Prof. M. Dafermos, Michaelmas 2016

1. Let $\alpha : I \to S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}$ is an eigenvector of $dN_{\alpha(t)}$ where N denotes the Gauss map. [*Hint: compare the normal n of α with the normal N of S.*]

2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or sphere. [*Hint: use the previous problem and the results of Problem 14 of the previous example sheet.*]

3. Let $f : S_1 \to S_2$ be an isometry between two surfaces.
(i) Let $\alpha : I \to S_1$ be a regular curve and $V(t)$ a parallel vector field along α. Show that $df_{\alpha(t)}(V(t))$ is a parallel vector field along $f \circ \alpha$. Show more generally that for V not necessarily parallel, $\frac{DV}{dt} = \frac{D(df_{\alpha(t)}V(t))}{dt}$.
(ii) Show that f maps geodesics to geodesics.

4. Consider the surface of revolution from Problem 10 of the last example sheet.
(i) Write down explicitly the differential equation for geodesics.
(ii) Establish *Clairaut's relation*: $f^2\dot{u}$ is constant along geodesics. Show that if θ is the angle that a geodesic makes with a parallel and r is the radius of the parallel at the intersection point, then Clairaut's relation says that $r\cos\theta$ is constant along geodesics.
(iii) Show that meridians are geodesics; when is a parallel a geodesic?

5. Let S_1 and S_2 be surfaces with Gauss curvature K_{S_1}, K_{S_2}, respectively. Suppose $f : S_1 \to S_2$ is a diffeomorphism with $K_{S_2}(f(x)) = K_{S_1}(x)$. Is f an isometry?
Now suppose that f maps geodesics of S_1 to geodesics of S_2. Is f an isometry?

6. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

7. Let S be a connected surface and let p be such that all geodesics through p are closed, i.e. all geodesics through p extend to smooth maps $\gamma : \mathbb{S}^1 \to S$. Show that S is compact.

8. Let S be a regular surface without umbilical points. Prove that S is a minimal surface iff the Gauss map $N : S \to S^2$ satisfies

$$\langle dN_p(v_1), dN_p(v_2)\rangle = \lambda(p)\langle v_1, v_2\rangle$$

for all $p \in S$ and all $v_1, v_2 \in T_pS$, were $\lambda(p) \neq 0$ is a number depending only on p. By considering stereographic projection deduce that isothermal coordinates exist around a nonplanar point in a minimal surface.

9. Let $D \subset \mathbb{C}$, f and g complex functions and $\phi : D \to \mathbb{R}^3$ the parametrization defined by the Weierstrass representation. Show that ϕ is an immersion iff f vanishes only at the poles of g and the order of its zero at each pole is exactly twice the order of the pole of g.

10. Find D, f, g representing the catenoid $\phi(u, v) = (a\cosh v\cos u, a\cosh v\sin u, av)$ and helicoid $\phi(u, v) = (a\sinh v\cos u, a\sinh v\sin u, au)$.

11. Show that the Gaussian curvature of the minimal surface determined by a Weierstrass

representation defined by functions f and g is

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2.$$

Show that either $K = 0$ identically, or the zeros of K are isolated. [*There is a way of doing this problem almost entirely without calculation. Think about the relation between g and the Gauss map, together with the fact that stereographic projection is conformal.*]

12. The Weierstrass representation is not unique: if $\phi_{(f,g)} : D \to \mathbb{R}^3$ is the associated parametrization and $\alpha : W \to D$ is a bijective holomorphic map, then $\phi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and thus can be expressed as a Weierstrass parametrization corresponding to different f and g. Show that by choosing $\alpha(z) = g^{-1}(z)$, by restricting to a suitable open set around a point which is not a pole of g and for which g' is not zero, we may assume that (f, g) is of the form (F, id). We denote such a parametrization by ϕ_F.

13. Show that the minimal surfaces given by $\phi_{e^{-i\theta}F}$ for θ real are all locally isometric. With an appropriate choice of F, show that the catenoid and the helicoid are locally isometric.

14. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ with is not a meridian intersects itself an infinite number of times. [*Hint: use Clairaut's relation. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic.*]

15. Let S_1 and S_2 be surfaces and $p_1 \in S_1$, $p_2 \in S_2$.

(i) Suppose S_1 and S_2 are locally isometric around p_1 and p_2. Show that there exists a geodesic $\gamma_1(t)$ emanating from p_1 and $\gamma_2(t)$ emanating from p_2 so that $K_{S_1}(t) = K_{S_2}(t)$, and such that, for all θ, if $\alpha_1(t)$, $\alpha_2(t)$ are geodesics emanating from p_1, p_2 respectively, making an angle θ from γ_1, γ_2, respectively, then $K_{S_1}(t) = K_{S_2}(t)$ along $\alpha_1(t)$ and $\alpha_2(t)$. Here K_{S_i} denotes the Gauss curvature of S_i and geodesics are considered parametrized by arc length.

(ii) Show conversely that if the above property is satisfied then p_1 and p_2 are locally isometric. Deduce that if S_1 and S_2 have constant curvature $K_{S_1} = K_{S_2}$, then S_1 and S_2 are locally isometric. [*Hint: consider geodesic polar coordinates and solve an ode for G.*]

For comments, email `M.Dafermos` in `dpmms`.