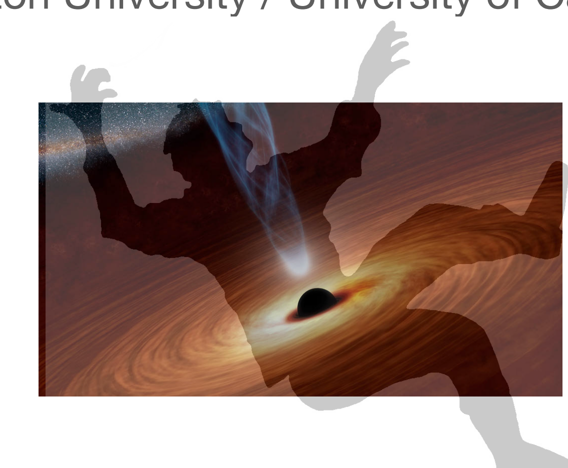
The geometry and analysis of black hole spacetimes

Mihalis Dafermos
Princeton University / University of Cambridge



Plan of the lectures

Lecture 1. *General Relativity and Lorentzian geometry*

Lecture 2. *The geometry of Schwarzschild black holes*

Lecture 3. *The analysis of waves on Schwarzschild exteriors*

Lecture 4. *The geometry of Kerr black holes and the strong cosmic censorship conjecture*

Lecture 5. *The analysis of waves on Kerr black hole interiors*

Lecture 6. *Nonlinear C^0 stability of the Kerr Cauchy horizon*

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Lecture 1

General Relativity and Lorentzian geometry

Spacetime

space $\mathbb{R}^3$+ time $\mathbb{R}$ **spacetime:** $\mathscr{M}^4$

Lorentzian manifold $(\mathscr{M}, g)$

Lorentzian metric $g = g_{\mu\nu} dx^\mu dx^\nu$

g is symmetric $(g_{\mu\nu} = g_{\nu\mu})$ with signature $(-,+,+,+)$

i.e. at each point p, can choose basis $e_0, \ldots, e_3 \in T_p\mathscr{M}$

s.t. $g(e_0, e_0) = -1,\ g(e_i, e_j) = \delta_{ij},\ g(e_0, e_i) = 0,\ i = 1,\ldots,3$

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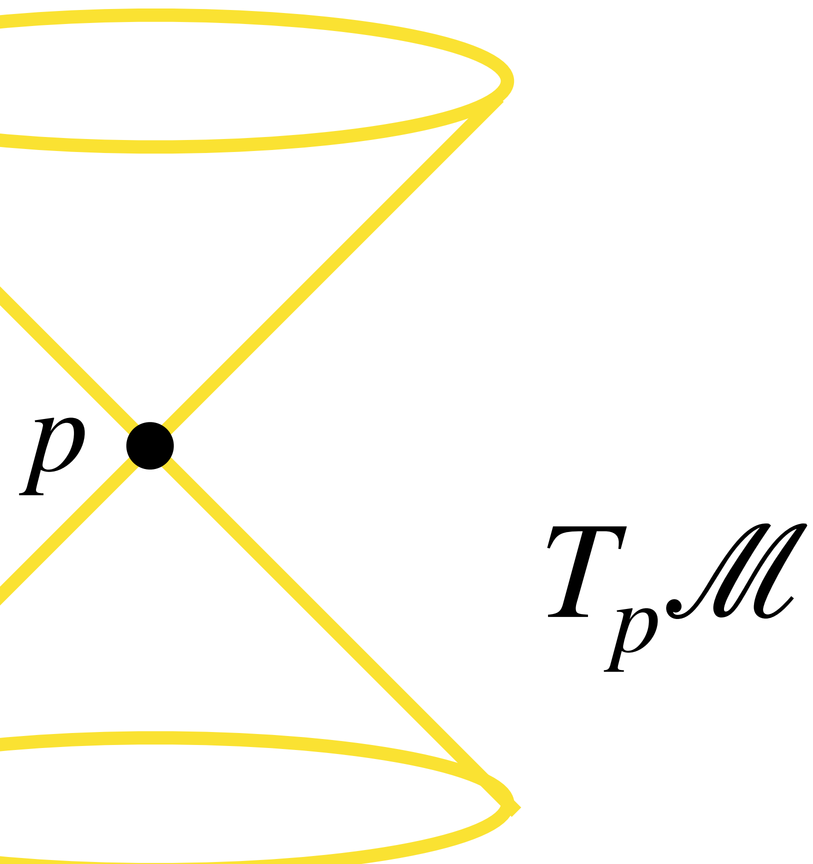
The null cone

$(\mathscr{M}, g)$



vectors $v \in T_p\mathscr{M}$ can be classified as

- timelike if $g(v, v) < 0$
- null if $g(v, v) = 0$
- spacelike if $g(v, v) > 0$

(timelike and null: "causal")

The null cone

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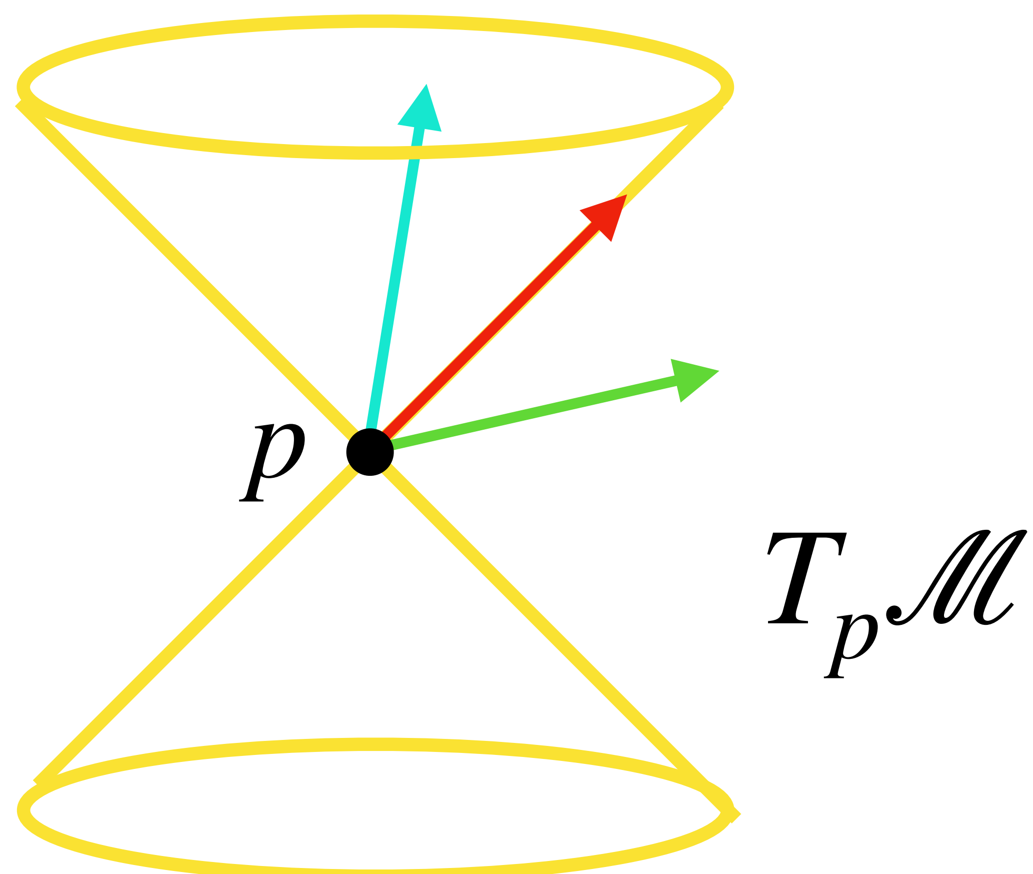

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"causal" (timelike or null)

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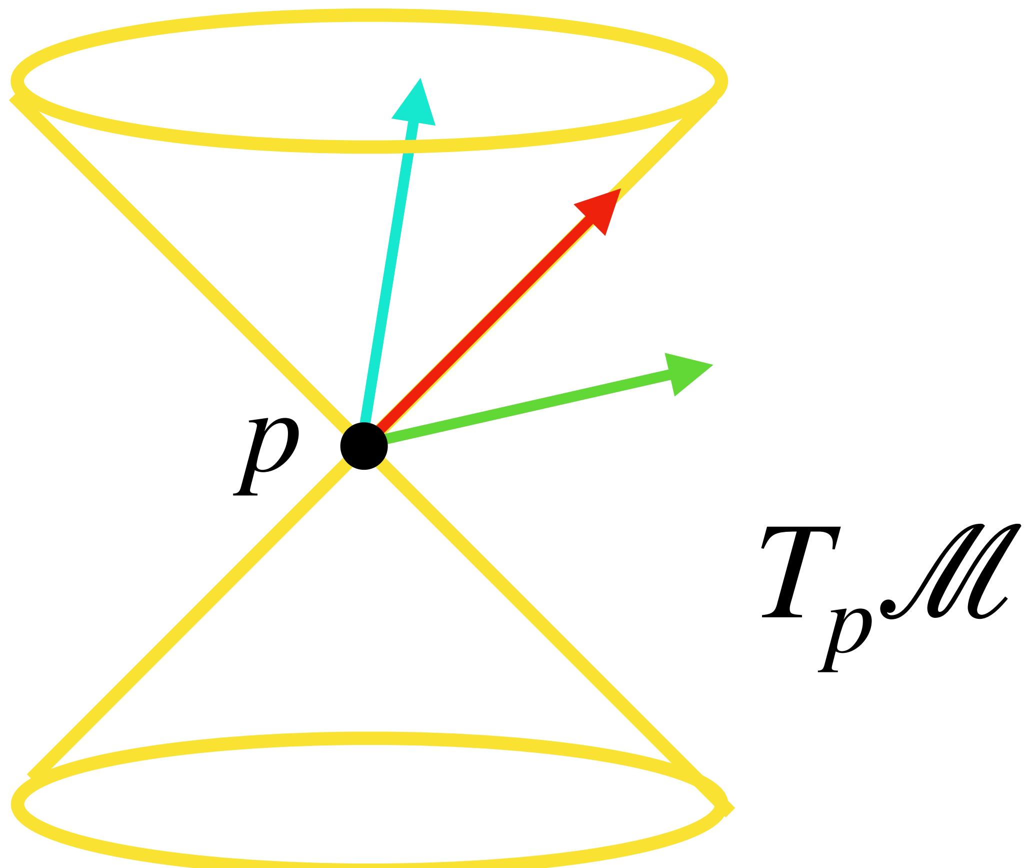

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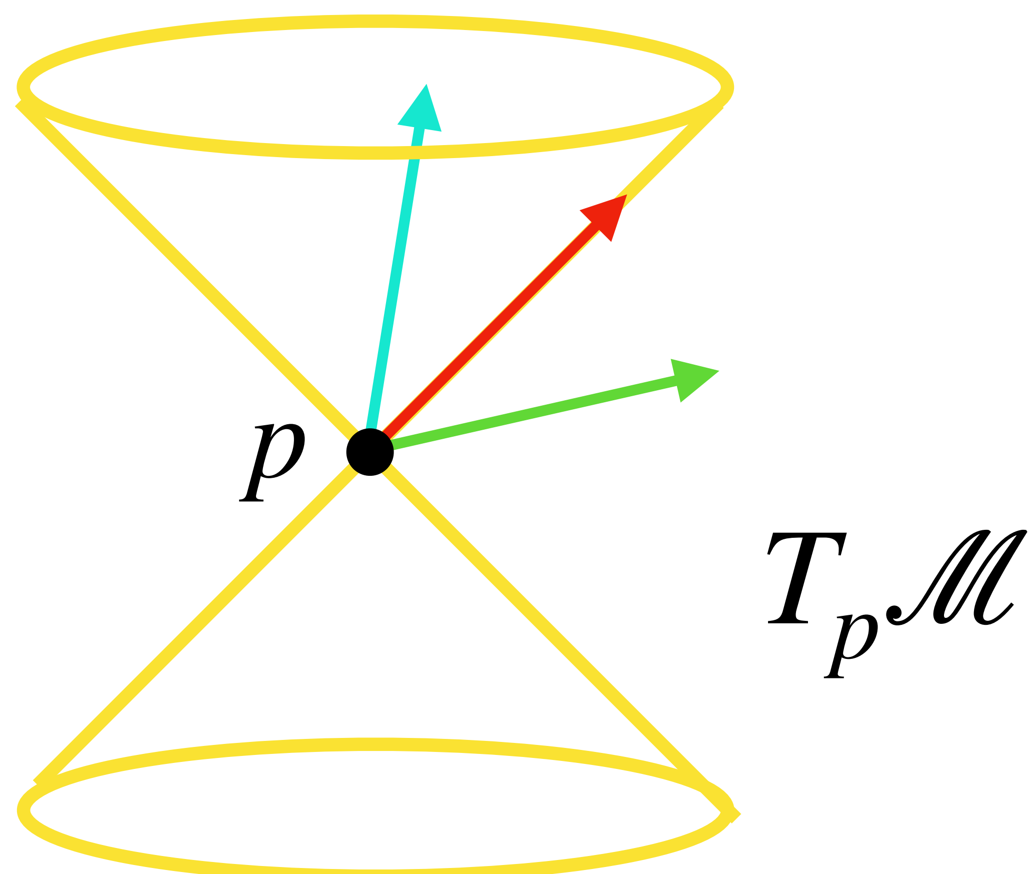

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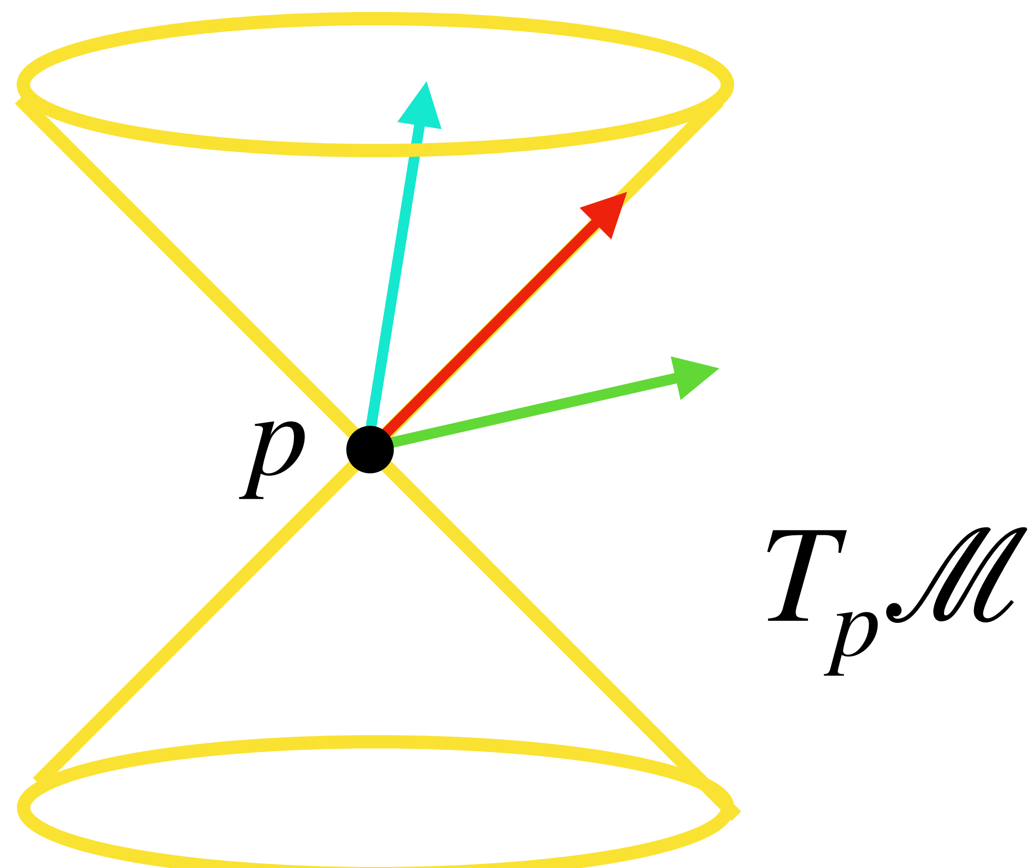

