

Prof. M. Dafermos, Michaelmas 2016

1. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arc-length, with curvature $k(s) \neq 0$ for all $s \in I$. Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\langle \dot{\alpha} \wedge \ddot{\alpha}, \ddot{\alpha} \rangle}{|k(s)|^2}.$$

2. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a regular curve parametrized by arc length with $\tau(s) \neq 0$ and $\dot{k}(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (\dot{R})^2 T^2$$

is constant, where $R = 1/k$ and $T = 1/\tau$. [Hint: To prove that the condition is necessary you need to differentiate three times $|\alpha(s)|^2$. To prove sufficiency, differentiate $\alpha + Rn - \dot{R}Tb$.]

3. Consider a closed plane curve inside a disk of radius r . Prove that there exists a point on the curve at which the curvature has absolute value $\geq 1/r$.

4. Let \overline{AB} be a straight line segment in the plane with endpoints A and B and let ℓ be a fixed number strictly bigger than the length of \overline{AB} . Show that the curve joining A and B with length ℓ and such that together with \overline{AB} bounds the largest possible area is an arc of a circle passing through A and B . [You may suppose that the isoperimetric inequality holds for piecewise smooth boundaries.]

5. (i) Show that of all n -sided (simple) polygons of equal perimeter, the n -sided regular polygon encloses the most area.

(ii) Show that if $n > m$, then the n -sided regular polygon encloses more area than the m -sided regular polygon of equal perimeter.

Deduce a version of the isoperimetric inequality for a suitably general class of curves.

6. Let $\phi : U \rightarrow S$ be a parametrization of a surface S in \mathbb{R}^3 . Show that

$$|\phi_u \wedge \phi_v| = \sqrt{EG - F^2}.$$

7. Let $\alpha : [0, \ell] \rightarrow \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature. Suppose α has no self-intersections, $\alpha(0) = \alpha(\ell)$ and it induces a smooth map from \mathbb{S}^1 to \mathbb{R}^3 (i.e α is a smooth simple closed curve). Let $r > 0$ and consider $\phi : [0, \ell] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ given by:

$$\phi(s, v) = \alpha(s) + r(n(s) \cos v + b(s) \sin v)$$

where $n = n(s)$ and $b = b(s)$ are the normal and binormal vectors of α . The image T of ϕ is called the *tube* of radius r around α . It can be shown that for r sufficiently small T is a surface. Prove that the area of T is $2\pi r\ell$.

8. (i) Let S be a surface that can be covered by connected coordinate neighbourhoods V_1 and V_2 . Assume that $V_1 \cap V_2$ has two connected components W_1 and W_2 , and that the Jacobian of the change of coordinates is positive on W_1 and negative on W_2 . Prove that S is not orientable.

(ii) Let $\phi : [0, 2\pi] \times (-1, 1) \rightarrow \mathbb{R}^3$ given by:

$$\phi(u, v) = ((s - v \sin(u/2)) \sin u, (s - v \sin(u/2)) \cos u, v \cos(u/2)).$$

The image of ϕ is the Möbius strip. By considering the parametrizations given by ϕ restricted to $(0, 2\pi) \times (-1, 1)$ and

$$\psi(\bar{u}, \bar{v}) = ((2 - \bar{v} \sin(\pi/4 + \bar{u}/2)) \cos \bar{u}, -(2 - \bar{v} \sin(\pi/4 + \bar{u}/2)) \sin \bar{u}, \bar{v} \cos(\pi/4 + \bar{u}/2)),$$

$(\bar{u}, \bar{v}) \in (0, 2\pi) \times (-1, 1)$, show that the Möbius strip is not orientable.

9. Show that the mean curvature H at $p \in S$ is given by $H = \frac{1}{\pi} \int_0^\pi k_n(\theta) d\theta$, where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

10. Consider a surface of revolution parametrized by $\phi : (0, 2\pi) \times (a, b) \rightarrow \mathbb{R}^3$, where

$$\phi(u, v) = (f(v) \cos u, f(v) \sin u, g(v)).$$

Suppose f never vanishes and that $(f(v), g(v))$ is parametrized by arc length. Compute the Gaussian curvature K and the mean curvature H .

11. (i) Determine an equation for the *tractrix*, which is the curve such that the length of the segment of the tangent line between the point of tangency and some fixed line L in the plane—not meeting the curve—is a constant equal to 1.

(ii) Rotate the tractrix about the line L to obtain a surface of revolution (called the *pseudosphere*). Compute its Gaussian curvature.

12. Let S be a compact orientable surface in \mathbb{R}^3 . Show that the Gauss map is surjective and that it hits almost every direction the same number of times modulo 2. [You may use the Jordan–Brouwer separation theorem.] Show that S always has an elliptic point.

13. Let p be a point of a surface S such that the Gaussian curvature $K(p) \neq 0$ and let V be a small connected neighborhood of p where K does not change sign. Define the *spherical area* $A_N(B)$ of a domain B contained in V as the area of $N(B)$ if $K(p) > 0$ or as minus the area of $N(B)$ if $K(p) < 0$, where N here denotes the Gauss map. Show that

$$K(p) = \lim_{A \rightarrow 0} \frac{A_N(B)}{A(B)}$$

where $A(B)$ is the area of B and the limit is taken through a sequence of domains B_n that converge to p in the sense that any sphere around p contains all B_n for all n sufficiently large. (Note: This is the way Gauss introduced K .)

14. Show that if S is a connected surface in \mathbb{R}^3 such that every point is umbilic, then S is part of a plane or a sphere. [Hint: use that in a parametrization $\phi(u, v)$, $N_{uv} = N_{vu}$.]

15. Let S be a connected surface in \mathbb{R}^3 . Suppose that for each $p \in S$, there exists a Euclidean rotation ρ of \mathbb{R}^3 centered at p , such that $\rho^n \neq 1$ for $1 \leq n \leq 5$, and such that $\rho(S) = S$. Show that S is a plane or sphere.

16. Given an example of two curves $\alpha : [0, 1] \rightarrow \mathbb{R}^3$, $\tilde{\alpha} : [0, 1] \rightarrow \mathbb{R}^3$, parametrized by arc length, such that $k(s) = \tilde{k}(s)$, and $\tau(s) = \tilde{\tau}(s)$ whenever $k(s) \neq 0$, but such that α and $\tilde{\alpha}$ are not related via a Euclidean motion.

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