Prof. M. Dafermos, Michaelmas 2016

1. Using geodesic polar coordinates, show that given $p \in S$, we can express the Gussian curvature as

$$K(p) = \lim_{r \to 0} \frac{3(2\pi r - L)}{\pi r^3},$$

where L is the length of the geodesic circle of radius r. [*Hint: Taylor expansion.*]

2. Find the geodesic curvature of a parallel of latitude on the 2-sphere.

3. Prove that on a surface of constant Gaussian curvature, the geodesic circles have constant geodesic curvature. Moreover, prove this in two ways: by direct computation, and by showing that rotation in a geodesic circle is a local isometry (cf. Problem 15 from the last example sheet).

4. Let S be a connected surface and $f, g : S \to S$ two isometries. Assume that $f(p) = g(p)$ and $df_p = dg_p$ for some $p \in S$. Show that $f(q) = g(q)$ for all $q \in S$.

5. Let p be a point on a surface S. Complete the outline in lecture to show that there exists an open set V containing p such that if $\gamma : [0, 1] \to V$ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and $\alpha : [0, 1] \to S$ is a regular curve joining p to q, then

$$\ell(\gamma) \leq \ell(\alpha)$$

with equality iff α is a reparametrization of γ.

6. Let S be a compact orientable surface which is not diffeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative and zero.

7. Let S be a compact oriented surface with positive Gaussian curvature and let $N : S \to \mathbb{S}^2$ be the Gauss map. Let γ be a simple closed geodesic in S, and let A and B be the regions which have γ as a common boundary. Show that $N(A)$ and $N(B)$ have the same area.

8. Let S be an orientable surface with Gaussian curvature $K \leq 0$. Show that 2 geodesics γ_1 and γ_2 which start from a point $p \in S$ will not meet again at a point q in such a way that the images of γ_1 and γ_2 form the boundary of a domain homeomorphic to a disk.

9. Let S be a surface homeomorphic to a cylinder with negative Gaussian curvature. Show that S has at most one simple closed geodesic. Does the result remain true if "negative" is replaced by "nonpositive"?

10. Let $\phi : U \to S$ be an orthogonal parametrization around a point p. Let $\alpha : [0, \ell] \to \phi(U)$ be a simple closed curve parametrized by arc-length enclosing a domain R. Fix a unit vector $w_0 \in T_{\alpha(0)}(S)$ and consider $W(t)$ the parallel transport of w_0 along α. Let $\psi(t)$ be a differentiable determination of the angle from ϕ_u to $W(t)$. Show that

$$\psi(\ell) - \psi(0) = \int_R K dA.$$

Let S now be a connected surface. Use the above to show that if the parallel transport between any two points does not depend on the curve joining the points, then the Gaussian curvature of S vanishes identically.

11. Let S_t be a family of smooth oriented surfaces, where t ranges in an interval $I \subset \mathbb{R}$ containing 0, such that, around each point $p \in S_t$, there exists a family of local parametrizations $x_t(u,v), y_t(u,v), z_t(u,v)$ of S_t such that $x(t,u,v)$, etc. are smooth maps and

$$(\dot{x}_t, \dot{y}_t, \dot{z}_t) = 2H_t(x_t, y_t, z_t)N_t(x_t, y_t, z_t),$$

where $\cdot$ denotes differentiation with respect to t, H_t denotes the mean curvature of S_t, N_t denotes the normal of S_t, and it is assumed that $H_t \neq 0$ and N_t is continuous in t. We say that S_t *evolves under mean curvature flow.*

Show that the map $\phi_t : S_0 \to S_t$ defined by taking $(x_0, y_0, z_0) \mapsto (x_t, y_t, z_t)$ is well-defined, i.e. it does not depend on the parametrizations. Show that the map $\phi : I \times S_0 \to S_t$ is smooth.

Now let γ_0 be a closed geodesic in S_0, and define $\gamma_t = \phi_t \circ \gamma$. Let $L(\gamma_t)$ denote the length. Assume moreover that the Gaussian curvature satisfies $K \geq 0$ along γ_0. Show that

$$\frac{d}{dt}L(\gamma_t)|_{t=0} \leq -\frac{4\pi^2}{L(\gamma_0)}.$$

What can you infer from this?

