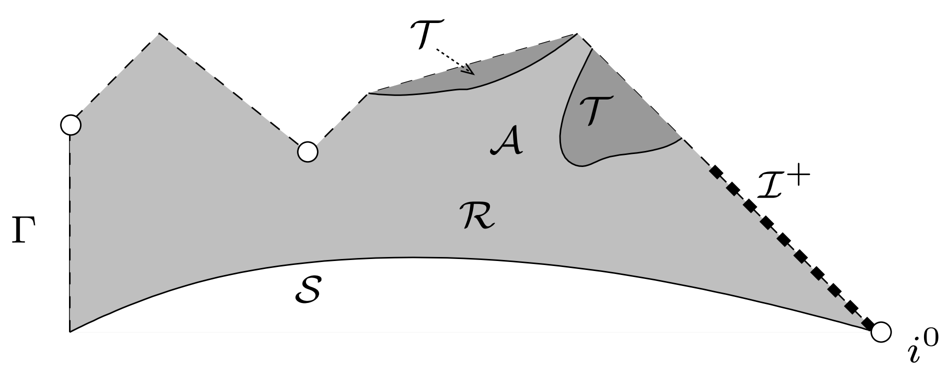
the marginally trapped region $\mathcal{A}$ *by*

$$\mathcal{A} = \{p \in \mathcal{Q} : \partial_u r < 0, \partial_v r = 0\}$$

and the trapped region $\mathcal{T}$ *by*

$$\mathcal{T} = \{p \in \mathcal{Q} : \partial_u r < 0, \partial_v r < 0\}$$

Proposition 4 *$J^-(\mathcal{I}^+) \subset \mathcal{R}$. If $(u, v) \in \mathcal{A}$, then $(u, v') \in \mathcal{T} \cup \mathcal{A}$ for all $(u, v') \in \mathcal{Q}$ with $v' > v$. If $(u, v) \in \mathcal{T}$, then $(u, v') \in \mathcal{T}$ for all $(u, v') \in \mathcal{Q}$ with $v' > v$.*



The calculations correctly:

$$
\begin{aligned}
\partial_u \partial_v r &= -\frac{1}{4r}\Omega^2(1 + 4\Omega^{-2}\partial_u r \partial_v r) \\
&\quad + rT_{uv} && (1) \\
\partial_u \partial_v \log \Omega &= -T_{uv} \\
&\quad + \frac{1}{4r^2}\Omega^2(1 + 4\Omega^{-2}\partial_u r \partial_v r) \\
&\quad - \frac{1}{4}\Omega^2 g^{AB} T_{AB} && (2) \\
\partial_u(\Omega^{-2}\partial_u r) &= -r\Omega^{-2}T_{uu} && (3) \\
\partial_v(\Omega^{-2}\partial_v r) &= -r\Omega^{-2}T_{vv} && (4)
\end{aligned}
$$

The Hawking mass

Define $m = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r \partial_v r)$.

Compute:

$$\partial_v m = 2r^2\Omega^{-2}(T_{uv}\partial_v r - T_{vv}\partial_u r) \qquad (5)$$

$$\partial_u m = 2r^2\Omega^{-2}(T_{uv}\partial_u r - T_{uu}\partial_v r) \qquad (6)$$

Thus

Proposition 5 *Suppose* $T_{uv} \geq 0$, $T_{vv} \geq 0$, $T_{uu} \geq 0$. *Then, in* $\mathcal{R} \cup \mathcal{A}$, $\partial_v m \geq 0$, $\partial_u m \leq 0$.

$\mathcal{S}$ is asymptotically flat implies $\partial_v r > 0$ on $\mathcal{S}$ in a neighborhood of i_0, and $m \leq M_0 = \sup_{\mathcal{S}} m$.

Thus, by (6), $m \leq M_0$ in $J^-(\mathcal{I}^+)$.

By (5), m extends to a (not necessarily differentiable) function $M(u)$ on $\mathcal{I}^+$: If $(u, v) \in \mathcal{I}^+$, then

$$M(u) = \lim_{v' \to v} m(u, v')$$

Proposition 6 *The function $M(u)$ is monotonically non-increasing in u, and $M(u) \leq M_0$.*

We call $M(u)$ the *Bondi mass* at retarded time u, and we call $M_f = \liminf M(u)$ the *final Bondi mass.*

Let us define the "mass ratio" $\mu = \frac{2m}{r}$, and also $\nu = \partial_u r$, $\lambda = \partial_v r$. We compute that

$$-\frac{1}{4}\Omega^2(1-\mu) = \lambda\nu.$$

Thus, as $\Omega^2 > 0$, and $\nu < 0$, we have that $1-\mu$ and λ have the same sign.

