Analysis of Partial Differential Equations
Example sheet 3 (Chapter 3)

Prof. M. Dafermos, Prof. C. Mouhot

1. In the lectures, we have seen that $H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $s > d/2$. Give an example of a function $f \in H^1(\mathbb{R}^2)$ but $f \notin L^\infty(\mathbb{R}^2)$.

2. (Poincaré–Wirtinger) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain. Show that there exists a constant $C(\Omega)$, depending on the domain, such that

$$\forall\, u \in H^1(\Omega), \qquad \int_\Omega |u - \bar{u}|^2 \,\mathrm{d}x \le C(\Omega) \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \tag{1}$$

where $\bar{u} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u(x)\,\mathrm{d}x$.

3. (Hardy) Let $\Omega \subset \mathbb{R}^d$ be a C^1 bounded domain. Define the distance function $d : \Omega \to \mathbb{R}_+$ by $d(x) := \mathrm{dist}(x, \partial\Omega)$. Show that there exists a constant $C > 0$ such that

$$\left\| \frac{u}{d} \right\|_{L^2(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)} \text{ for all } u \in H^1_0(\Omega). \tag{2}$$

4. (Rellich) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. Prove that any sequence uniformly bounded in $H^1(\Omega)$ is relatively compact in $L^2(\Omega)$ i.e., if $\{u_n\} \subset H^1(\Omega)$ is a sequence such that $\|u_n\|_{H^1(\Omega)} \le C$ for some constant C independent of n, then there exists a subsequence $\{u_{\varphi(n)}\}$ (with $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing) and a limit function $u \in L^2(\Omega)$ such that

$$\left\| u_{\varphi(n)} - u \right\|_{L^2(\Omega)} \to 0 \qquad \text{as} \quad n \to \infty.$$

5. (Riesz–Fréchet–Kolmogorov) Let $\Omega \subset \mathbb{R}^d$ be open.

- First consider $\omega \subset\subset \Omega$ i.e., ω open with $\bar{\omega} \subset \Omega$. Consider a bounded $\mathcal{G} \subset L^2(\Omega)$. Suppose

$$\begin{aligned} &\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \delta < \mathrm{dist}(\omega, \partial\Omega) \text{ such that} \\ &\forall\, h \in \mathbb{R}^d,\ |h| < \delta \ \text{ and } \forall\, u \in \mathcal{G}, \quad \int_\omega |u(x+h) - u(x)|^2 \,\mathrm{d}x \le \varepsilon. \end{aligned} \tag{3}$$

Prove that $\mathcal{G}|_\omega$ is relatively compact in $L^2(\omega)$. (Notation: $\mathcal{G}|_\omega$ denotes the elements of $\mathcal{G}$ restricted to ω).

- Second, assume that $\Omega = \mathbb{R}^d$ and consider again a bounded $\mathcal{G} \subset L^2(\mathbb{R}^d)$. Suppose

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, h \in \mathbb{R}^d,\ |h| < \delta \ \text{ and } \forall\, u \in \mathcal{G}, \quad \int_{\mathbb{R}^d} |u(x+h) - u(x)|^2 \,\mathrm{d}x \le \varepsilon. \tag{4}$$

In addition to (4), suppose also that

$$\forall\, \varepsilon > 0,\ \exists\, \omega \subset\subset \mathbb{R}^d \text{ such that } \|u\|_{L^2(\mathbb{R}^d \setminus \omega)} < \varepsilon \ \forall\, u \in \mathcal{G}. \tag{5}$$

Then, deduce that $\mathcal{G}$ is relatively compact in $L^2(\mathbb{R}^d)$.