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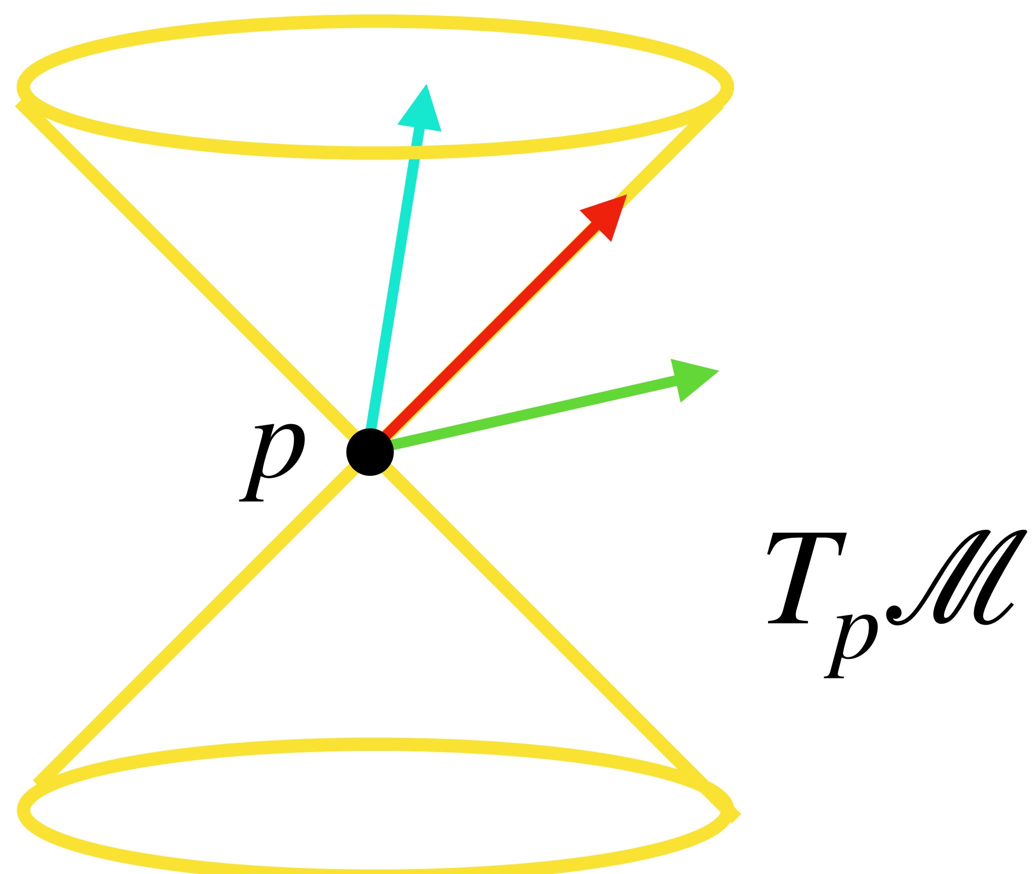

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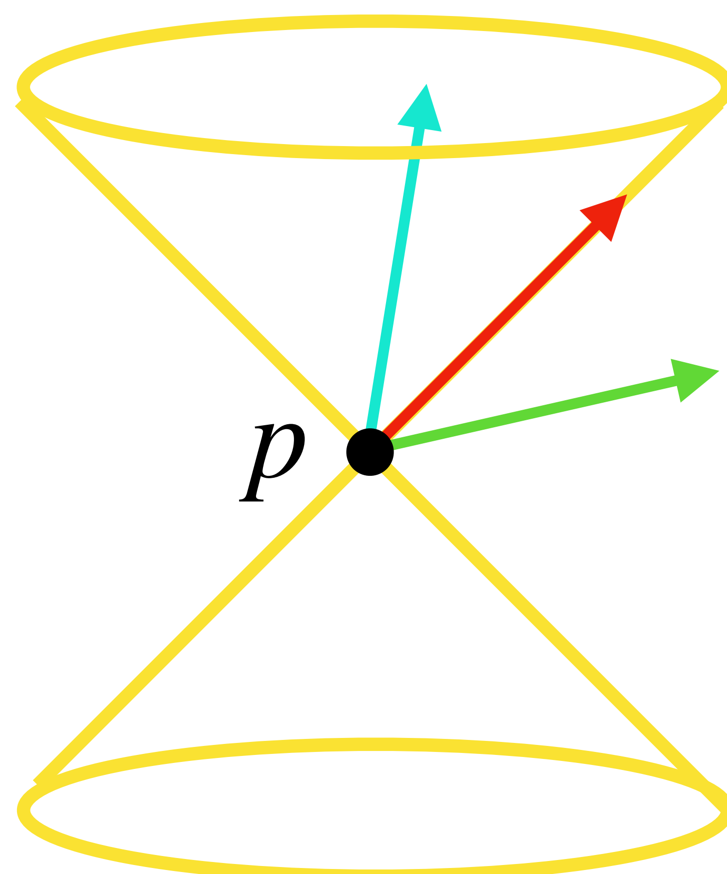

vectors $v \in T_p\mathscr{M}$ can be classified as

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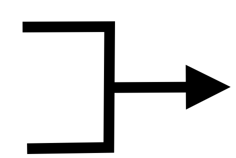
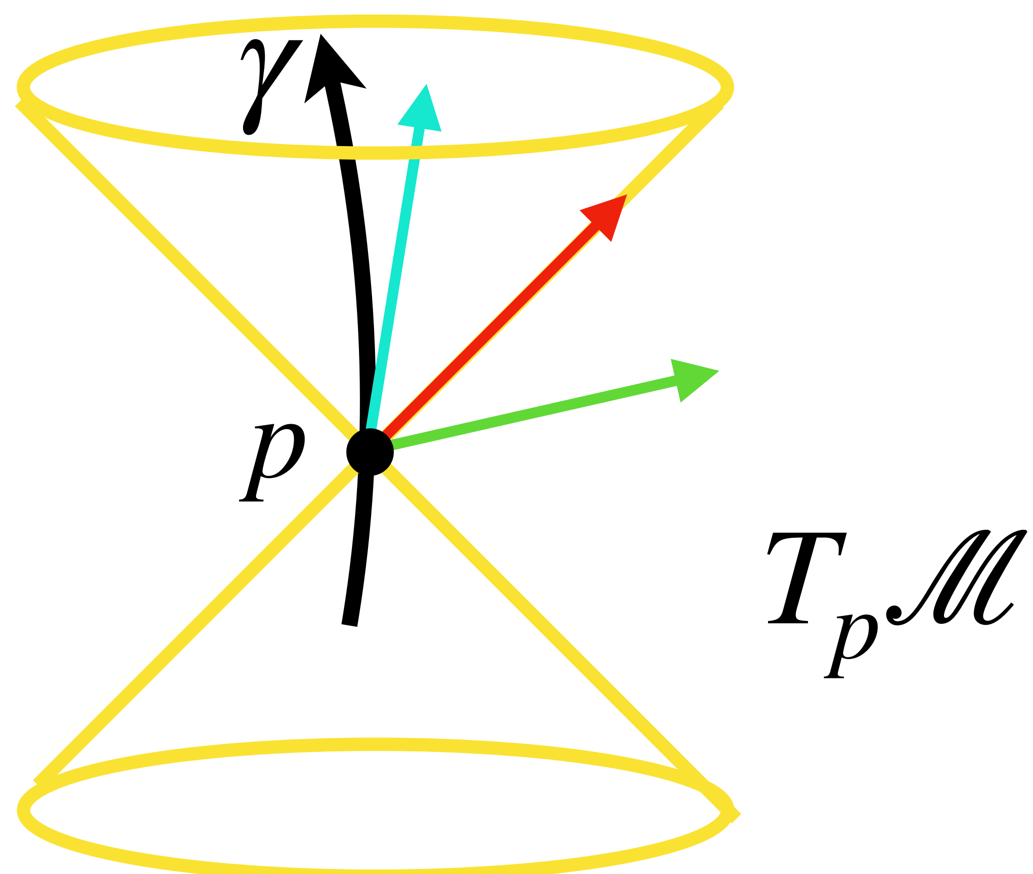
- null if $v \neq 0$ and $g(v, v) = 0$
- spacelike if $v = 0$ or $g(v, v) > 0$



Causal curves

$(\mathcal{M}, g)$



curves γ inherit these names from their tangent vector $\dot{\gamma}$

so γ is **timelike**, …, if $\dot{\gamma}$ is timelike, i.e. if $g(\dot{\gamma}, \dot{\gamma}) < 0$, …

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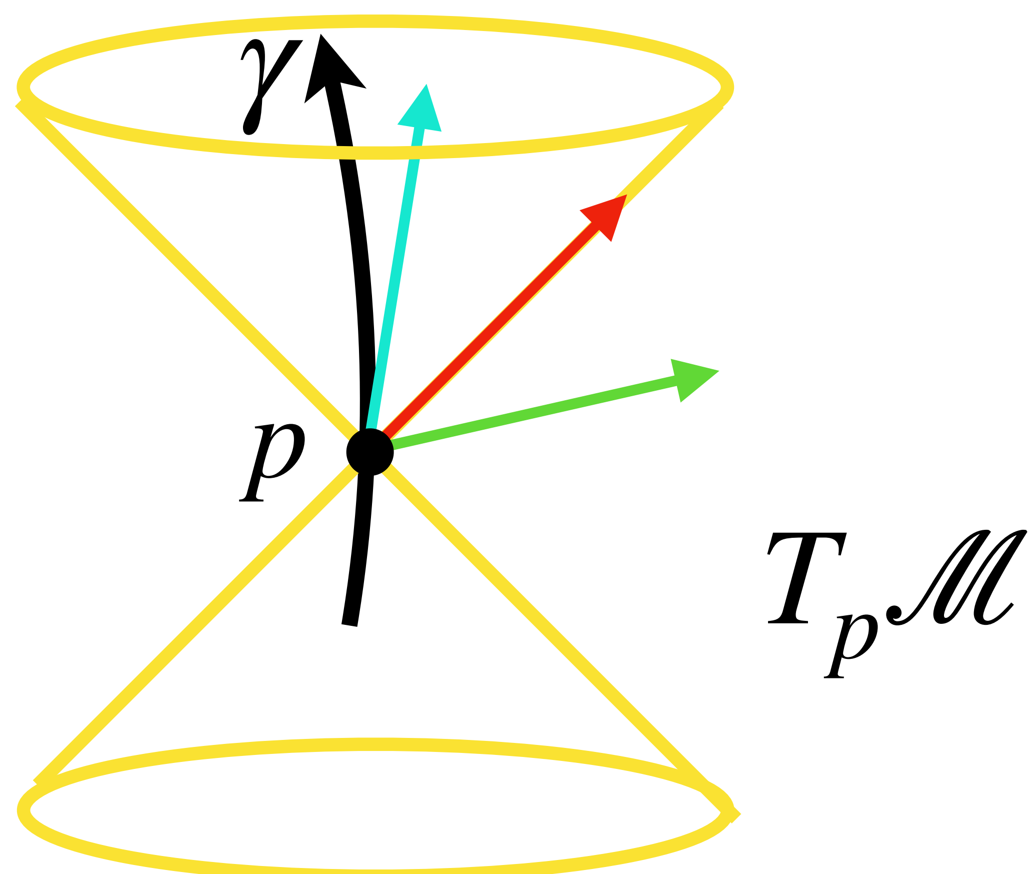

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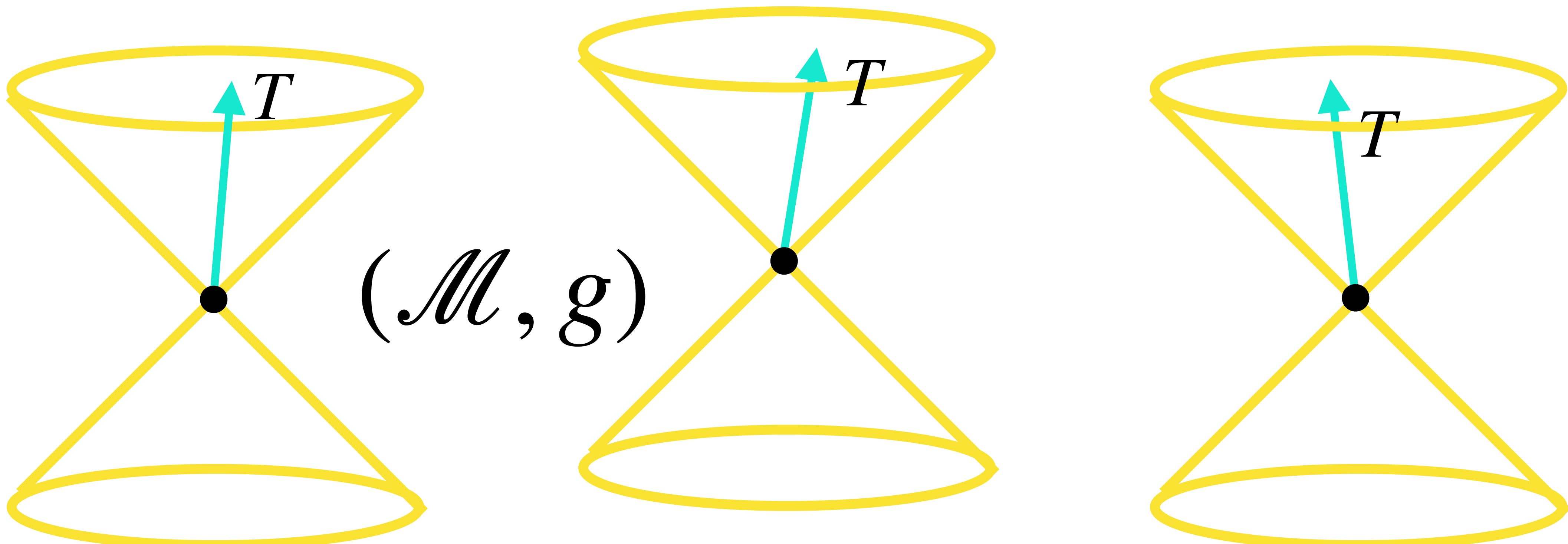
Time orientation



A **time orientation** is a global continuous timelike vectorfield T.
Given T, a causal vector $v \in T_p\mathscr{M}$ is said to be