By regularity of Γ, $m = 0$ on Γ, in fact $\mu = 0$ on Γ.

Proposition 7 *If $\mathcal{S} \cap \Gamma \neq \emptyset$, then $m \geq 0$ on $\mathcal{S} \cap \mathcal{R}$. Suppose $\mathcal{S} \cap \mathcal{A} \neq \emptyset$, $p \in \mathcal{S} \cap \mathcal{A}$, and $X = \mathcal{S} \cap \{v \geq v(p)\}$ satisfies*

$$X \subset \mathcal{A} \cup \mathcal{R}.$$

Then any $q \in X$ satisfies $m(q) \geq \frac{1}{2} r(p)$. Consequently, $M_0 \geq \frac{1}{2} r(p)$.

The last statement of the above proposition is an example of what is known as a *Penrose inequality.*

Proposition 8 *We have $m \geq 0$ in $\mathcal{R}$. In particular, $M_f \geq 0$.*

Proposition 9 *Suppose* $T_{uu} = T_{vv} = T_{uv} = 0$. *Then* $\mathcal{Q}$ *is Schwarzschild or Minkowski space.*

Proposition 10 *Suppose* $\mathcal{S}' \subset \mathcal{Q}$ *is spacelike and* $m = C$, $\lambda > 0$, *along* $\mathcal{S}'$. *Then* $m = C$ *for all* $p \in \mathcal{Q}$ *such that*

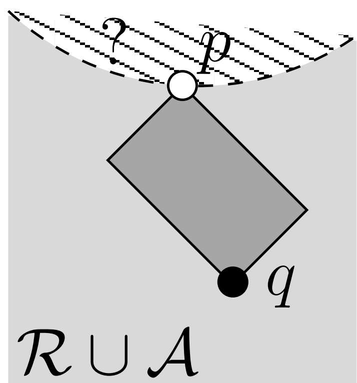
$$J^-(p) \cap J^+(\mathcal{S}') \subset D^+(\mathcal{S}') \cap \{\lambda \geq 0\}.$$

Moreover, $J^-(p) \cap J^+(\mathcal{S}')$ *is isometric to a piece of Schwarzschild.*

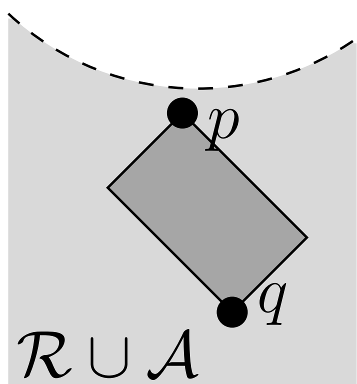
The extension assumption

Let $p \in \overline{\mathcal{R}} \setminus \overline{\Gamma}$, and $q \in \overline{\mathcal{R}} \cap I^{-}(p)$ such that

$$J^{-}(p) \cap J^{+}(q) \setminus \{p\} \subset \mathcal{R} \cup \mathcal{A}:$$



Then $p \in \mathcal{R} \cup \mathcal{A}$.

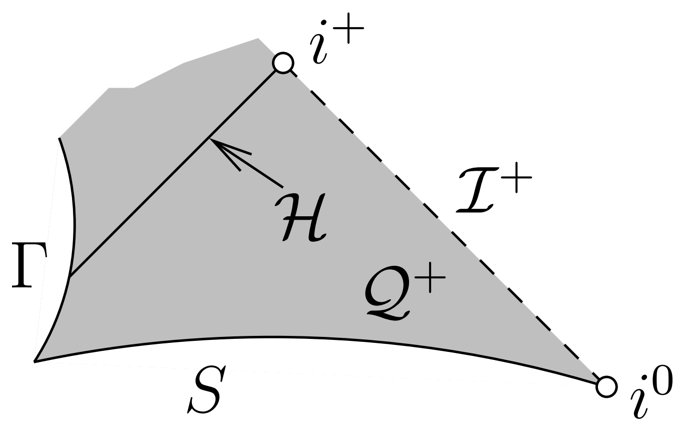


In the evolutionary context, this assumption can be stated informally as the proposition that a "first singularity" emanating from the non-trapped region can only arise from $\overline{\Gamma}$.

