Prof. M. Dafermos, Michaelmas 2016

1. Let X, Y be manifolds. Show that $X \times Y$ is a manifold of dimension $\dim X \times Y = \dim X + \dim Y$.

2. Let X be a submanifold of Y and suppose that X and Y have the same dimension. Show that X is an open subset of Y.

3. Let B_r denote the open ball $\{x \in \mathbb{R}^k : |x| < r\}$. Show that the map $\phi : B_r \to \mathbb{R}^k$

$$x \mapsto \frac{rx}{\sqrt{r^2 - |x|^2}}$$

is a diffeomorphism. (This implies that local parametrizations can always be chosen with $\mathbb{R}^k$ as domain.)

4. (i) Is the union of two coordinates axes in $\mathbb{R}^2$ a manifold?
(ii) Prove that the hyperboloid in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = a$ is a manifold for $a > 0$. What happens for $a \leq 0$? Find the tangent space at the point $(\sqrt{a}, 0, 0)$.
(iii) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary $(a > 0)$.

5. Prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not diffeomorphic if $n \neq m$. Prove that dimension is well defined, i.e. an n-manifold $X \subset \mathbb{R}^N$ is not an m-manifold for $m \neq n$.

6. Let X, Y be manifolds. A *submersion* is defined to be a smooth map $f : X \to Y$ such that df_x is surjective for all $x \in X$. The *canonical submersion* is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for $k \geq l$ defined by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots x_l).$$

(i) Let f be a submersion with $y = f(x)$. Show that there exist local coordinates around x and y such that f in these coordinates is the canonical submersion $\mathbb{R}^k \to \mathbb{R}^l$, where k and l are the dimensions of X and Y, respectively.
(ii) Show that submersions are *open maps*, i.e. they map open sets to open sets.
(iii) If X is compact and Y connected, show that every submersion is surjective.
(iv) Are there submersions of compact manifolds into Euclidean spaces?

7. Let $f : X \to Y$ be a smooth map and $y \in Y$ a regular value of f. Show that the tangent space to $f^{-1}(y)$ at a point $x \in f^{-1}(y)$ is given by the kernal of $df_x : T_xX \to T_yY$.

8. (i) Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4$.
(ii) Show that the orthogonal group $O(n)$ is a compact manifold and that its tangent space at $A \in O(n)$ is given by the set of all $n \times n$ real matrices for which $AH^t + HA^t = 0$.

9. For which values of a does the hyperboloid $x^2+y^2-z^2 = 1$ intersect the sphere $x^2+y^2+z^2 = a$ transversally? What does the intersection look like for different values of a?

10. Let $f : X \to X$ be a smooth map. f is called a *Lefschetz map* if given any fixed point x of f, $df_x : T_xX \to T_xX$ does not have 1 as an eigenvalue. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

11. Prove the following theorem due to Frobenius: let A be an $n \times n$ matrix all of whose entries are nonnegative. Then A has a nonnegative real eigenvalue. [Hint: consider the set $\{(x_1, \ldots, x_n) \in S^{n-1} : x_i \geq 0 \forall i\}$ and apply the (topological) Brouwer fixed point theorem.]

12. A manifold is said to be *contractible* if the identity map is homotopic to a constant map. Show that compact manifolds of dimension 2 or greater are not contractible.

13. Let X be a compact manifold and Y a connected manifold of the same dimension.
(i) Suppose $f : X \to Y$ has $\deg_2(f) \neq 0$. Prove that f is onto.
(ii) If Y is not compact, prove that $\deg_2(f) = 0$ for all smooth $f : X \to Y$.

14. Suppose $f : X \to \mathbb{S}^k$ is smooth, where X is compact and $0 < \dim X < k$. Let $Z \subset \mathbb{S}^k$ be a compact submanifold of dimension $k - \dim X$. Show that $I_2(f, Z) = 0$. (Thus degrees are the only interesting intersection numbers on spheres.) [Hint: Sard's theorem.]

15. (i) Prove that the boundary of a manifold with boundary is a manifold.
(ii) Show that the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ is *not* a manifold with boundary.

16. (i) Let $\lambda : \mathbb{R} \to \mathbb{R}$ be given by $\lambda(x) = e^{-x^{-2}}$ for $x > 0$ and $\lambda(x) = 0$ for $x \leq 0$. You know from Analysis I that λ is smooth. Show that $\tau(x) = \lambda(x - 1)\lambda(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere ($a < b$).
(ii) Show that

$$\phi(x) = \frac{\int_{-\infty}^{x} \tau}{\int_{-\infty}^{\infty} \tau}$$

is smooth, $\phi(x) = 0$ for $x < a$, $\phi(x) = 1$ for $x > b$, and $0 < \phi(x) < 1$ for $x \in (a, b)$.
(iii) Finally, construct a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ that equals 1 on the ball of radius a, zero outside the ball of radius $b > a$, and such that $f(\mathbb{R}^n) \subset [0, 1]$.

These functions are very useful for gluings. As an illustration, suppose $f_0, f_1 : X \to Y$ are smooth homotopic maps. Show that there exists a smooth homotopy $\tilde{F} : X \times [0, 1] \to Y$ such that $\tilde{F}(x, t) = f_0$ for all $t \in [0, 1/4]$ and $\tilde{F}(x, t) = f_1(x)$ for all $t \in [3/4, 1]$. Conclude that smooth homotopy is an equivalence relation.

17. (Morse functions) Let X be a k-dimensional manifold and $f : X \to \mathbb{R}$ a smooth function. A critical point $x \in X$ is said to be *nondegenerate* if, in local coordinates x^i around $\mathbf{x}$, the *Hessian* matrix $(\partial_i \partial_j f)$ has non-vanishing determinate. If all critical points are non-degenerate f is said to be a *Morse function*.
(i) Show that the condition $\det(\partial_i \partial_j f) \neq 0$ is independent of the local parametrization.
(ii) Suppose now that X is an open subset of $\mathbb{R}^k$. Given $\mathbf{a} \in \mathbb{R}^k$, define a function $f_{\mathbf{a}} : \mathbb{R}^k \to \mathbb{R}$ by

$$f_{\mathbf{a}}(\mathbf{x}) = f(\mathbf{x}) + \mathbf{x} \cdot \mathbf{a}$$

where $\mathbf{x} \cdot \mathbf{a}$ denotes the Euclidean inner product. Show that $f_{\mathbf{a}}$ is a Morse function for a dense set of $\mathbf{a}$. [Hint: consider $\nabla f : X \to \mathbb{R}^k$.]

With a bit more work one can show that the same result is true if X is an arbitrary manifold. In other words, a "generic" smooth function is in fact Morse.
(iii) Show that the determinant function on $M(n)$ is Morse if $n = 2$ but not if $n > 2$.

For comments, email `M.Dafermos` in `dpmms`.