- **future pointing** if $g(v, T(p)) < 0$
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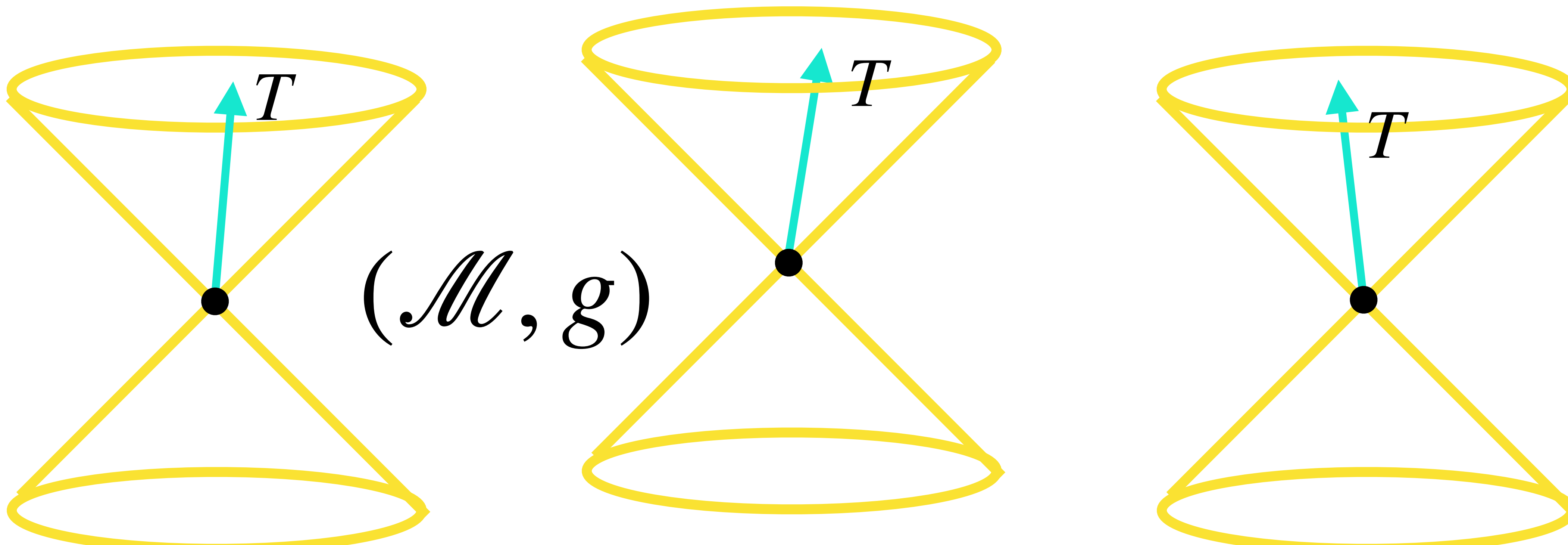

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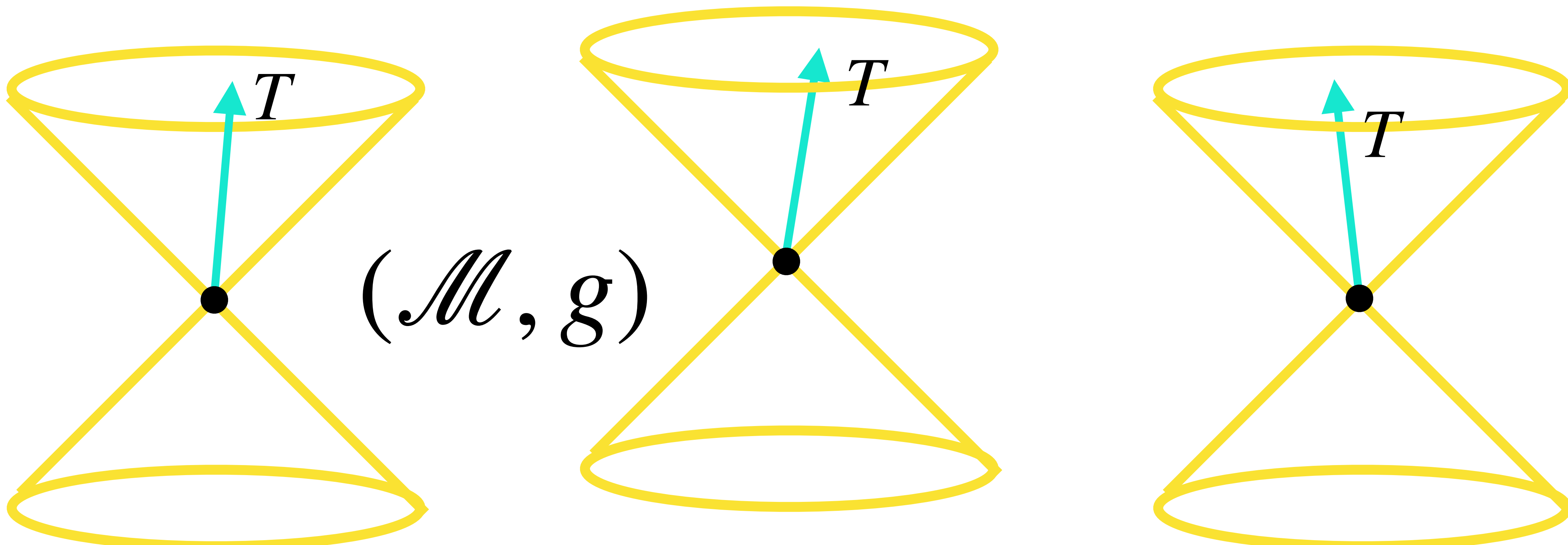

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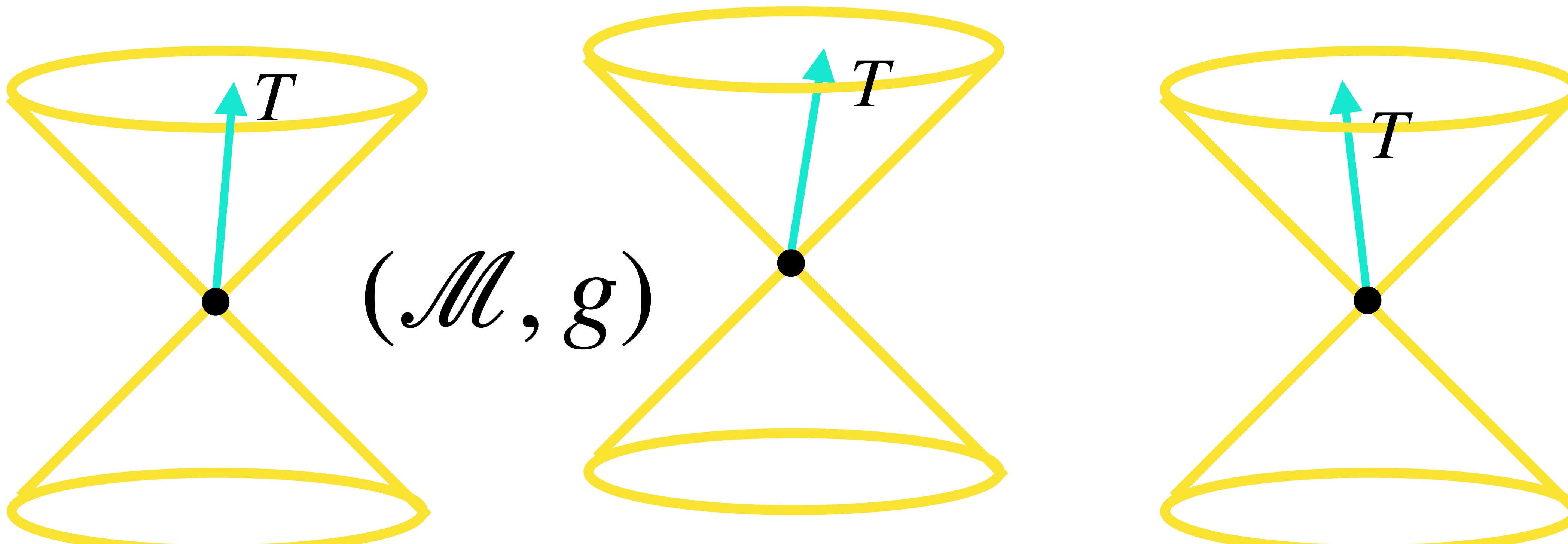

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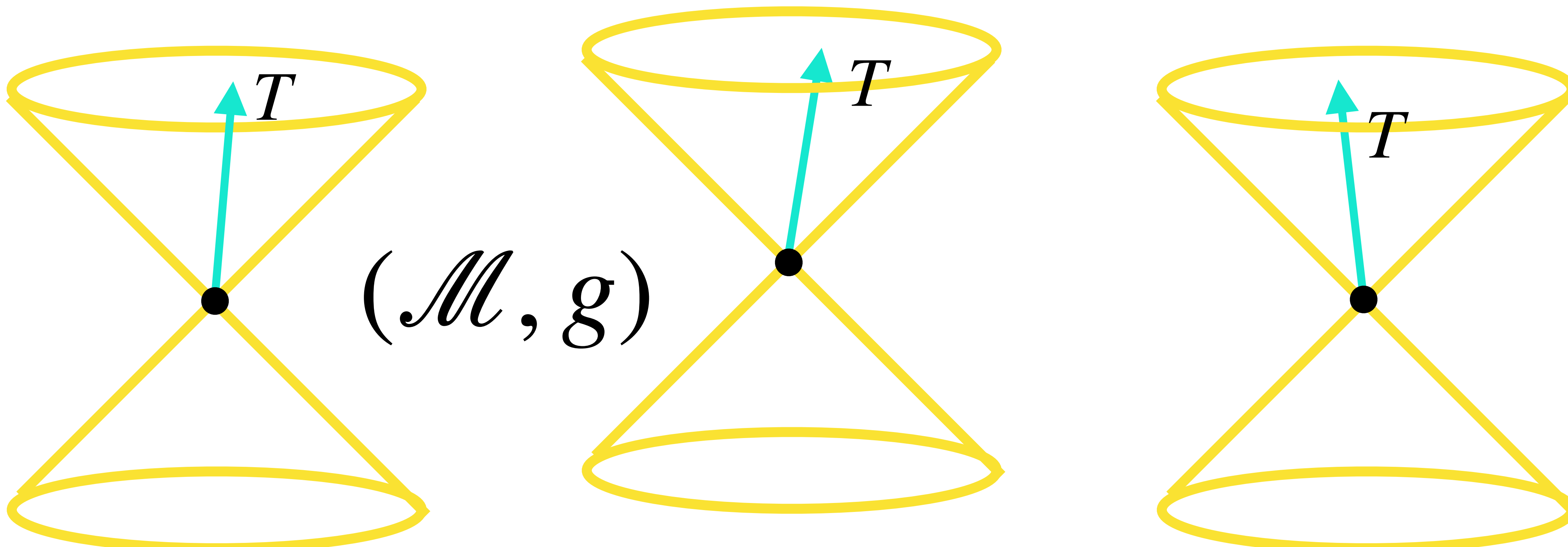

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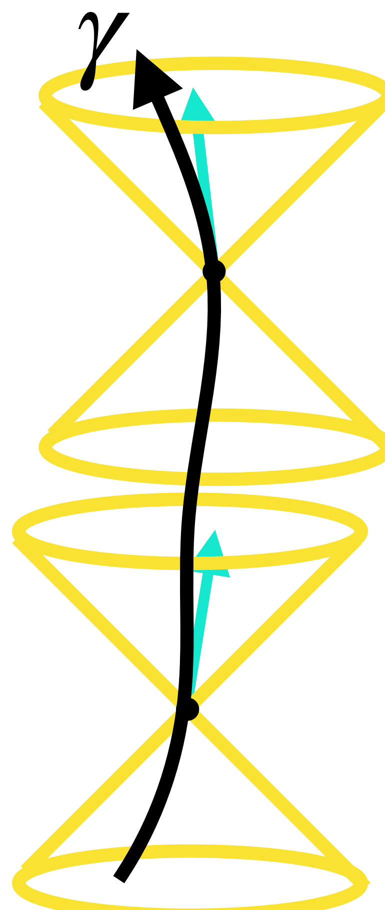
Kinematics



$(\mathscr{M}, g)$

A basic kinematic principle is that all physical particles traverse future-directed causal curves γ in spacetime $(\mathscr{M}, g)$.

Freely falling massive (resp. massless) particles traverse future-directed timelike (resp. null) **geodesics:** $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$

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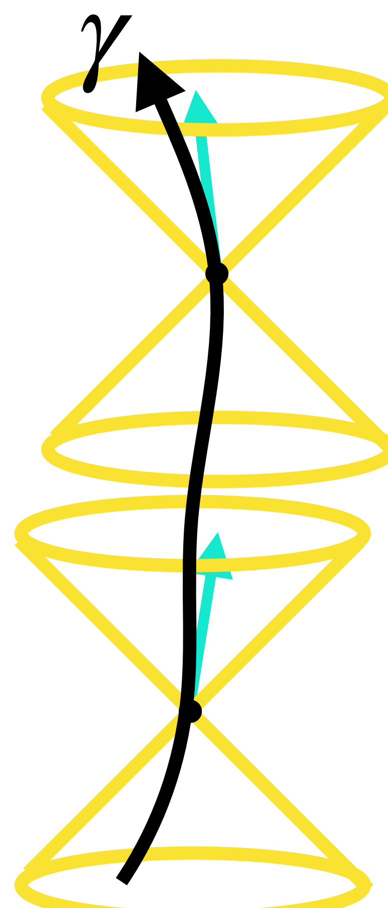

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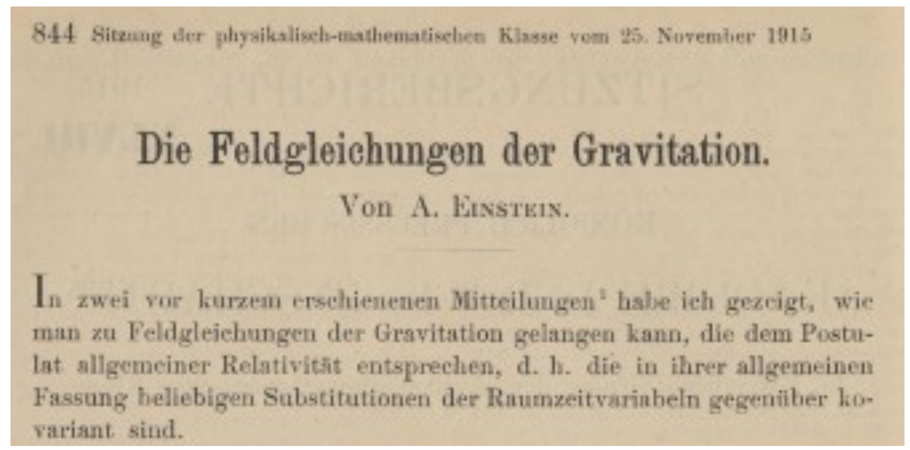
The Einstein equations