Now assume $\mathcal{I}^+$ is non-empty, and assume $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$ is also non-empty. Claim: If $\mathcal{A} \cup \mathcal{T} \neq \emptyset$, then $\mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$.

Define $\mathcal{H}^+ = \mathcal{Q} \cap \partial J^-(\mathcal{I}^+)$.

Theorem 5 *If $\mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$, then the set $J^-(\mathcal{I}^+)$ looks like*



i.e. $\mathcal{H}^+$ terminates at i^+. Moreover, along $\mathcal{H}^+$, we have the Penrose inequality $r \leq 2M_f$. In particular $i^+ \notin \mathcal{I}^+$.

Theorem 6 *If* $\mathcal{Q} \setminus J^{-}(\mathcal{I}^{+}) \neq \emptyset$*, then* $\mathcal{I}^{+}$ *is future complete.*

This is the statement that the affine length of ingoing null rays, measured by an affine parameter normalized appropriately on an *outgoing* null ray, tends to infinity.

The "weak cosmic censorship" conjecture is the statement that $\mathcal{I}^{+}$ be complete for generic initial data for suitable Einstein-matter systems.

Thus, restricted to spherical symmetry, for systems satisfying the extension criterion, it follows that to prove weak cosmic censorship, it suffices to prove that generically a marginally trapped surface forms.

Suppose $\mathcal{A} \cup \mathcal{T}$ is non-empty. We can define the *outermost apparent horizon* $\mathcal{A}'$ to be $\{(u,v) \in \mathcal{A} : (u',v) \in \mathcal{R}, \forall u' < u\}$.

Proposition 11 *If $\mathcal{A} \cup \mathcal{T} \neq \emptyset$, then $\mathcal{A}'$ is a (possibly disconnected) achronal curve. There exists a v' such that for all $v > v'$, the constant v ray intersects $\mathcal{A}'$. Moreover, there exists a v'' such that for $v > v''$, $r \leq 2M_f$ on $\mathcal{A}'$.*

Examples: scalar fields and collisionless matter

References

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The massless scalar field

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}$$

$$\Box_g \phi = g^{\mu\nu}\phi_{;\mu\nu} = 0$$

Under spherical symmetry we compute:

$$T_{uv} = 0,$$

$$T_{vv} = (\partial_v \phi)^2, T_{uu} = (\partial_u \phi)^2,$$

$$T_{AB} = -g_{AB}g^{uv}\partial_u\phi\partial_v\phi.$$

Setting $\theta = r\partial_v\phi$, $\zeta = r\partial_u\phi$, the wave equation can be written

$$\partial_u \theta = -\frac{\zeta\lambda}{r} \tag{7}$$

$$\partial_v \zeta = -\frac{\theta\nu}{r} \tag{8}$$

We would like to show that the extension hypothesis holds for this matter model.

In general, showing this has two parts:

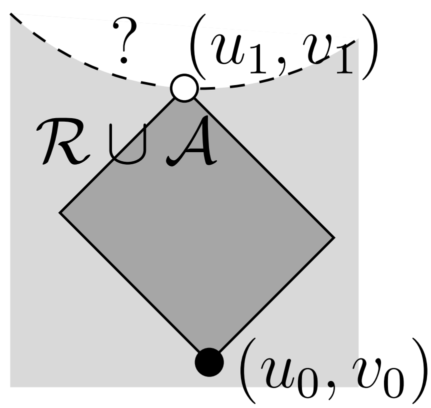
1. proving *a priori* estimates in $J^-(p)$, and
2. proving a local existence theorem in a suitable norm.

The norm of the local existence theorem dictates the strength of the *a priori* estimates that must be proven.

We will turn first to the estimates.

A priori estimates

Proposition 12 *Let* $(u_1, v_1) \in \overline{\mathcal{Q}}$ *be such that there exists* $u_0 < u$, $v_0 < v$ *such that* $[u_0, u_1] \times [v_0, v_1] \setminus (u_1, v_1) \subset \mathcal{R} \cup \mathcal{A}$.



Then there exists a constant $C > 0$ *such that*

$$|\phi| \le C, |\partial_u \phi| \le C, |\partial_v \phi| \le C, \tag{9}$$

$$|\log r| \le C, |\log -\partial_u r| \le C, \tag{10}$$

$$|\log \Omega^2| \le C, \tag{11}$$

in $[u_0, u_1] \times [v_0, v_1] \setminus (u_1, v_1)$.

