Analysis of Partial Differential Equations
Example sheet I (Chapter 1)

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1. (The Picard–Lindelöf / Cauchy–Lipschitz theorem) Let $\mathbf{F} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and locally Lipschitz in the second variable i.e., assume that for each $\bar{\mathbf{u}} \in \mathbb{R}^d$, there exist constants $\delta > 0$ and $L > 0$ such that

$$|\mathbf{u} - \bar{\mathbf{u}}| < \delta \implies |\mathbf{F}(\cdot, \mathbf{u}) - \mathbf{F}(\cdot, \bar{\mathbf{u}})| \leq L|\mathbf{u} - \bar{\mathbf{u}}|.$$

Consider the following initial value problem:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d. \tag{1}$$

(a) Prove the existence and uniqueness of a *maximum* C^1 solution $\mathbf{u} : (-T_c^-, T_c^+) \to \mathbb{R}^d$ of (1) on $(-T_c^-, T_c^+)$ with $\infty \geq T_c^-, T_c^+ > 0$ i.e., a solution with the property that if $\tilde{\mathbf{u}}(-\tau^-, \tau^+) : (-T_c^-, T_c^+) \to \mathbb{R}^d$ is any other C^1 solution of (1) with $0 < \tau^\pm$, then $\tau^\pm \leq T_c^\pm$ and $\mathbf{u}|_{(-\tau^-, \tau^+)} = \tilde{\mathbf{u}}$.
(b) Prove moreover that if $T_c^+ < +\infty$, then for every $R > 0$ there exists a $t_R < T_c^+$ such that $|\mathbf{u}(t)| > R$ for all $t \geq t_R$.

2. Show that the following initial value problems have infinitely many C^1 solutions $u : [0, \infty) \to \mathbb{R}$:

$$\begin{cases} u'(t) = \sqrt{|u(t)|} \\ u(0) = 0, \end{cases} \tag{2}$$

$$\begin{cases} u'(t) = \dfrac{4\,t\,u(t)}{u(t)^2 + t^2}, \\ u(0) = 0. \end{cases} \tag{3}$$

Describe how the set of such solutions $u : [0, \infty) \to \mathbb{R}$ to (2) can be naturally classified into TWO different types while the set of such solutions $u : [0, \infty) \to \mathbb{R}$ to (3) can be naturally classified into FIVE different types.

3. Let $u : (T_c^-, T_c^+) :\to \mathbb{R}$ be the unique maximal C^1 solution to the following initial value problem:

$$\begin{cases} u'(t) = u(t)^2, \\ u(0) = u_0 > 0. \end{cases} \tag{4}$$

(a) Show that $T_c^+ < \infty$ and compute it. What about T_c^-?
(b) What happens when $u(t)^2$ is replaced by $-u(t)^2$ in (4)?

4. Consider the following initial value problem for a second order ODE:

$$\begin{cases} u''(t) + \sin(u(t)^2) = 0 \\ (u(0), u'(0)) = (u_0, u_1) \in \mathbb{R}^2. \end{cases} \tag{5}$$

Argue, using problem 1, that there exists a unique maximal C^2 solution $u : (T_c^-, T_c^+) \to \mathbb{R}$ of (5). Show that the solution is global, i.e. $T_c^\pm = \infty$.

5. (The Gronwall lemma) For some $T > 0, C \geq 0$, let $u, v \in C^1([0,T);[0,\infty))$ be such that:

$$\forall\, t \in [0,T), \quad u(t) \leq C + \int_0^t v(s)u(s)\,\mathrm{d}s.$$

Show that u satisfies

$$\forall\, t \in [0,T), \quad u(t) \leq C \exp\left(\int_0^T v(s)\,\mathrm{d}s\right).$$

[*Hint*: Set $w(t) := C + \int_0^t v(s)u(s)\,\mathrm{d}s$ and check $w'(t) - v(t)w(t) \leq 0$ for all $t \in [0,T)$.]