844 Sitzung der physikalisch-mathematischen Klasse vom 25. November 1915

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

$$\mathrm{Ric}_{\mu\nu}[g] - \frac{1}{2} g_{\mu\nu} R[g] = 8\pi T_{\mu\nu}$$

- *cf.* the Newtonian theory governed by $\Delta\phi = 4\pi\mu$
- $\mathrm{Ric}_{\mu\nu}$ is the **Ricci curvature** $\mathrm{Ric}_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\alpha\mu} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}$
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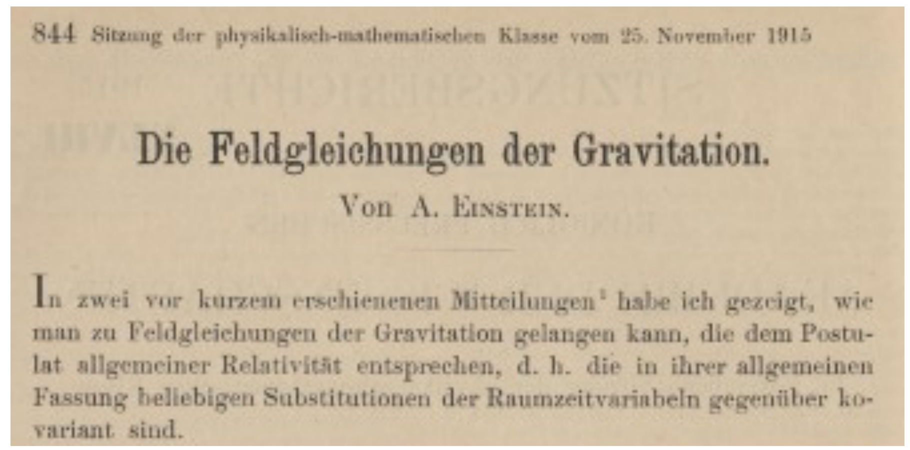

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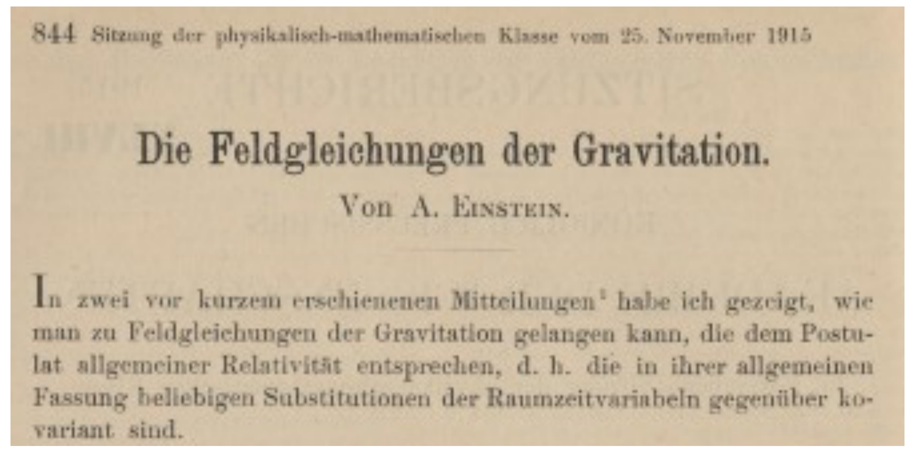
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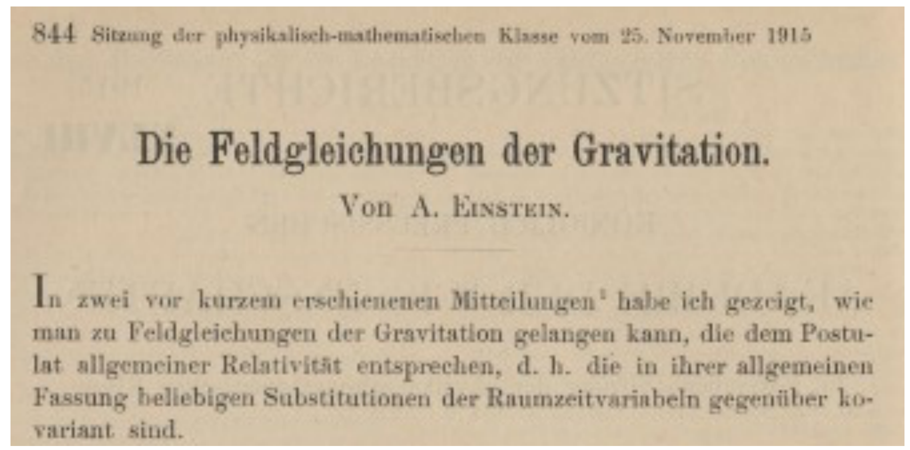
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The vacuum equations

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- The vacuum Einstein equations constitute a nonlinear system of hyperbolic equations with a *well posed* initial value problem (Choquet-Bruhat 1952)
- This is in stark constrast to the Newtonian theory!

$$\Delta\phi = 0, \phi(r) \to 0 \text{ as } \; r \to \infty \implies \phi = 0$$

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Minkowski spacetime

Raum und Zeit[1]).

Von Hermann Minkowski in Göttingen.

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Harmonic coordinates

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linearising around Minkowski space we obtain that the components $\dot{g}_{\mu\nu}$ of the linearised metric satisfy

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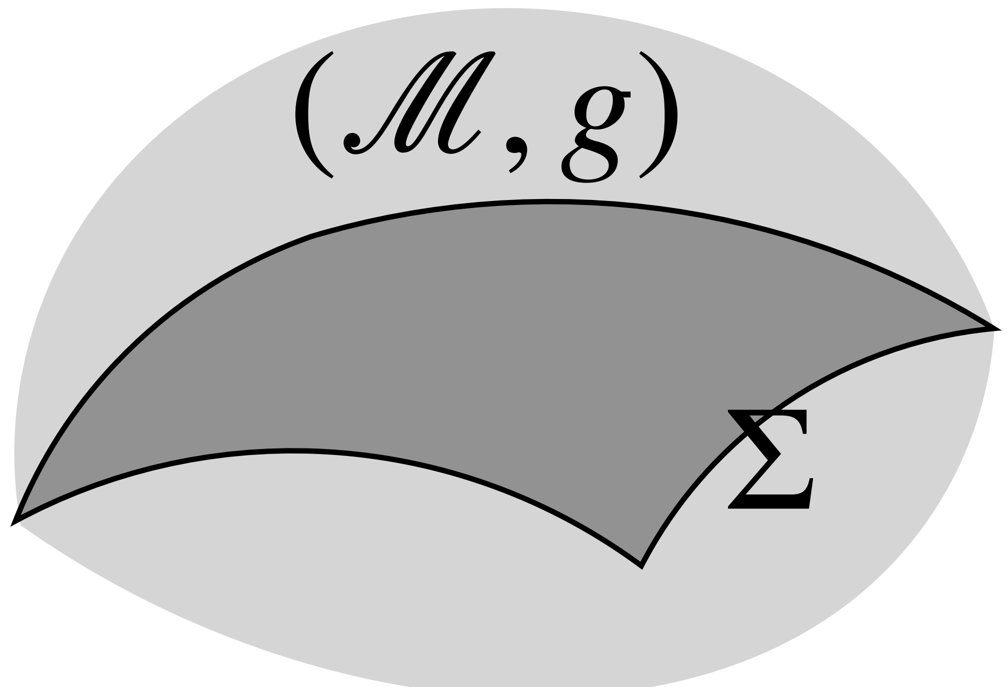
Cauchy hypersurfaces



- A hypersurface Σ is **spacelike** if its normal n is timelike, equivalently induced metric $\bar{g}$ is Riemannian.
- Σ is a **Cauchy hypersurface** if all *inextendible* causal curves γ intersect Σ precisely once.
- $(\mathscr{M}, g)$ is **globally hyperbolic** if it admits a Cauchy Σ.
- Example: $\Sigma = \{t = 0\}$ in Minkowski space $\mathbb{R}^{3+1}$

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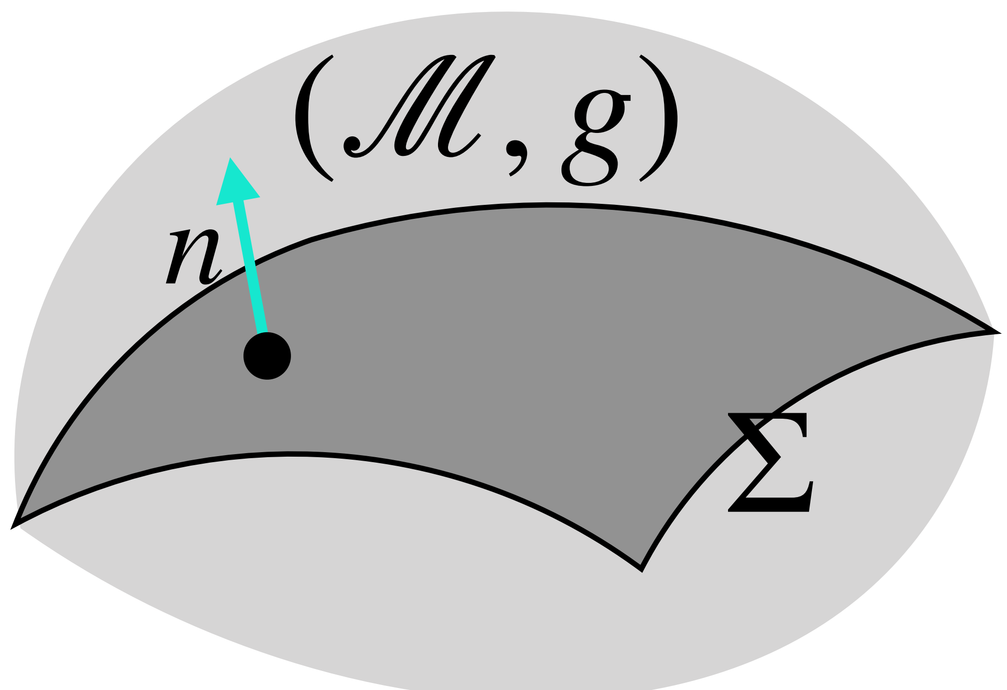

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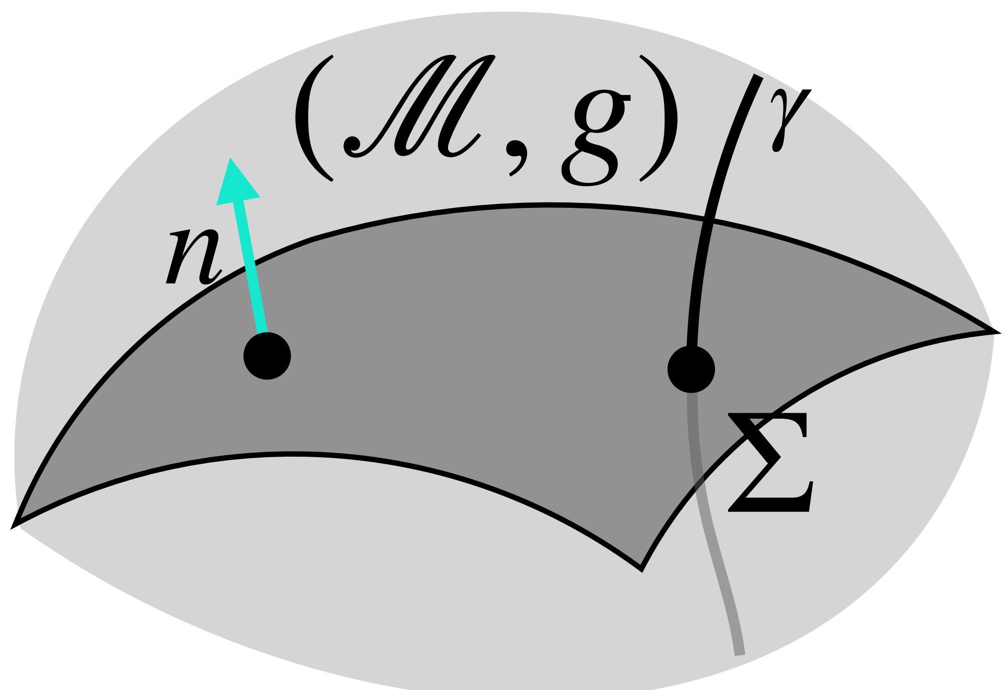

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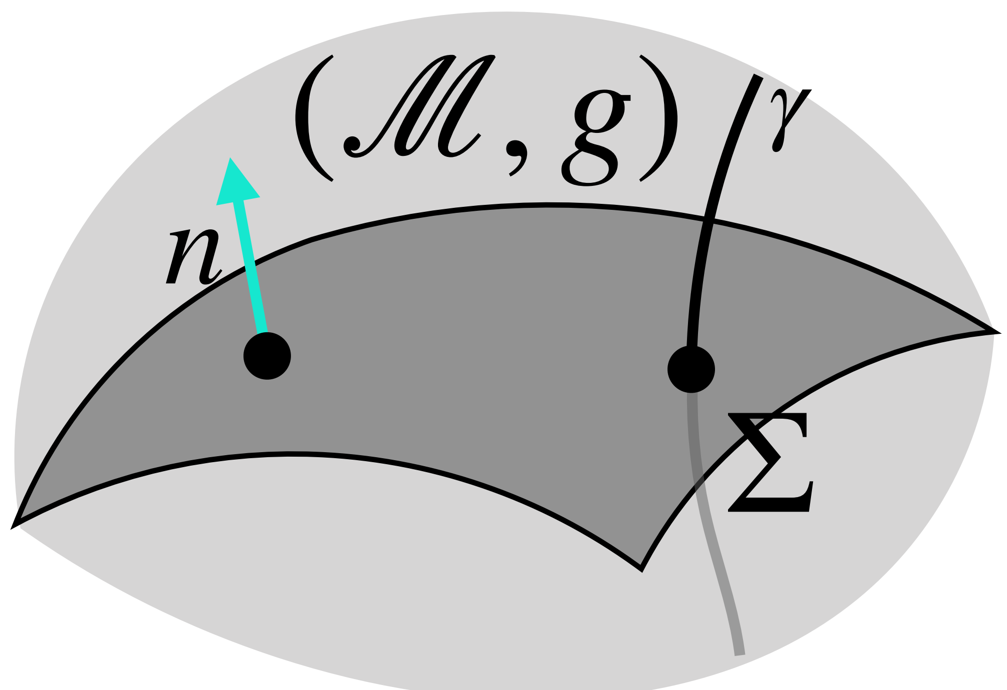

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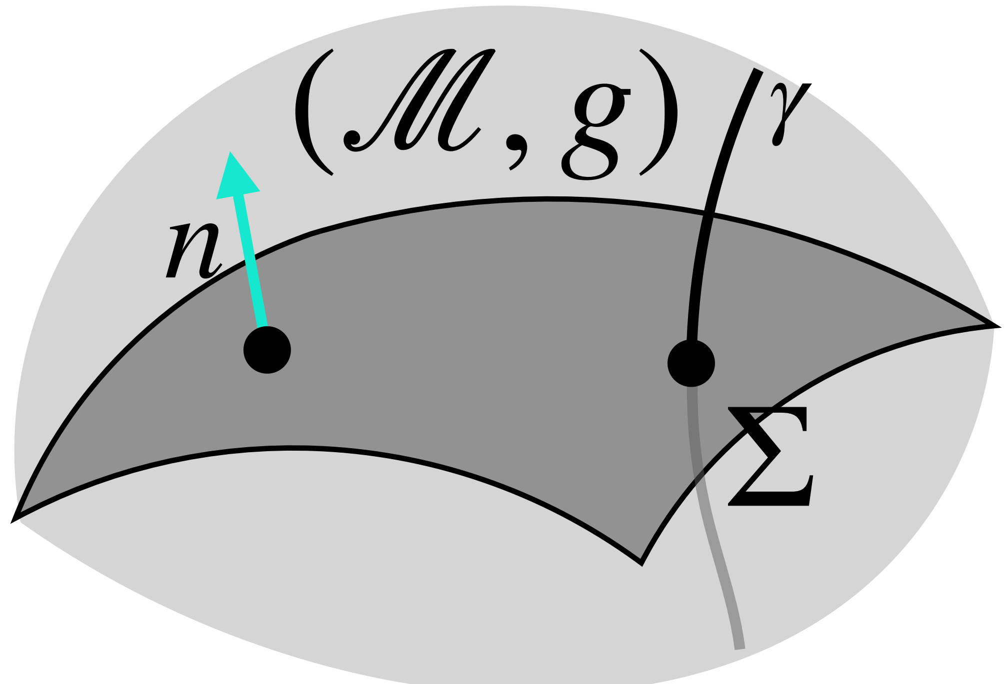