6. (Fredholm Alternative) Let $u \in H^1(\Omega)$ be a weak solution of the following Neumann problem:

$$\begin{cases} b(x) \cdot \nabla u - \nabla \cdot (A(x)\nabla u) = f & \text{in } \Omega, \\ -A(x)\nabla u \cdot n = g & \text{on } \partial\Omega. \end{cases} \tag{6}$$

where $f \in L^2(\Omega)$, $g \in H^1(\Omega)$. Here, $A(x)$ denotes a symmetric matrix of measurable coefficients $A^{ij}(x)$ such that there exist $\alpha_0 > 0$, $\alpha_1 > 0$ with

$$\alpha_0|\xi|^2 \leq A^{ij}(x)\xi_i\xi_j \leq \alpha_1|\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^d \tag{7}$$

and $b(x)$ is a vector of coefficients $b_i(x) \in L^\infty(\Omega)$ which satisfies $\nabla \cdot b = 0$ weakly in Ω and $b \cdot n = 0$ on $\partial\Omega$. Prove that (6) has a unique weak solution modulo an additive constant if and only if the source terms satisfy the following compatibility condition:

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\partial\Omega} g(x)\,\mathrm{d}\sigma(x), \tag{8}$$

where $\mathrm{d}\sigma(x)$ is the surface measure on $\partial\Omega$.
[*Hint: Employ the Poincaré–Wirtinger inequality and use Lax–Milgram in the quotient space* $H^1(\Omega)/\mathbb{R}$.]

7. (Mean Value Theorem) Let $\Omega \subset \mathbb{R}^d$ be open. A function $u \in C^2(\Omega)$ is said to be *harmonic* if $\Delta u = 0$ in Ω. Suppose u is harmonic in Ω. Let $x_0 \in \Omega$ and $r > 0$ so that the closed ball $\bar{B}(x_0, r) \subset \Omega$. Show that:

$$u(x_0) = \frac{1}{r^{d-1}\omega_d} \int_{S(x_0,r)} u(y)\mathrm{d}\sigma(y), \tag{9}$$

where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$ and $S(x_0, r)$ is the sphere of radius r centered at x_0.

8. (Liouville's Theorem) Prove that every bounded harmonic function on the whole space $\mathbb{R}^d$ is constant. Deduce that any harmonic function $v(x)$ on the whole space $\mathbb{R}^d$ which satisfies $|v(x)| \to 0$ as $x \to \infty$ vanishes identically.

9. (Dirichlet Principle) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain. For a source term $f \in L^2(\Omega)$, show that solving for $u \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

is the same as solving for $u \in H_0^1(\Omega)$ the following minimisation problem:

$$F(u) = \inf_{v \in H_0^1(\Omega)} F(v), \tag{11}$$

where

$$F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x.$$

10. (Helmholtz decomposition) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain. Suppose $b(x) \in (L^2(\Omega))^d$ is a vector field in Ω. Show that there exists $u \in H_0^1(\Omega)$ and $v \in (L^2(\Omega))^d$ such that

$$b(x) = \nabla u(x) + v(x)$$

with

$$\nabla \cdot v = 0 \text{ weakly in } \Omega \quad \text{and} \quad \int_\Omega \nabla u \cdot v\,\mathrm{d}x = 0.$$

11. (Caccioopoli) Let $\Omega \subset \mathbb{R}^d$ be open. Let $x_0 \in \Omega$ and $0 < \rho < \bar{\rho}$ be such that the ball $B(x_0, \bar{\rho}) \subset \Omega$. Suppose $u \in H^1(\Omega)$ satisfies

$$-\Delta u + \mathbf{b} \cdot \nabla u + c\, u = 0 \text{ in } \Omega, \tag{12}$$

where $\mathbf{b} \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Show that there exists a constant C such that

$$\int_{B(x_0,\rho)} |\nabla u|^2 \,\mathrm{d}x \leq \frac{C}{(\bar{\rho} - \rho)^2} \int_{B(x_0,\bar{\rho})} |u|^2 \,\mathrm{d}x. \tag{13}$$