6. (Approximation of solutions to ODE) Let $F \in C^1(\mathbb{R};\mathbb{R})$. Let $T > 0$ and let $u, v \in C^1([0,T];\mathbb{R})$ be solutions of the same ODE:

$$u'(t) = F(u(t)), \qquad v'(t) = F(v(t)).$$

Setting $u_0 = u(0)$, $v_0 = v(0)$, show that

$$|u(t) - v(t)| \leq |u_0 - v_0| e^{C_T t}. \tag{6}$$

Fix constants $\varepsilon_1, \varepsilon_2 > 0$, and assume now that $u, v \in C^1([0,T];\mathbb{R})$ only satisfy the inequalities

$$|u'(t) - F(u(t))| \leq \varepsilon_1, \qquad |v'(t) - F(v(t))| \leq \varepsilon_2.$$

Show that

$$|u(t) - v(t)| \leq |u_0 - v_0| e^{C_T t} + (\varepsilon_1 + \varepsilon_2)\frac{e^{C_T t} - 1}{C_T}.$$

7. (Osgood uniqueness Theorem) Let I be an interval of $\mathbb{R}$, and $\mathbf{F} : I \times \mathbb{R}^d \to \mathbb{R}^d$ a continuous function. Let Ω be an open subset of $\mathbb{R}^d$, $t_0 \in I$, $\mathbf{u}_0 \in \Omega$. We suppose that

$$\forall\, (t, \mathbf{y}_1, \mathbf{y}_2) \in I \times \Omega \times \Omega, \quad |\mathbf{F}(t,\mathbf{y}_1) - \mathbf{F}(t,\mathbf{y}_2)| \leq \omega(|\mathbf{y}_1 - \mathbf{y}_2|) \tag{7}$$

where $\omega \in C([0,\infty),\mathbb{R})$ is a non-decreasing function which satisfies

$$\omega(0) = 0; \quad \forall\, \sigma > 0, \quad \omega(\sigma) > 0; \quad \text{and} \quad \forall\, \alpha > 0, \quad \int_0^\alpha \frac{1}{\omega(\sigma)}\,\mathrm{d}\sigma = +\infty. \tag{8}$$

Let $\mathbf{u}_1, \mathbf{u}_2 : I \to \Omega$ be two differentiable functions which are solutions to the following initial value problem:

$$\begin{cases} \mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \\ \mathbf{u}(t_0) = \mathbf{u}_0. \end{cases}$$

(a) Show that $\mathbf{u}_1 = \mathbf{u}_2$.
(b) Give an example of a non-decreasing function $\omega \in C(\mathbb{R}_+,\mathbb{R}_+)$ which satisfies (8) but for which the condition (7) is weaker than being locally Lipschitz.

8. (Cauchy–Peano theorem) Let $\mathbf{F} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be merely continuous, and consider the initial value problem:

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^d.$$

Prove the existence of a *maximal* C^1 solution $\mathbf{u} : (-T_c^-, T_c^+) \to \mathbb{R}^d$ with $T_c^-, T_c^+ > 0$, i.e. C^1 solution with the property that if $\tilde{\mathbf{u}} : (-\tilde{T}_c^-, \tilde{T}_c^+) :\to \mathbb{R}^d$ is another C^1 solutions with $\tilde{\mathbf{u}}|_{(-T_c^-, T^+)} = \mathbf{u}$, with $\tilde{T}_c^\pm \geq T_c^\pm$, then $\tilde{T}_c^\pm = T_c^\pm$. Illustrate the non-uniqueness of $\mathbf{u}$ by the examples of problem 2. (Compare with the characterization of the *maximum* solution of problem 1.)
[*Hint.* From the fundamental theorem of calculus the ODE can be reframed as $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{F}(s, \mathbf{u}(s))\,\mathrm{d}s$. Set $\mathbf{u}_0(t) = \mathbf{u}_0$ to be constant and define the *Picard iterates*:

$$\mathbf{u}_{n+1}(t) = \mathbf{u}_0 + \int_0^t \mathbf{F}(s, \mathbf{u}_n(s))\,\mathrm{d}s. \in \mathbb{R}^d, \quad n \geq 0,$$

Note that $\mathbf{u}_n(0) = \mathbf{u}_0$ for all $n \geq 0$. Prove that restricted to appropriate $(-\epsilon, \epsilon)$ the sequence $\mathbf{u}_n$ is uniformly bounded and uniformly equicontinuous. Recall and use the Arzéla-Ascoli theorem to obtain a solution to the integral equation on that interval. Now maximalise.]