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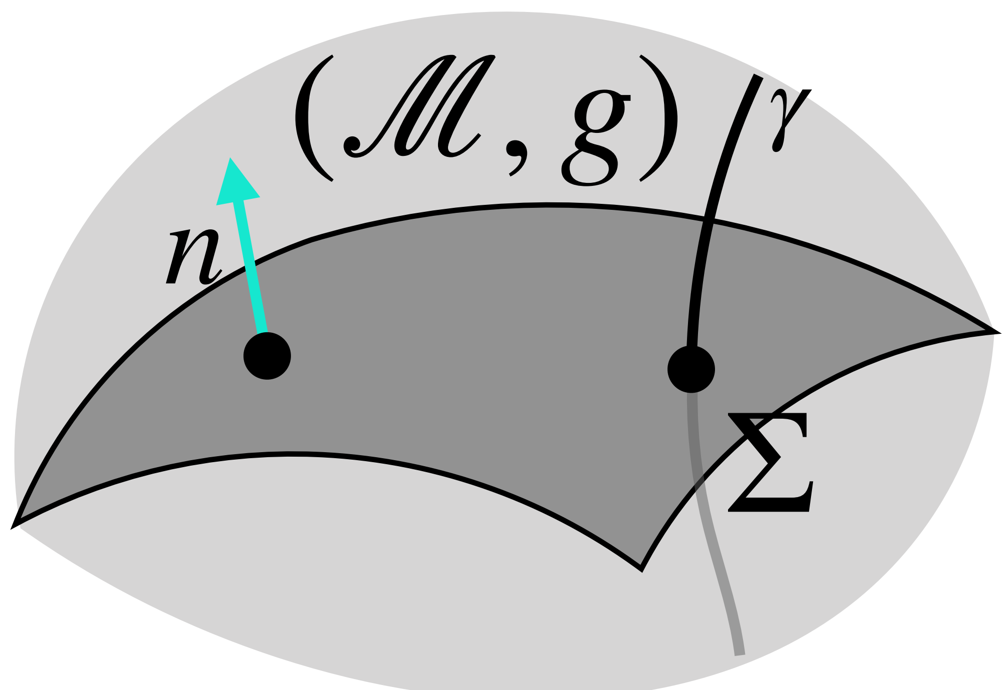
Aside: the wave equation



- Let $(\mathscr{M}, g)$ be globally hyperbolic with Cauchy hypersurface Σ
- Given smooth initial data ϕ_0, ϕ_1 on Σ, there exists a unique smooth **global** solution ϕ of $\Box_g \phi = 0$ s.t.

$$\phi\,|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi\,|_\Sigma = \phi_1$$

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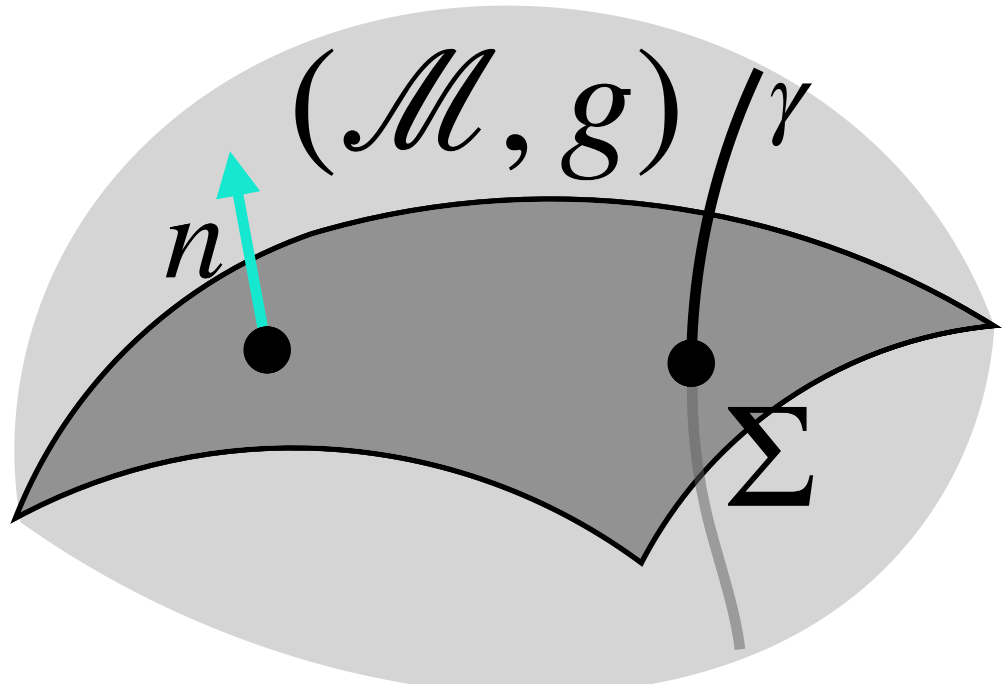

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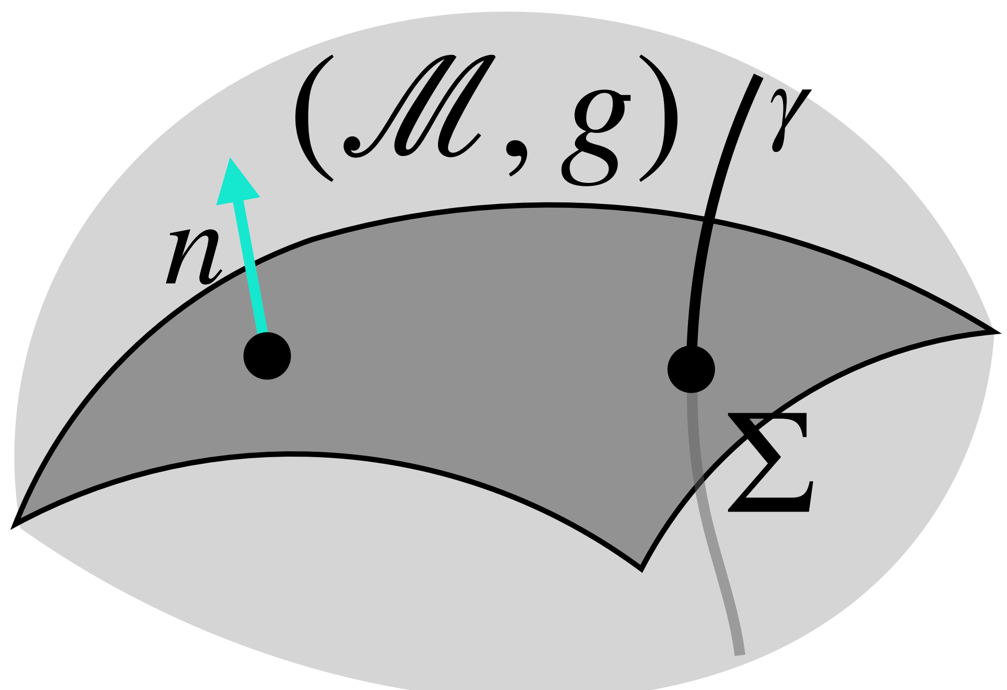

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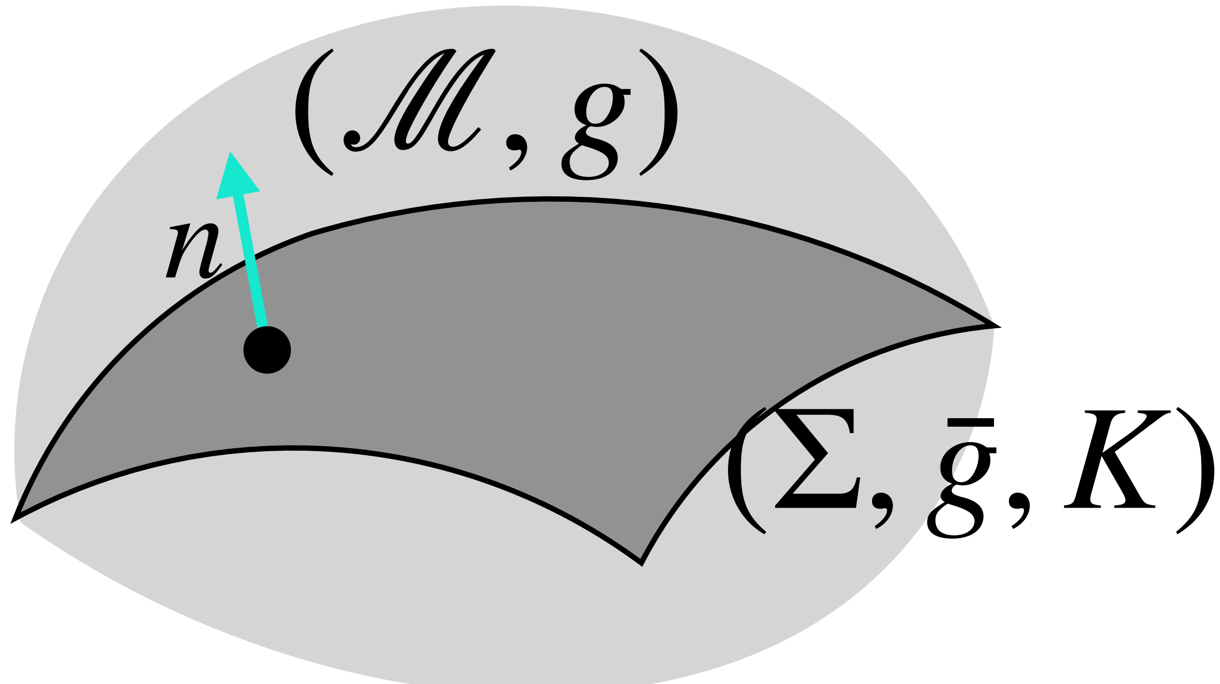

Let Σ be a spacelike hypersurf. in $(\mathscr{M}, g)$ satisfying $\mathrm{Ric}(g) = 0$.
Then the *Gauss* and *Codazzi* equations of classical differential geometry imply the following **Einstein constraint equations**

$$R(\bar{g}) + (K^a{}_a)^2 - K^a{}_b K^b{}_a = 0, \qquad \bar{\nabla}_b K^b{}_a - \bar{\nabla}_a K^b{}_b = 0$$

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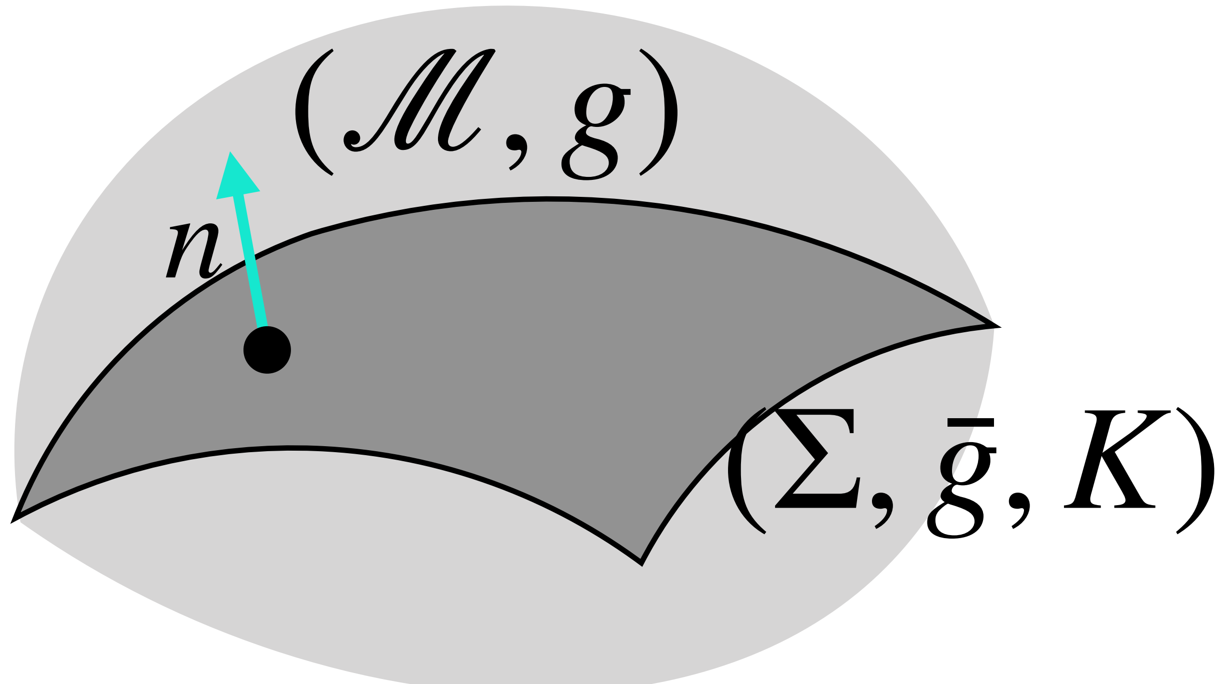

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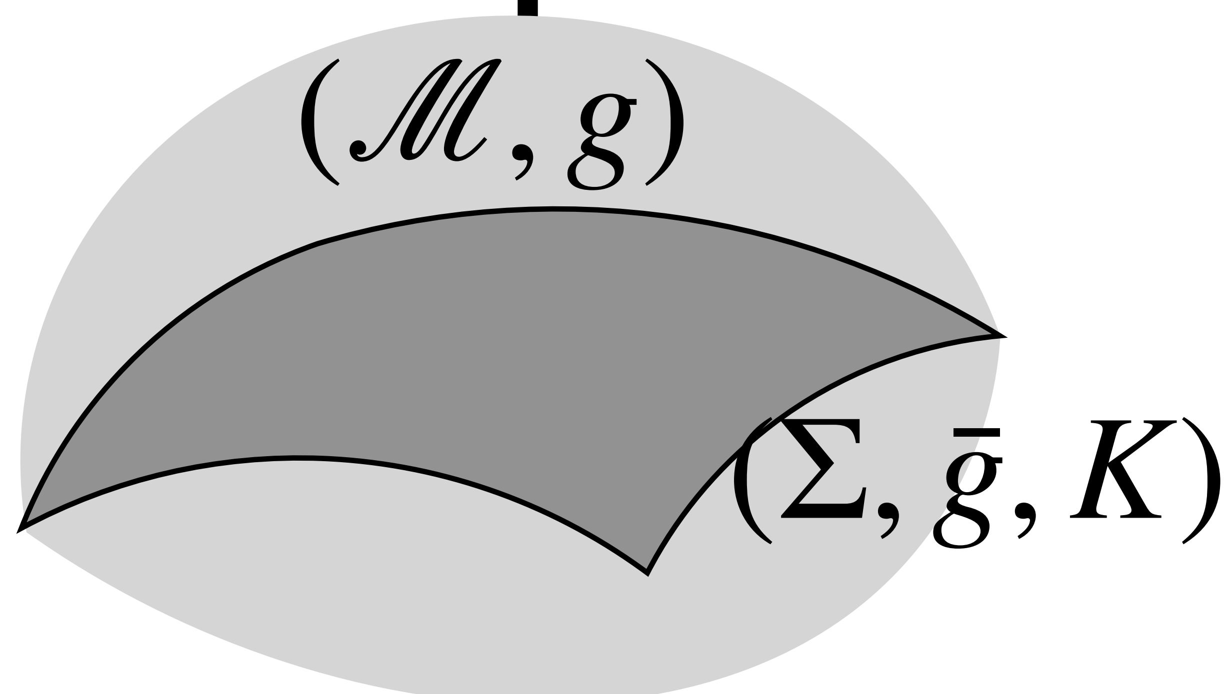
The maximal Cauchy development



Theorem. (Choquet-Bruhat–Geroch 1969) Let $(\Sigma, \bar{g}, K)$ satisfy smoothly the constraints. Then there exists spacetime $(\mathscr{M}^4, g)$ s.t.