Take $\mathbf{b} = 0$ and $c = 0$ in (12). Deduce from (13) that

$$\forall\, k \in \mathbb{N}, \quad \|u\|^2_{H^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k) \|u\|^2_{L^2(B(x_0,\bar{\rho}))} \tag{14}$$

and

$$\forall\, k \in \mathbb{N}, \quad \|u\|^2_{C^k(B(x_0,\rho))} \leq C(\rho, \bar{\rho}, k) \|u\|^2_{L^2(B(x_0,\bar{\rho}))}. \tag{15}$$

What can we infer from (15)?
[*Hint: Use 'Cut-off functions' as was done in lectures for the interior regularity results.*]

12. (Higher order boundary regularity) Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain and let P be a second order uniformly elliptic operator in divergence form

$$Pu = -\partial_i(a^{ij}(x)\partial_j u) + b^i(x)\partial_i u + cu(x)$$

Prove higher order boundary regularity, i.e. if $u \in H^{m+1} \cap H^1_0$ is a weak solution of $Pu = f$, with $\partial\Omega \in C^\infty$, $a^{ij} \in C^{m+1}(\overline{\Omega})$, $b^i \in C^{m+1}(\overline{\Omega})$, $c \in C^{m+1}(\overline{\Omega})$, $f \in H^m(\Omega)$ then $u \in H^{m+2}(\Omega)$, and $\|u\|_{H^{m+2}(\Omega)} \leq C(\|u\|_{H^{m+1}(\Omega)} + \|f\|_{H^m(\Omega)})$. What does C depend on? Deduce that if $a^{ij} \in C^\infty(\overline{\Omega})$, $b^i \in C^\infty(\overline{\Omega})$, $c \in C^\infty(\overline{\Omega})$, $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.

13. (Maximum Principle - Divergence Form) Let $A(x)$ be a symmetric matrix (i.e., $A_{ij} = A_{ji}$) with measurable coefficients such that there exist $\alpha_0 > 0$, $\alpha_1 > 0$ with

$$\alpha_0|\xi|^2 \leq A_{ij}(x)\xi_i\xi_j \leq \alpha_1|\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d. \tag{16}$$

Also, let $c(x) \in L^\infty(\Omega)$ and $c(x) \geq \lambda > 0$. Suppose $u \in H^1(\Omega)$ verifies in the weak sense:

$$-\nabla \cdot (A(x)\nabla u) + c\, u \geq 0 \text{ on } \Omega, \tag{17}$$

i.e. suppose that

$$\int_\Omega A(x)\nabla u \cdot \nabla\phi \,\mathrm{d}x + \int_\Omega c\, u\phi \,\mathrm{d}x \geq 0, \quad \forall\, \phi \in C_0^\infty(\Omega) \text{ with } \phi \geq 0 \text{ in } \Omega. \tag{18}$$

Show that

$$\inf_{x\in\Omega} u(x) = \inf_{x\in\partial\Omega} u(x).$$

[*Hint: Use the density of $C_0^\infty(\Omega)$ in $H^1_0(\Omega)$. Take $-(u - \inf_{x\in\partial\Omega} u(x))^-$ as test function. We have employed the following notation: $h^- := \min(0, -h)$.*]

14. (Nonlinear Equations) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem

$$\triangle u = \epsilon|u|^p + 1, \qquad u|_{\partial\Omega} = 0 \tag{19}$$

for $\frac{d}{d-2} > p > 1$.

(**14.a**) Prove, for sufficiently small ϵ, the existence of a unique weak solution u of (19) in $H^1_0(\Omega)$ as follows: Let $B : L^2(\Omega) \to H^1_0(\Omega)$ denote the map taking f to the unique weak solution of $\triangle u = f + 1$. Let $Q : H^1_0(\Omega) \to L^2$ denote the map given by $u \mapsto \epsilon u^p$. (Show that indeed, under the restriction of p, this defines a bounded map on the spaces claimed.) Show that $B \circ Q$ defines a contraction on $H^1_0(\Omega)$ and argue that a fixed point of $B \circ Q$ is a weak solution of (19).

(**14.b**) Show that in fact $u \in C^\infty(\overline{\Omega})$ and u vanishes on $\partial\Omega$.