Proof. Consider the segments $\{u_0\} \times [v_0, v_1] \cup [u_0, u_1] \times \{v_0\}$. By compactness, we have

$$0 < r_0 \leq r \leq R_0 < \infty \tag{12}$$

on these segments, and

$$0 \leq m < M_0 < \infty. \tag{13}$$

By monotonicity we have $\partial_v r \geq 0$, $\partial_u r \leq 0$, $\partial_v m \geq 0$, $\partial_u m \leq 0$, and thus the estimates (12), (13) apply everywhere in $[u_0, u_1] \times [v_0, v_1] \setminus (u_1, v_1)$.

Now consider the quantity $\kappa = -\frac{1}{4}\Omega^2\nu^{-1}$.

Again, by compactness, we have that $0 < c_0 \leq \kappa \leq C_0 < \infty$ on $\{u_0\} \times [v_0, v_1]$. By monotonicity, we have $\kappa > 0$ and $\partial_u \kappa \leq 0$, and thus

$$0 < \kappa \leq K_0$$

everywhere in $[u_0, u_1] \times [v_0, v_1] \setminus (u_1, v_1)$.

Consider the quantity ν on $\{v_0\} \times [v_0, v_1]$.

By compactness, we have $\nu_0 \leq |\nu| \leq N_0$ on $\{v_0\} \times [v_0, v_1]$. We can write the equation $\partial_u \partial_v r = \cdots$ as

$$\partial_v \nu = -\frac{\mu}{r} \kappa \nu,$$

and thus

$$\begin{aligned} \nu_1 &= \nu_0 e^{-K_0 r_0^{-1}(v_1 - v_0)} \\ &\leq |\nu| \leq N_0 e^{K_0 r_0^{-1}(v_1 - v_0)} = N. \end{aligned}$$

Note also the upper bound for λ provided by

$$\lambda = \kappa(1 - \mu) \leq K_0.$$

We turn to ζ. Again, by compactness, there exists a constant such that $|\zeta| \leq Z_0$ on $[u_0, u_1] \times \{v_0\}$.

Now, consider the equation (7). Integrating in u, we have

$$\zeta(u,v) = \zeta(u,v_0) + \int_{v_0}^{v} \frac{-\theta\nu}{r} d\bar{v}$$

and thus

$$\begin{aligned}
|\zeta(u,v)| &\leq |\zeta(u,v_0)| + \int_{v_0}^{v} \left|\frac{\theta\nu}{r}\right| d\bar{v} \\
&\leq |\zeta(u,v_0)| \\
&\quad + \sqrt{\int_{v_0}^{v} \theta^2\kappa^{-1} d\bar{v} \int_{v_0}^{v} \kappa\nu^2 r^{-2} d\bar{v}} \\
&\leq Z_0 \\
&\quad + N r_0^{-1} \sqrt{K_0(v_1 - v_0)} \\
&\quad \cdot \sqrt{\int_{v_0}^{v} \theta^2\kappa^{-1} d\bar{v}}
\end{aligned}$$

On the other hand, considering the equation,

$$\partial_v m = \frac{1}{2}\theta^2\kappa^{-1}$$

We have that

$$\int_{v_0}^{v} \frac{1}{2}\theta^2\kappa^{-1} d\bar{v} \leq m(v) - m(v_0) \leq M_0.$$

So, plugging into the estimate for ζ, we obtain finally:

$$|\zeta(u,v)| \leq Z_0 + Nr_0^{-1}\sqrt{K_0(v_1 - v_0)}\sqrt{M} = Z.$$

Since ϕ is bounded $|\phi| \leq \Phi_0$ on $\{u_0\} \times [v_0, v_1]$, we estimate ϕ now by

$$\begin{aligned}
\phi(u,v) &= \phi(u_0,v) + \int_{u_0}^{u} \partial_u \phi d\bar{u} \\
&= \phi(u_0,v) + \int_{u_0}^{u} \frac{1}{r}(r\partial_u \phi) d\bar{u} \\
&= \phi(u_0,v) + \int_{u_0}^{u} \frac{1}{r}(\zeta) d\bar{u}
\end{aligned}$$

so thus

$$|\phi(u,v)| \leq \Phi_0 + (u_1 - u_0) r_0^{-1} Z.$$

As for θ, again, we have that $|\theta| \leq \Theta_0$ on $\{u_0\} \times [v_0, v_1]$. Now we can estimate θ integrating (7).