- $(\mathscr{M}, g)$ is globally hyperbolic, $\Sigma \to \mathscr{M}$ embeds iso. as Cauchy hypersurface with induced metric $\bar{g}$ and 2nd f.f. K
- $\mathrm{Ric}(g) = 0$
- The spacetime $(\mathscr{M}, g)$ is **maximal.** Any $(\widetilde{\mathscr{M}}^4, \widetilde{g})$ satisfying the above embeds iso. $(\widetilde{\mathscr{M}}, \widetilde{g}) \to (\mathscr{M}, g)$

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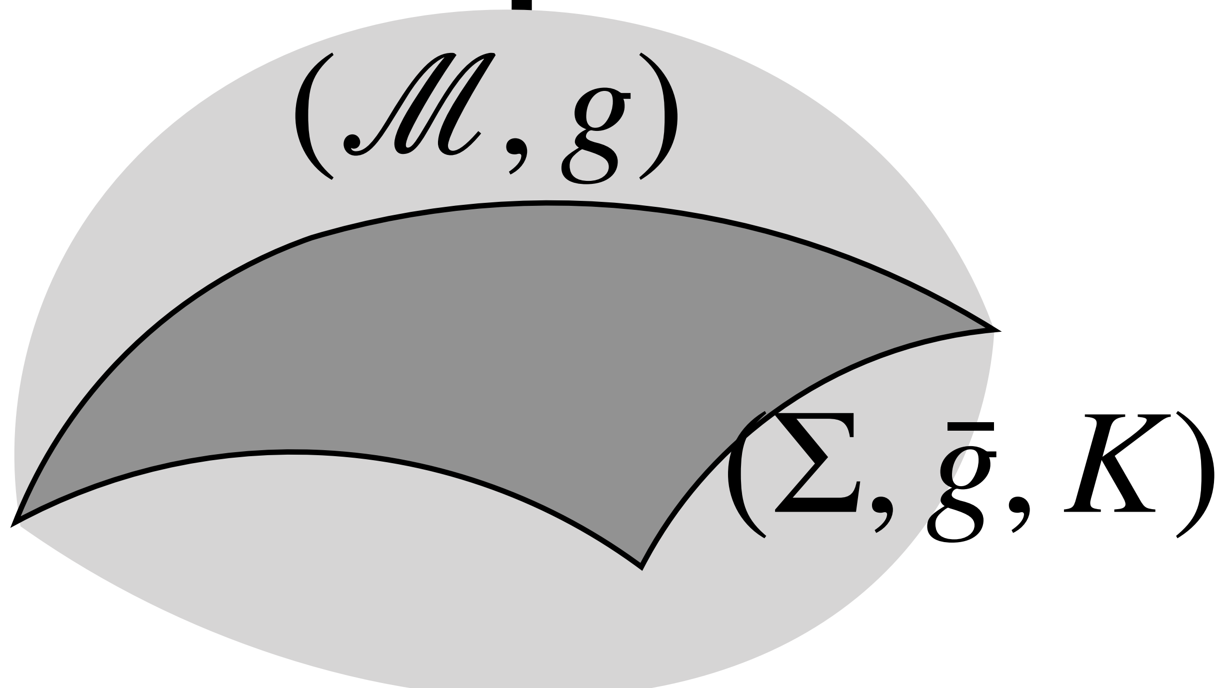

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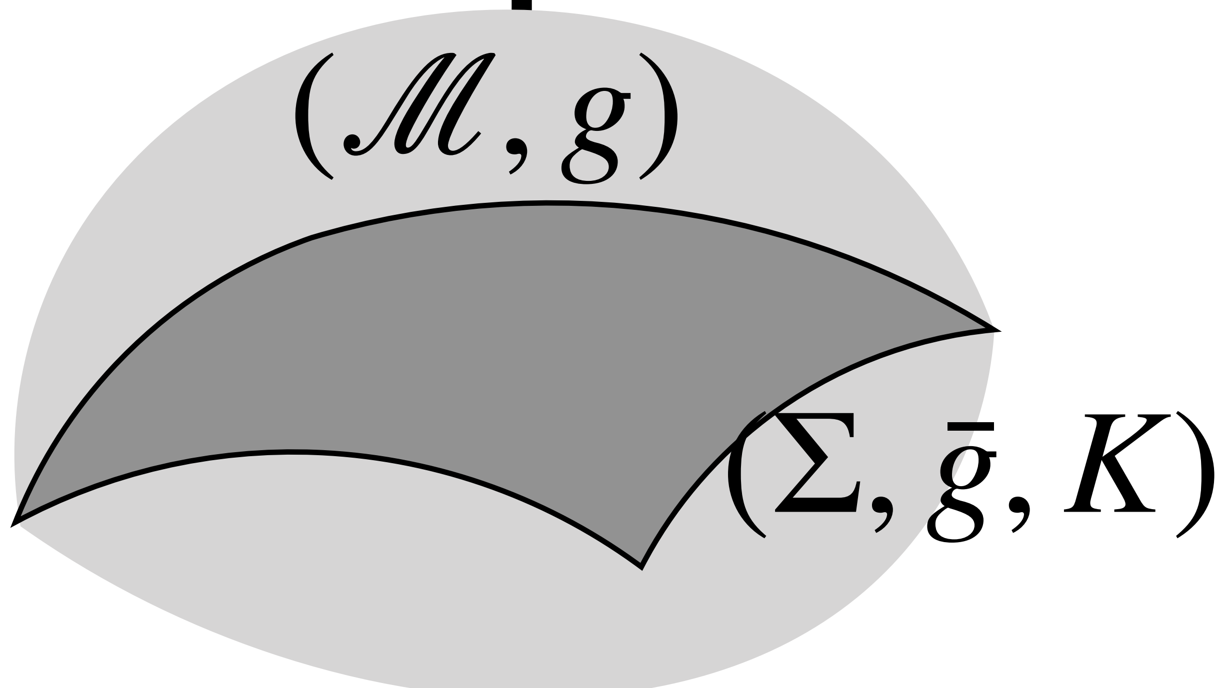

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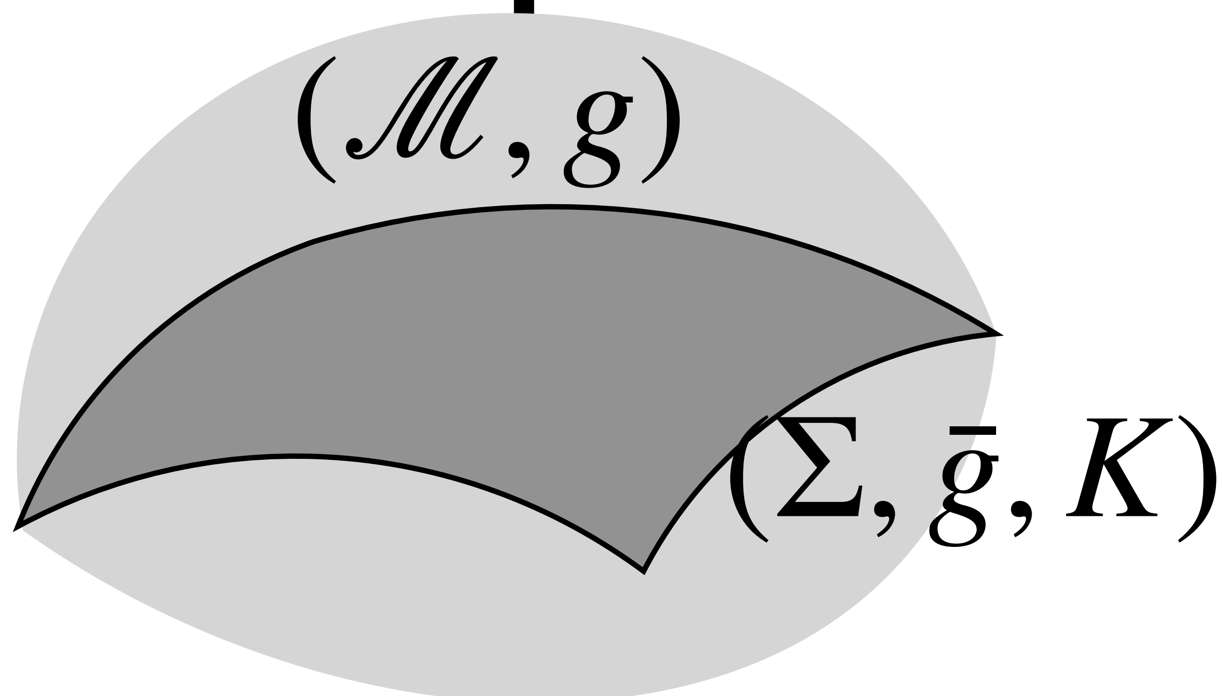

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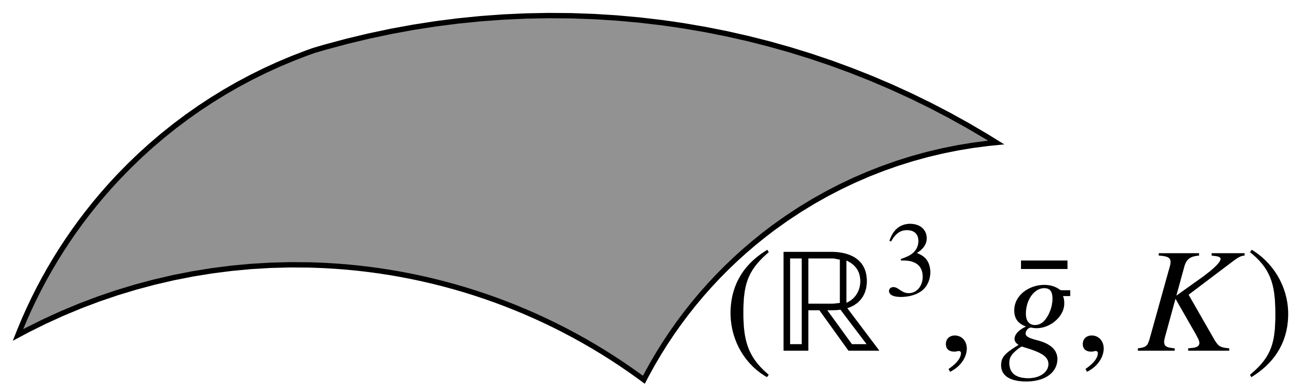
Nonlinear stability of Minkowski space



Theorem. (Christodoulou—Klainerman 1993) Let $(\mathbb{R}^3, \bar{g}, K)$ satisfy the constraints and be suitably close to trivial initial data $(\mathbb{R}^3, \bar{g}_{Eucl}, 0)$. Then the max. Cauchy development $(\mathscr{M}, g)$

- is geodesically complete
- remains globally close to $(\mathbb{R}^{3+1}, g_{\mathrm{Mink}})$, and
- approaches g_{Mink} along all causal geodesics

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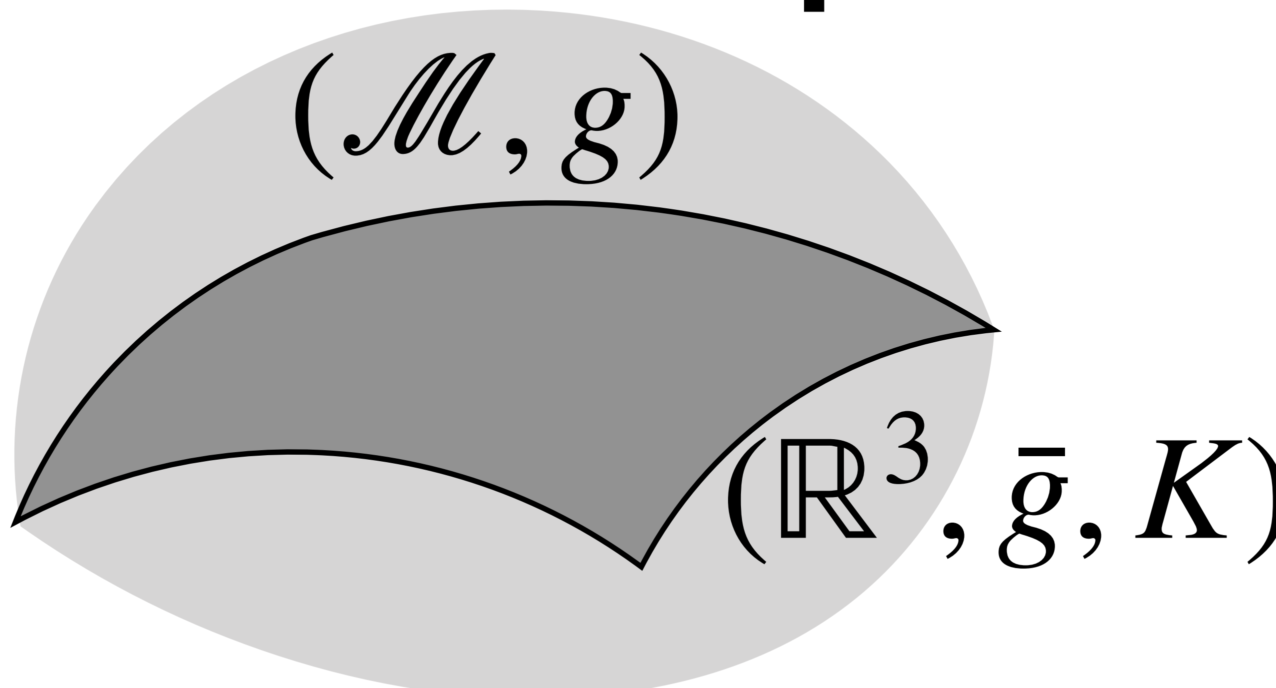