(**14.c**) Can you allow higher p? How high?

15. (Parabolic Equations) Let $\Omega \subset \mathbb{R}^d$ be open. We consider the initial-boundary value problem (IBVP):

$$\begin{cases} \dfrac{\partial u}{\partial t} + Pu = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = g & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{20}$$

where P is a second order partial differential operator in divergence form:

$$Pu := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u.$$

Define a time-dependent bilinear form for $u, v \in H_0^1(\Omega)$ and for a.e. $t \in [0,T]$:

$$B[u,v;t] := \sum_{i,j=1}^{d} \int_\Omega a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\mathrm{d}x + \sum_{i=1}^{d}\int_\Omega b_i(t,x)\frac{\partial u}{\partial x_i} v\mathrm{d}x + \int_\Omega c(t,x)uv\mathrm{d}x. \tag{21}$$

We give the following definition for a weak solution of the IBVP (20): A function $u \in L^2((0,T);H_0^1(\Omega))$ with $u' \in L^2((0,T);H^{-1}(\Omega))$ is a *weak solution* to (20) if

$$\begin{aligned} &\text{(i) } \langle u', v\rangle + B[u,v;t] = (f,v) \text{ for each } v \in H_0^1(\Omega) \text{ and for a.e. } t \in [0,T] \\ &\text{and (ii) } u(0) = g, \end{aligned} \tag{22}$$

where $\langle \cdot,\cdot\rangle$ is the dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ whereas $(\cdot,\cdot)$ is the standard L^2 inner product. [*If you prefer, you may restrict this problem to $f \in L^2$ and define (f,v) to be the L^2 inner product.*]
The idea is to construct approximate solutions to (20) by considering an orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ of $L^2(\Omega)$. Define approximations for $n \in \mathbb{N}$:

$$u_n(t) := \sum_{k=1}^{n} d_n^k(t)\varphi_k, \tag{23}$$

where the coefficient functions $d_n^k : [0,T] \to \mathbb{R}$ for $k = 1, \cdots, n$ are chosen such that:

$$d_n^k(0) = (g,\varphi_k) \text{ for } k = 1,\cdots,n \qquad \text{and} \qquad (u_n', \varphi_k) + B[u_n,\varphi_k;t] = (f,\varphi_k). \tag{24}$$

(**15.a**) Using an existence result from the theory of ODEs, show that for each $n \in \mathbb{N}$ there exists a unique function $u_n(t)$ of the form (23) satisfying (24).
[*Hint: Use the Cauchy–Lipschitz Theorem.*]

The next task is to consider the finite dimensional approximations u_n for $n \in \mathbb{N}$ and pass to the limit as $n \to \infty$. In order to apply some compactness results, we need to derive uniform (in n) estimates on $\{u_n\}$. The next question addresses this aspect.

(**15.b**) There exists a constant C depending only on $\Omega, T, a_{ij}, b_i, c$ such that

$$\|u_n\|_{L^\infty([0,T];L^2(\Omega))} + \|u_n\|_{L^2((0,T);H_0^1(\Omega))} + \|u_n'\|_{L^2((0,T);H^{-1}(\Omega))} \le C\Big(\|f\|_{L^2((0,T);L^2(\Omega))} + \|g\|_{L^2(\Omega)}\Big). \tag{25}$$

(**15.c**) Using the apriori estimates (25) and compactness results, arrive at the associated limit function with the approximate solutions (23). Show that the limit function is indeed a weak solution of (20) in the sense of (22) by passing to the limit in (24).
[*You may assume the following: If $v_n \rightharpoonup v$ in $L^2((0,T);H_0^1(\Omega))$ and $v_n' \rightharpoonup w$ in $L^2((0,T);H^{-1}(\Omega))$, then $w = v'$.*]

(**15.d**) Show that a weak solution of (20) is unique.
[*Hint: Assume that $f = g = 0$ in (20) and show that the solution $u = 0$.*]