$$\begin{aligned} |\theta|(u,v) &\leq |\theta(u_0,v)| + \int_{u_0}^{u} \left| \frac{\zeta\lambda}{r} \right| \\ &\leq \Theta_0 + r_0^{-1} Z K (u_1 - u_0) = \Theta. \end{aligned}$$

Now we have a *lower bound* for κ from noting that, by compactness, $0 < \kappa_0 \leq \kappa$ on $\{u_0\} \times [v_0, v_1]$, and then integrating the equation

$$\partial_u \kappa = \kappa \left(\frac{\zeta^2}{r\nu} \right)$$

to obtain

$$\kappa \geq \kappa_0 e^{-Z^2 r_0^{-1} \nu_1^{-1} (u_1 - u_0)},$$

In particular, since $\Omega^2 = -4\kappa\nu$, we have obtained upper and lower bounds on Ω^2. This completes the proof. $\square$

A local existence theorem for a characteristic initial value problem.

Proposition 13 *Let* $\bar\Omega, \bar r, \bar\phi$ *be* C^2 *functions defined on* $u_0 \times [v_0, v_1] \cup [u_0, u_1] \times v_0$ *satisfying the constraint equations* (3)–(4). *Let* C *be such that*

$$|\bar\phi(u_0, \cdot)|_{C^1} \leq C, |\bar\phi(\cdot, v_0)|_{C^1} \leq C,$$

$$|\log \bar r(\cdot, v_0)|_{C^1} \leq C,$$

$$|\log \bar\Omega^2(\cdot, v_0)| \leq C, |\log \bar\Omega^2(u_0, \cdot)| \leq C.$$

Then there exists a constant $\epsilon > 0$, **depending only on** C *such that, defining* $\tilde u_1 = \min(u_0 + \epsilon, u_1)$, $\tilde v_1 = \min(v_0 + \epsilon, v_1)$, *there exist unique* C^2 *functions* (Ω, r, ϕ) *on* $[u_0, \tilde u_1] \times [v_0, \tilde v_1]$ *coinciding with* $(\bar\Omega, \bar r, \bar\phi)$ *on the initial segments, and satisfying the Einstein-scalar field equations* (1)–(4). *If* $\bar\Omega, \bar r, \bar\phi$ *are initially* C^∞, *then* (Ω, r, ϕ) *are* C^∞ *in* $[u_0, \tilde u_1] \times [v_0, \tilde v_1]$.

Local existence theorems typically proceed as follows:

1. Reformulation of the problem as a fixed point problem for a map Φ in a complete metric space
2. Prove estimates for Φ
3. Apply the contraction mapping principle

Example. Local existence for o.d.e.'s

Contraction mapping principle in complete metric spaces (e.g. closed subsets of Banach spaces).

Let X be a complete metric space, and let $\Phi : X \to X$ be such that there exists a constant $\gamma < 1$ with $d(\Phi(x), \Phi(y)) \leq \gamma d(x, y)$ for all $x, y \in X$. Then Φ has a unique fixed point x_0, i.e. a unique point x_0 such that $\Phi(x_0) = x_0$.

Proof of Proposition 13.

Let $\mathcal{X}$ be the set of functions $\{(\Omega, r, \phi)\}$ defined on $[u_0, \tilde{u}_1] \times [v_0, \tilde{v}_1]$, where ϵ is still to be determined, such that $\Omega, r > 0$, Ω is C^0 and r, ϕ are C^1.

Define a distance

$$d((\Omega_1, r_1, \phi_1), (\Omega_2, r_2, \phi_2)) = \max(|\log \Omega_1 - \log \Omega_2|_{C^0}, |\log r_1 - \log r_2|_{C^1}, |\phi_1 - \phi_2|_{C^1}).$$

The distance function d makes $\mathcal{X}$ into a complete metric space. Let $\mathcal{X}_E$ be

$$\{(\Omega, r, \phi) \in \mathcal{X} : d(\Omega, r, \phi), (1, 1, 0)) \leq E\}$$