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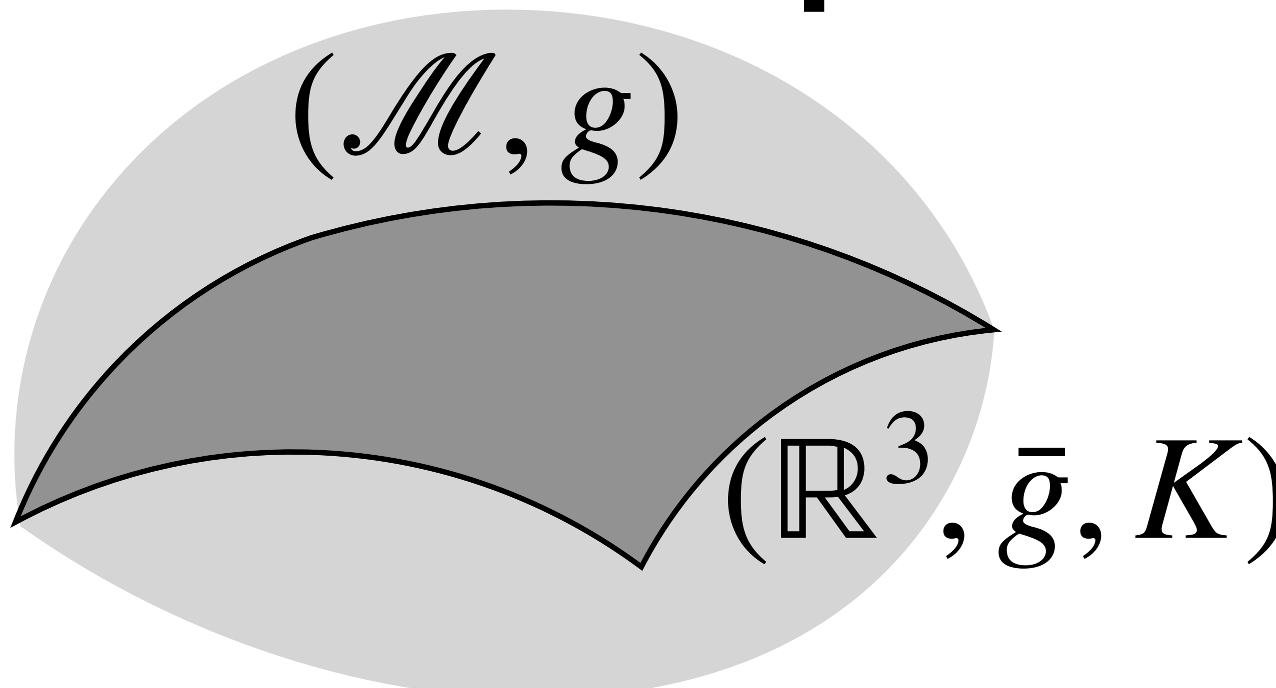

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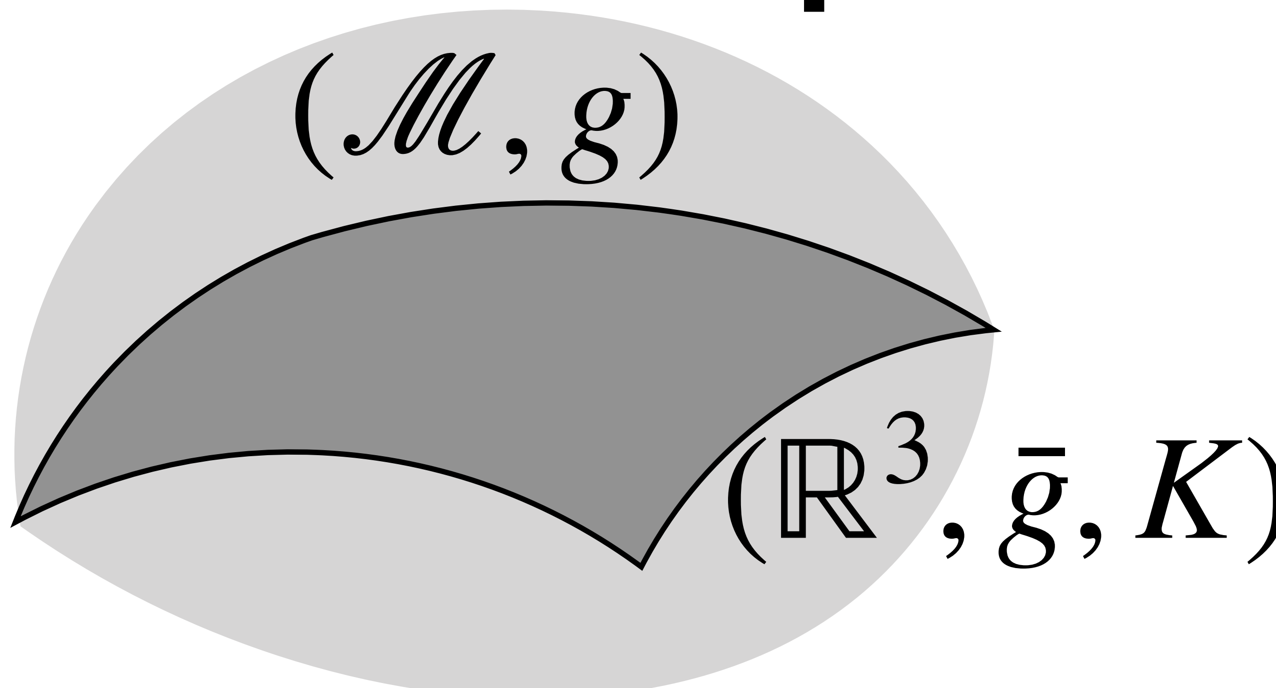

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The Global Nonlinear Stability of the Minkowski Space

Demetrios Christodoulou
and Sergiu Klainerman

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY
1993

Why so hard?

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References for Lecture 1

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Plan of the lectures

Lecture 1. *General Relativity and Lorentzian geometry*

Lecture 2. *The geometry of Schwarzschild black holes*

Lecture 3. *The analysis of waves on Schwarzschild exteriors*

Lecture 4. *The geometry of Kerr black holes and the strong cosmic censorship conjecture*

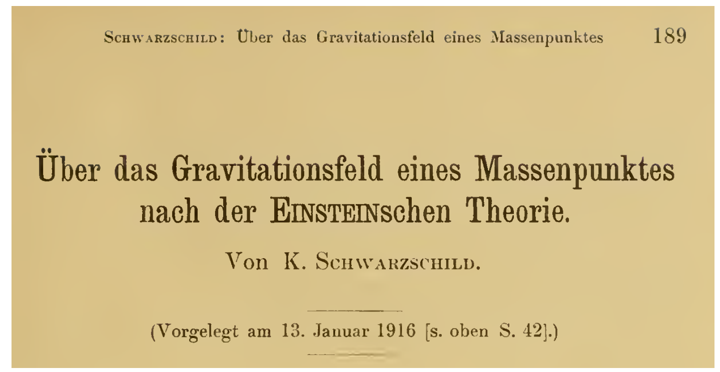
Lecture 5. *The analysis of waves on Kerr black hole interiors*

Lecture 6. *Nonlinear C^0 stability of the Kerr Cauchy horizon*

Lecture 2

The geometry of Schwarzschild black holes

Schwarzschild spacetime


Schwarzschild: Über das Gravitationsfeld eines Massenpunktes 189

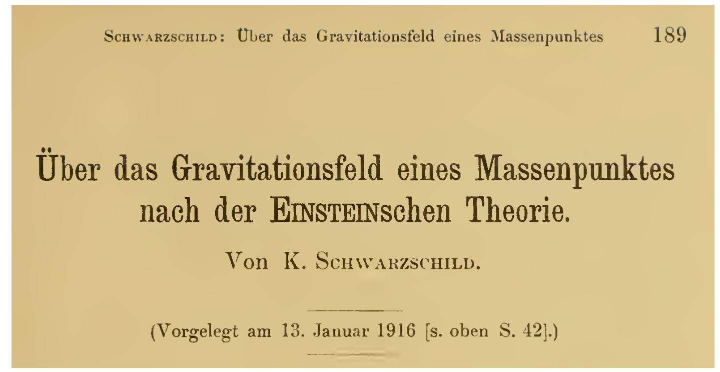
Über das Gravitationsfeld eines Massenpunktes
nach der Einsteinschen Theorie.

Von K. Schwarzschild.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

- discovered in 1915 by Karl Schwarzschild
- $g = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- spherically symmetric solution of vacuum equations $\mathrm{Ric}(g) = 0$
- singular? $r = 2M?$ $r = 0?$ $\theta = 0, \pi?$
- Lemaitre 1932 $t^* = t + 2M\log(r - 2M) \implies$

Schwarzschild spacetime



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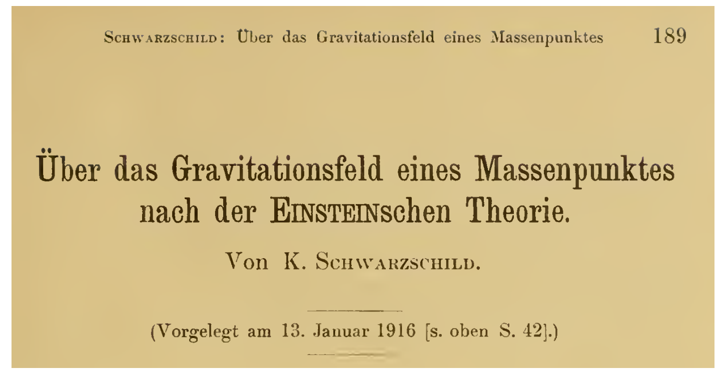
Schwarzschild spacetime


SCHWARZSCHILD: Über das Gravitationsfeld eines Massenpunktes 189

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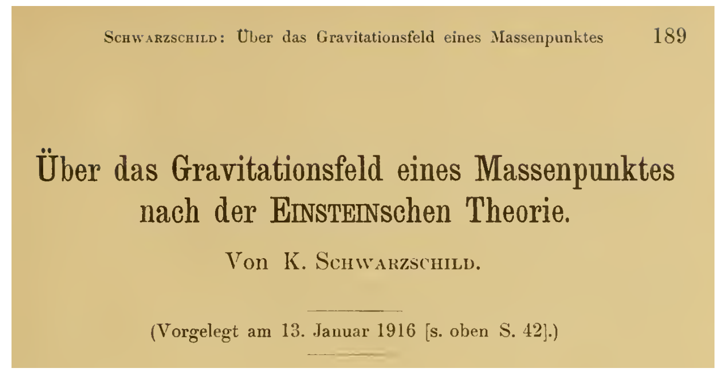
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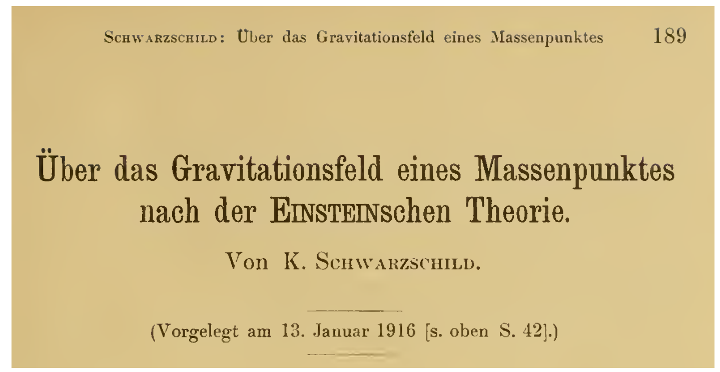
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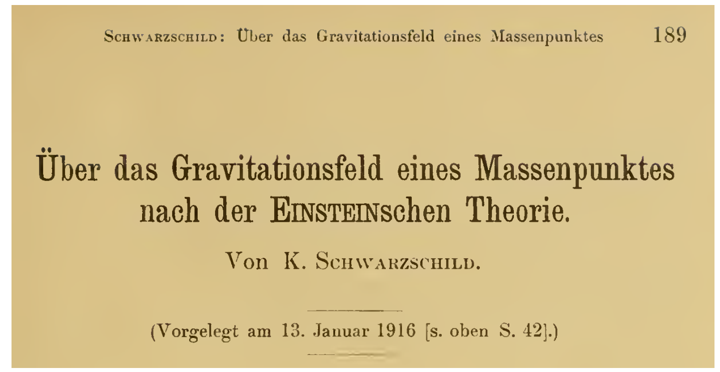

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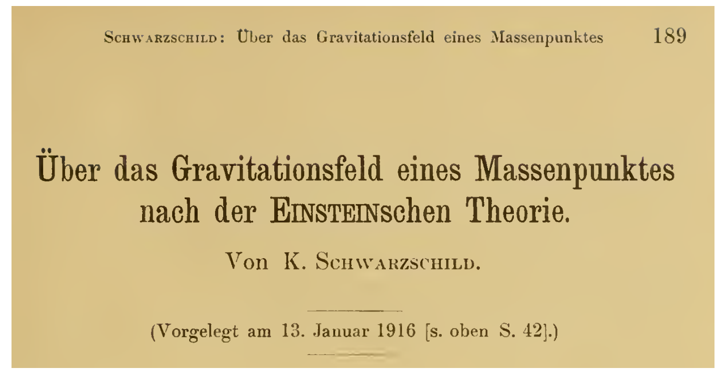
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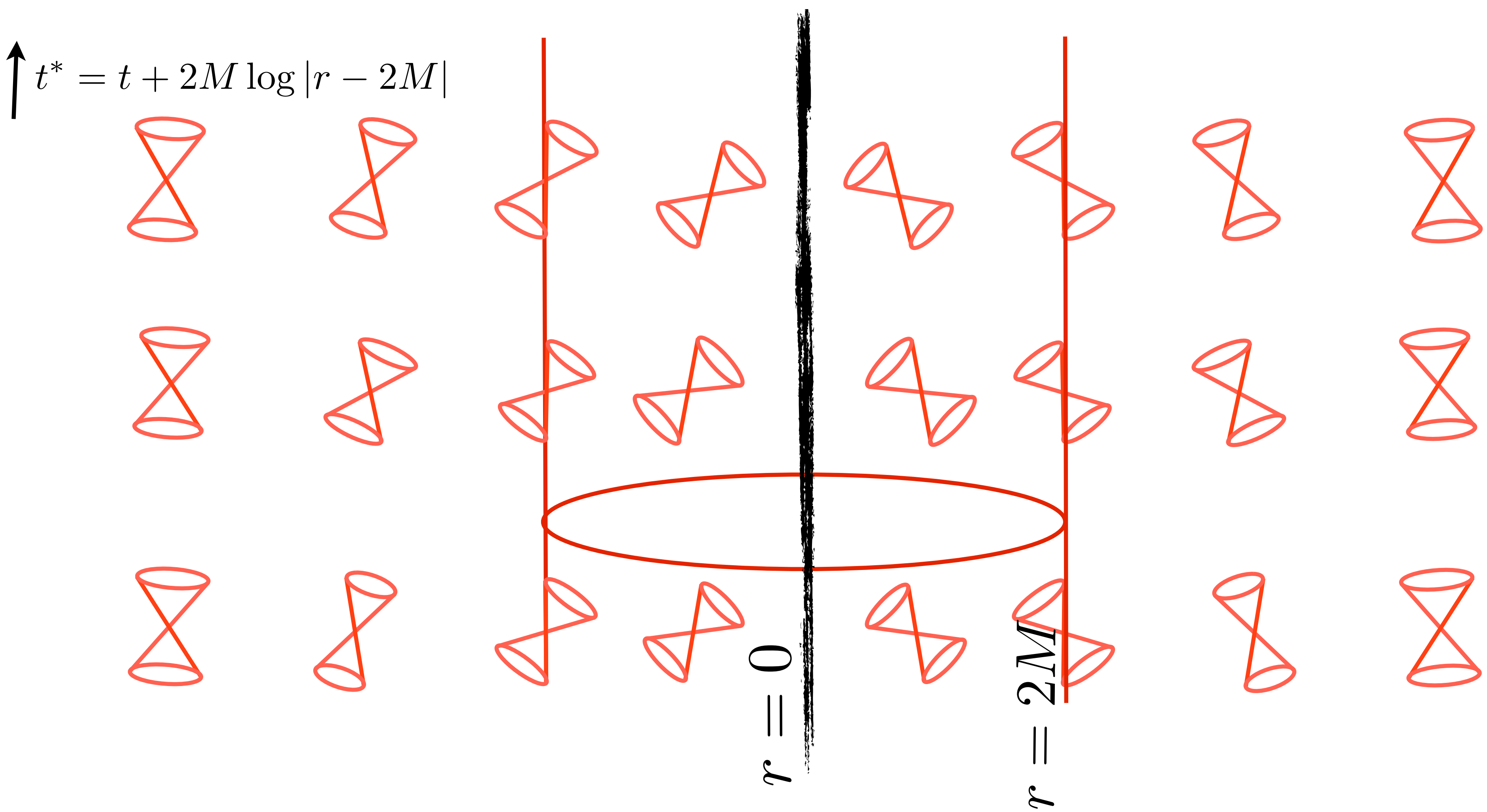
The geometry of Schwarzschild