The remaining two questions complete a circle of ideas in the course. They are non-examinable.

12. (The Poincaré–Hopf theorem) Let S be an oriented surface and $V : S \to \mathbb{R}^3$ a smooth vector field, that is, $V(p) \in T_pS$ for all $p \in S$. We say that p is *singular* if $V(p) = 0$. A singular point p is isolated if there exists a neighborhood of p in which V has no other zeros. The singular point p is *non-degenerate* if $dV_p : T_pS \to T_pS$ is a linear isomorphism. (Can you see why dV_p takes values in T_pS?) Show that if a singular point is non-degenerate, then it is isolated.

To each singular point p we associate an integer called the *index* of the vector field at p as follows: Let $\phi : U \to S$ be an orthogonal parametrization around p compatible with the orientation. Let $\alpha : [0, l] \to \phi(U)$ be a regular piecewise smooth simple closed curve so that p is the only zero of V in the domain enclosed by α. Let $\phi(t)$ be some differentiable determination of the angle from ϕ_u to $V(t) \doteq V \circ \alpha(t)$. Since α is closed, there is an integer I (the index) defined by

$$2\pi I \doteq \phi(l) - \phi(0).$$

(i) Show that I is independent of the choice of parametrization (Hint: use Problem 10). One can also show that I is independent of the choice of curve α, but this is somewhat harder. In addition, one can prove that if p is non-degenerate, then $I = 1$ if dV_p preserves orientation and $I = -1$ if dV_p reverses orientation.

(ii) Give examples of various vector fields on $\mathbb{R}^2$ with isolated singularities at the origin, computing their indices. Draw pictures.

(iii) Suppose now that S is compact and V is a smooth vector field with isolated singularities. Consider a triangulation of S such that

- every triangle is contained in the image of some orthogonal parametrization,
- every triangle contains at most one singular point
- the boundaries of the triangles contain no singular points and are positively oriented.

Show that

$$\sum_i I_i = \frac{1}{2\pi}\int_S K\,dA = \chi(S),$$

where I_i denote the indices of the singular points. Thus, you have proved that the sum of the indices of a smooth vector field with isolated singularities on a compact surface is equal to the Euler characteristic. This is known as the *Poincaré–Hopf theorem*. Conclude that a surface homeomorphic to $\mathbb{S}^2$ cannot be "combed".

Finally, suppose $f : S \to \mathbb{R}$ is a Morse function and consider the vector field given by the gradient of f, i.e. $\nabla f(p)$, defined in turn by the relation $\langle \nabla f(p), v\rangle = df_p(v)$ for all $v \in T_pS$. (Show that $\nabla f(p)$ is indeed well defined and only depends on the first fundamental form of S.) Use the Poincaré–Hopf theorem to show that $\chi(S)$ is the number of local maximum and minima minus the number of saddle points. Use this to find the Euler characteristic of a surface of genus two.

13. (The degree of the Gauss map) Let S be a compact oriented surface and let $N : S \to \mathbb{S}^2$ be its Gauss map. Let $y \in S^2$ be a regular value of N. Rather than counting the preimages of y modulo 2 as we did in the first lectures, we will count them with sign. Let $N^{-1}(y) = \{p_1, \ldots p_n\}$. Let $\varepsilon(p_i)$ be $+1$ if dN_{p_i} preserves orientation $K(p_i) > 0$, and -1 if dN_{p_i} reverses orientation ($K(p_i) < 0$). Now let

$$\deg(N) \doteq \sum_i \varepsilon(p_i).$$

As in the case of the degree mod 2, it can be shown that the sum on the right hand side is independent of the regular value and $\deg(N)$ turns out to be an invariant of the homotopy class of N.

Now choose $y \in \mathbb{S}^2$ such that both y and $-y$ are regular values of N. (Why can we do this?) Let V be the vector field on S given by

$$V(p) \doteq \langle y, N(p)\rangle N(p) - y.$$

(i) Show that the index of V at a zero p_i is $+1$ if dN_{p_i} preserves orientation and -1 if dN_{p_i} reverses orientation.

(ii) Show that the sum of the indices of V equals twice the degree of N.

(iii) Show that $\deg(N) = \chi(S)/2$.

For comments, email `M.Dafermos` in `dpmms`.