We define now a map $\Phi : \mathcal{X}_E \to \mathcal{X}_E$,
$\Phi(\Omega, r, \phi) = (\tilde{\Omega}, \tilde{r}, \tilde{\phi})$, where

$$\begin{aligned}\log \tilde{\Omega} &= \log \bar{\Omega}(u_0, v) + \log \bar{\Omega}(u, v_0) \\ &\quad - \log \bar{\Omega}(u_0, v_0) \\ &\quad + \int_{u_0}^{u} \int_{v_0}^{v} \left(\frac{1}{4r^2} \Omega^2 (1 + 4\Omega^{-2} \partial_u r \partial_v r) \right. \\ &\quad \left. - \partial_u \phi \partial_v \phi \right) d\bar{u} d\bar{v}\end{aligned} \tag{14}$$

$$\begin{aligned}\tilde{r} &= \bar{r}(u_0, v) + \bar{r}(u, v_0) - \bar{r}(u_0, v_0) \\ &\quad - \int_{u_0}^{u} \int_{v_0}^{v} \frac{1}{4r} \Omega^2 (1 + 4\Omega^{-2} \partial_u r \partial_v r) d\bar{u} d\bar{v}\end{aligned} \tag{15}$$

$$\begin{aligned}\tilde{\phi} &= \bar{\phi}(u_0, v) + \bar{\phi}(u, v_0) - \bar{\phi}(u_0, v_0) \\ &\quad - \int_{u_0}^{u} r^{-1} \int_{v_0}^{v} (\partial_v \phi \partial_u r) d\bar{v} d\bar{u}\end{aligned} \tag{16}$$

We first show, for E sufficiently large this is indeed a map $\mathcal{X}_E \to \mathcal{X}_E$.

It is more than clear that Ω is C^0 and positive. That r, ϕ are C^1 follows immediately by differentiating under the integral, in view also of regularity of initial data.

Estimating naively (14) we obtain

$$\log \tilde{\Omega} \leq 3C + \epsilon^2 \left(\frac{1}{4} e^{4E} + e^{4E} E^2 + E^2 \right).$$

Estimating naively (15) we have

$$\tilde{r} \leq 2e^C + \epsilon^2 \left(\frac{1}{4} e^{3E} + e^{3E} E^2 \right),$$

and

$$\tilde{r} \geq e^{-C} - \epsilon^2 \left(\frac{1}{4} e^{3E} + e^{3E} E^2 \right).$$

Estimating naively (16) we obtain

$$|\tilde{\phi}| \leq 3C + \epsilon^2 e^E E^2 e^E.$$

Differentiating (15) in u and v, we obtain

$$|\partial_u \tilde{r}| \leq Ce^C + \epsilon\left(\frac{1}{4}e^{4E} + e^{4E}E^2\right),$$

$$|\partial_v \tilde{r}| \leq Ce^C + \epsilon\left(\frac{1}{4}e^{4E} + e^{4E}E^2\right).$$

Differentiating (16) in in u we obtain

$$|\partial_u \tilde{\phi}| \leq 2C + e^E \epsilon E e^E E$$

and in v, we obtain

$$|\partial_v \tilde{\phi}| \leq C + \epsilon(e^{3E}E\epsilon E e^E E + e^E E e^E E).$$

Thus, if ϵ is small enough (how small depends only on C and E) then $\mathcal{X}_E$ is preserved by Φ.

Now we can similarly bound differences:

For ϵ sufficiently small we obtain that Φ is a contraction.

The contraction principle assures a fixed point $(\Omega, r, \phi) \in \mathcal{X}_C$.

Clearly, Ω, r, ϕ satisfy the evolution equations (1)–(2), and agree with $\bar{\Omega}$, $\bar{r}$, and $\bar{\phi}$ initially. In particular, Ω is in fact C^1.

But now since, setting $\partial_u r = \nu$, $\partial_v r = \lambda$,

$$\partial_v \nu = \frac{1}{4} r\Omega^2 + \nu(r^{-1}\lambda) = A + \nu B$$

where A and B are differentiable in u. Since ν can be expressed explicitly in terms of $\nu(\cdot, v_0)$, A and B, and since $\nu(\cdot, v_0)$ is initially differentiable in u, it follows that ν is differentiable in u everywhere in $[u_0, \tilde{u}_1] \times [v_0, \tilde{v}_1]$.

Similarly for λ. Thus r is in fact C^2. It follows now, via similar reasoning, that ϕ is C^2, and Ω is C^2.

Moreover, the last argument showed in fact that $\partial_u \partial_v \partial_v r$ and $\partial_v \partial_u \partial_u r$ are defined. Thus, we can compute

$$\partial_v(\partial_u(\Omega^{-2}\partial_u r)) \tag{17}$$

Claim. (17) equals $\partial_v(-r\Omega^{-2}T_{uu})$.

Thus, the constraint equations (3) and (4) are also satisfied since they are satisfied initially!!

So we have indeed a C^2 solution in $[u_0, u_1] \times [v_0, v_1]$. Higher regularity follows similarly if it is assumed initially. □