Construction of dynamical vacuum black holes without symmetries

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A fundamental open problem in general relativity is that of the stability of the Kerr black holes, the celebrated 2 parameter family of solutions to the *Einstein vacuum equations*

$$\mathrm{Ric}(g) = 0. \tag{1}$$

According to the *black hole stability conjecture*, small perturbations of Kerr initial data with sub-extremal Kerr parameters $|a_0| < M_0$ would asymptote in time under evolution by (1) to a nearby member of the Kerr family (with parameters $|a| < M$).

At present, not only is this stability conjecture not resolved, but there are no known examples of dynamic black holes with smooth event horizons which asymptote in time to a Kerr solution. It is this more basic question whose resolution is discussed in the present talk:

Theorem ([7])**.** *Given suitable smooth scattering data on the horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$, asymptoting to the induced Kerr geometry with parameters $|a| \leq M$, then there exists a corresponding smooth vacuum black hole spacetime $(\mathcal{M}, g)$ asymptotically approaching in its exterior region the Kerr solution with parameters a and M.*

In particular,

Corollary. *There exist black hole spacetimes with smooth horizons which are not exactly Schwarzschild or Kerr.*

As is suggested by the statement of the above theorem, the black hole spacetimes are constructed by prescribing "scattering data" on the event horizon $\mathcal{H}^+$ and on null infinity $\mathcal{I}^+$, and solving *backwards* as a characteristic initial value problem for (1). More precisely, they are constructed by taking the limit of a finite problem where null infinity is replaced by a far-away light cone and the two null hypersurfaces are supplemented with a late time spacelike piece. Note that this problem is well posed in the smooth category by work of Rendall [11]. See also [10].

For global estimates, one needs a formulation of the Einstein equations which captures both the hyperbolicity *per se* and the asymptotics towards null infinity. Moreover, some version of the null condition must be captured, as one cannot construct solutions even just in a neighbourhood of a point in null infinity for general non-linear equations with quadratic nonlinearities. We adopt thus a formulation where the hyperbolic aspects of the Einstein equations are captured at the level of the Bianchi equations, but to close the system these must be coupled with transport and elliptic equations for the metric and spin coefficients. This formulation first appeared in [3] and has been much exploited to understand global properties of (1), see for instance [2, 9].

Without getting technical, let us briefly motivate a certain assumption on the scattering data that plays a fundamental role in the proof. To understand this, one should remark first that the simplest scattering problems on *Schwarzschild* can of course be formulated with respect to the degenerate ∂_t-energy. See [8]. Thus, for those problems, the notion of scattering data is defined simply by the finiteness of the corresponding flux at $\mathcal{H}^+$ and $\mathcal{I}^+$. For non-linear problems, this is insufficient near infinity as one must consider weighted energies–this already requires imposing some decay along $\mathcal{I}^+$. But even for the linear problem of the fixed wave equation

$$\Box_g \psi = 0 \tag{2}$$

on *Kerr*, the ∂_t energy would already be inappropriate near the horizon as it does not yield a positive definite quantity. This is the well-known phenomenon of *superradiance*. For non-linear problems, one expects to have to control a non-degenerate energy, analogous to the J^N-flux introduced in [4] (see also [5]), exploiting the celebrated *red-shift effect*. To control such a non-degenerate energy when solving backwards, however **the red-shift effect is seen as a blue-shift effect**. Thus, to counterbalance this, one must impose suitably fast exponential decay on the scattering data.

It is interesting to recall that in the forward problem, understanding (2) on extremal Kerr $|a| = M$ is considerably more difficult than the subextremal case $|a| < M$, in view of the fact that the red-shift degenerates, and in fact, solutions of (2) exhibit a mild instability exactly on the horizon. See recent work of Aretakis [1]. (It is for this reason that we in particular exclude the extremal case from the stability conjecture.) In view of the comments of the previous paragraph, however, it should not be surprising that the extremal case is *not* excluded from the above theorem. In view of the fact that the red-shift now would appear as an obstacle, its degeneration is not such a bad thing, and thus the extremal case is if anything easier!

Details can be found in [7].

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