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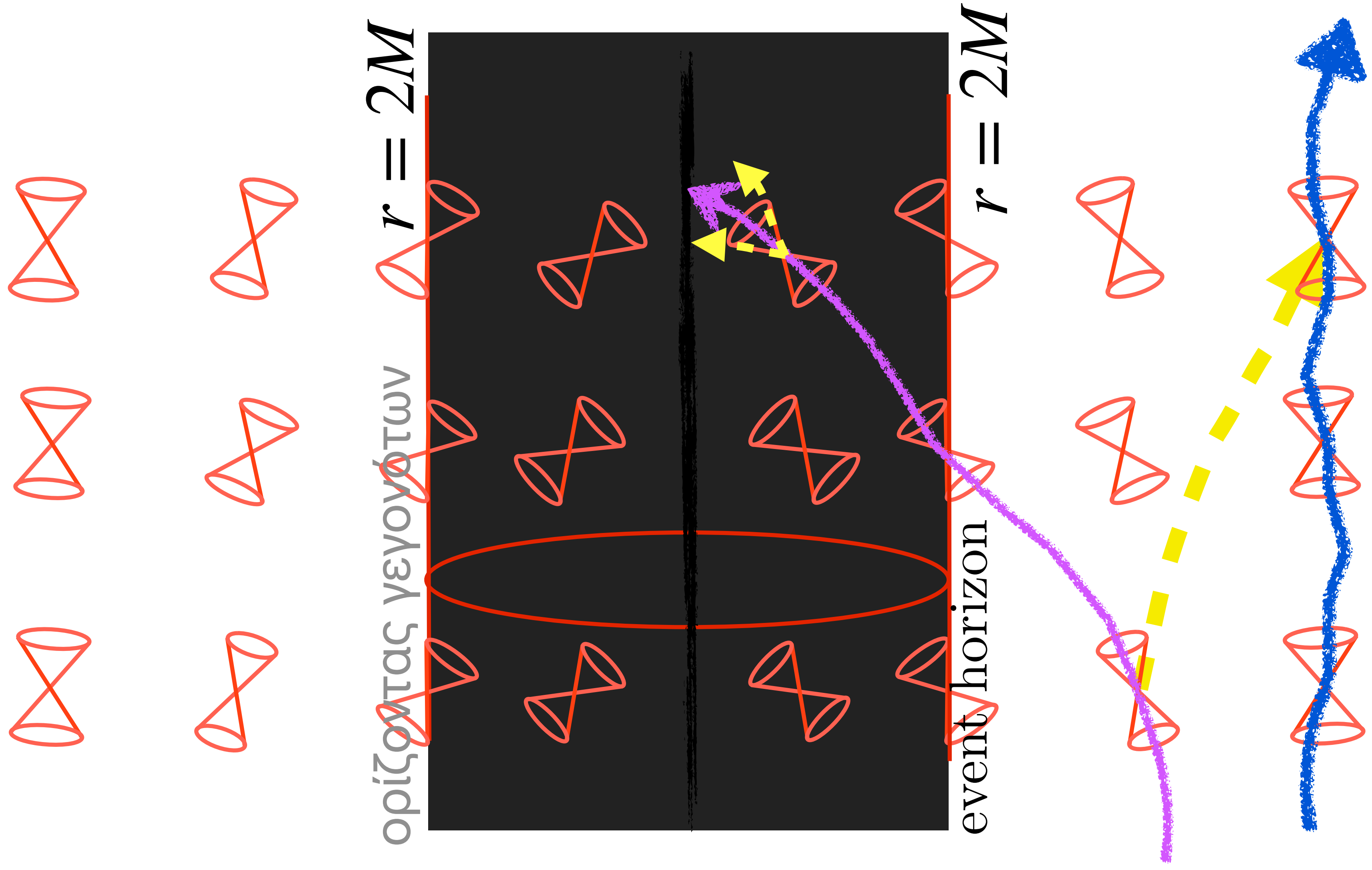
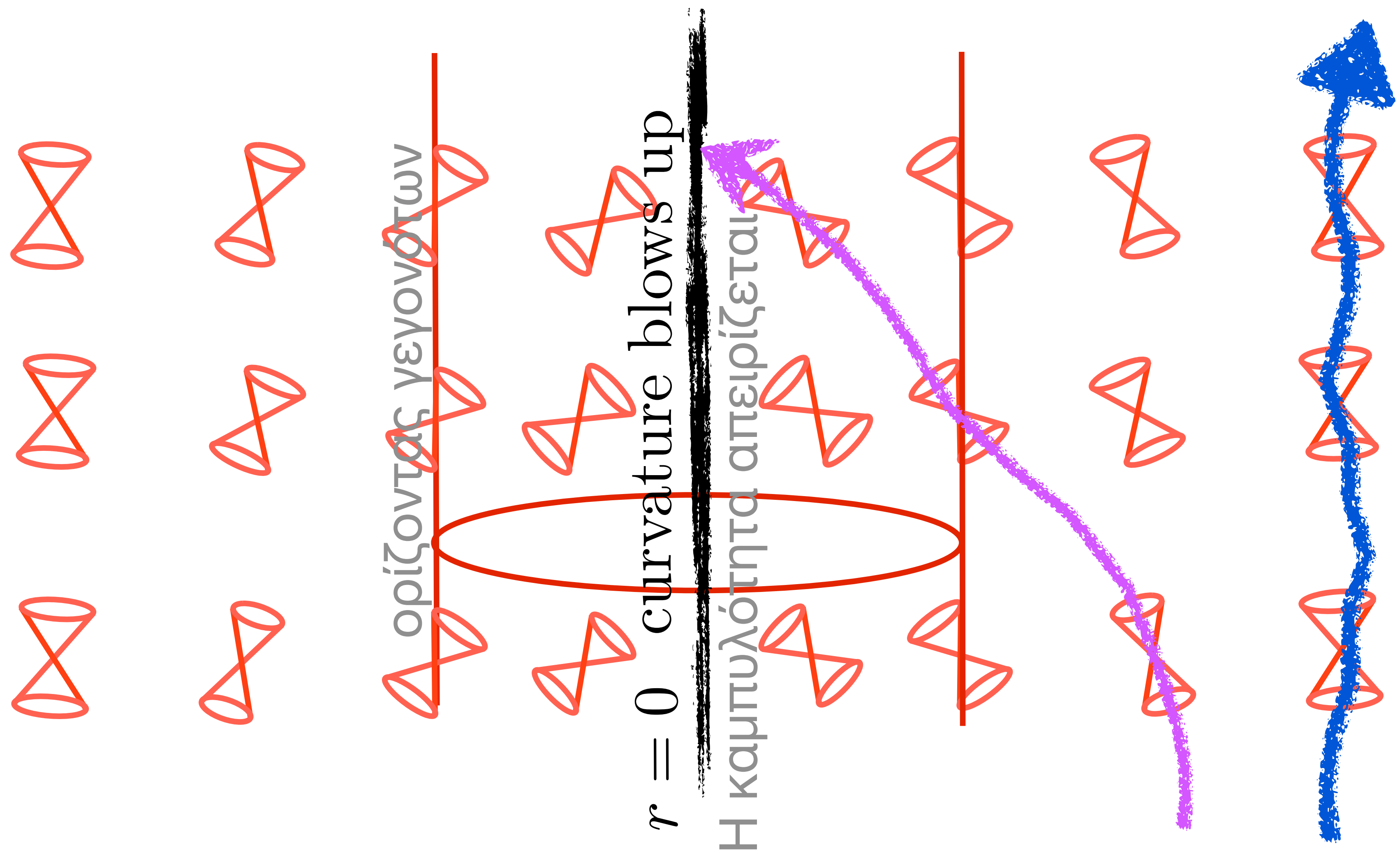
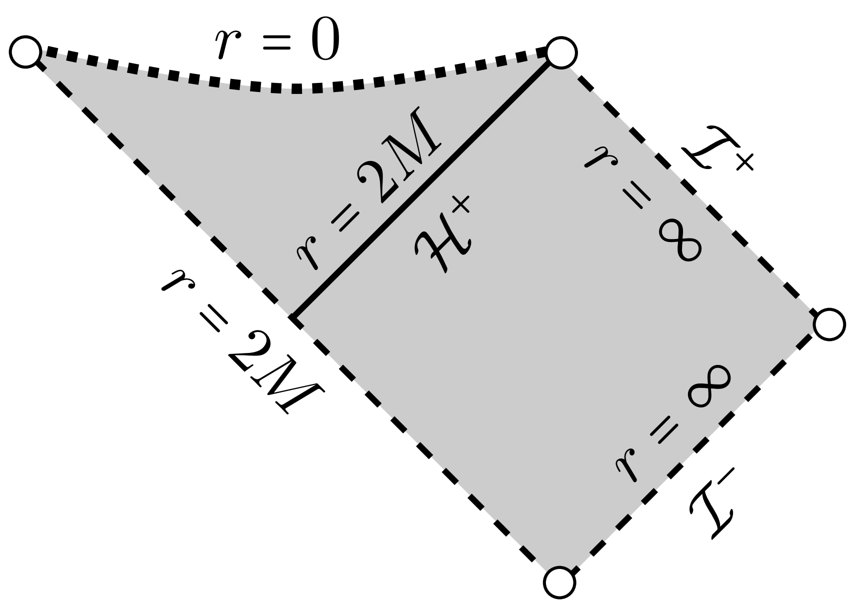
The black hole



Falling into the black hole



The Penrose diagram



- The singularity $r = 0$ is spacelike.
- The metric is inextendible beyond $r = 0$, not only as a C^2, but as a continuous (C^0) Lorentzian metric. (J. Sbierski 2016)
- Observers falling into the black hole are eventually torn apart by tidal deformations.

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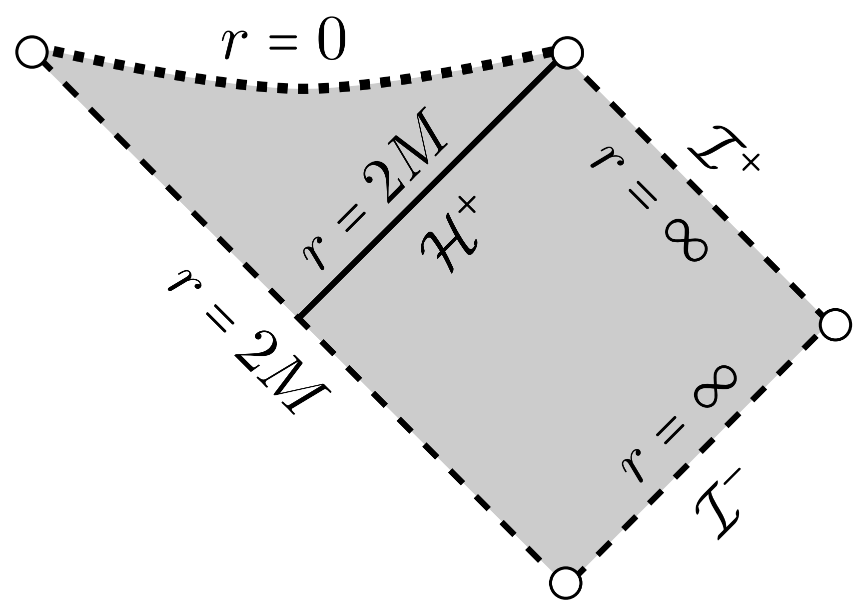

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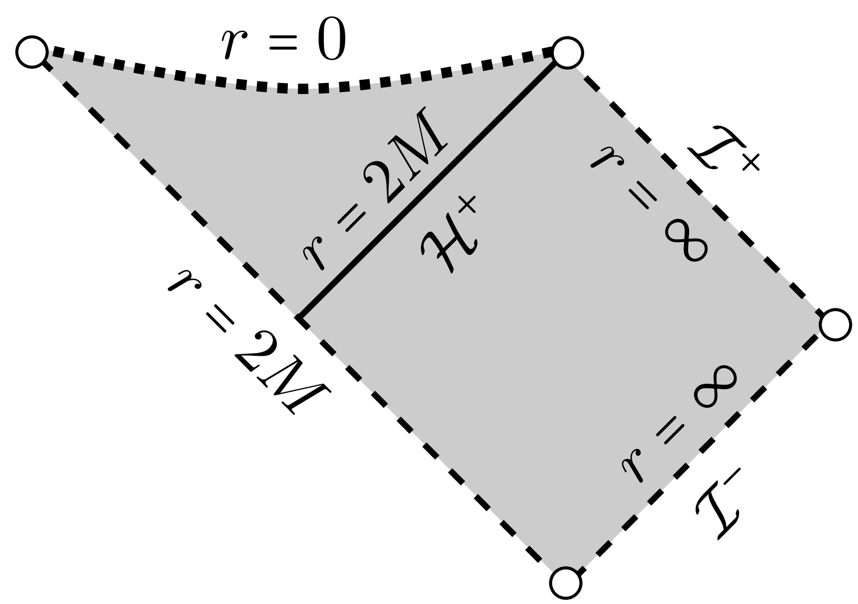

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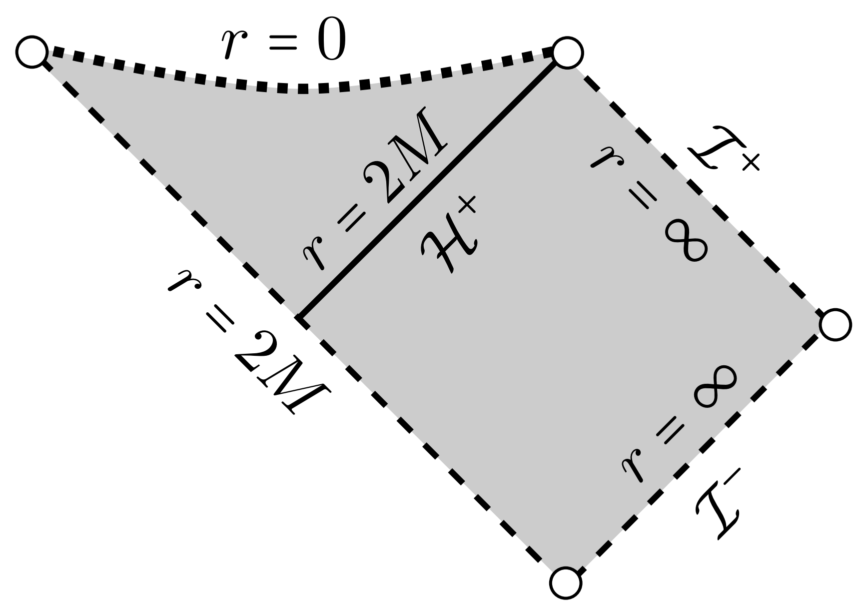

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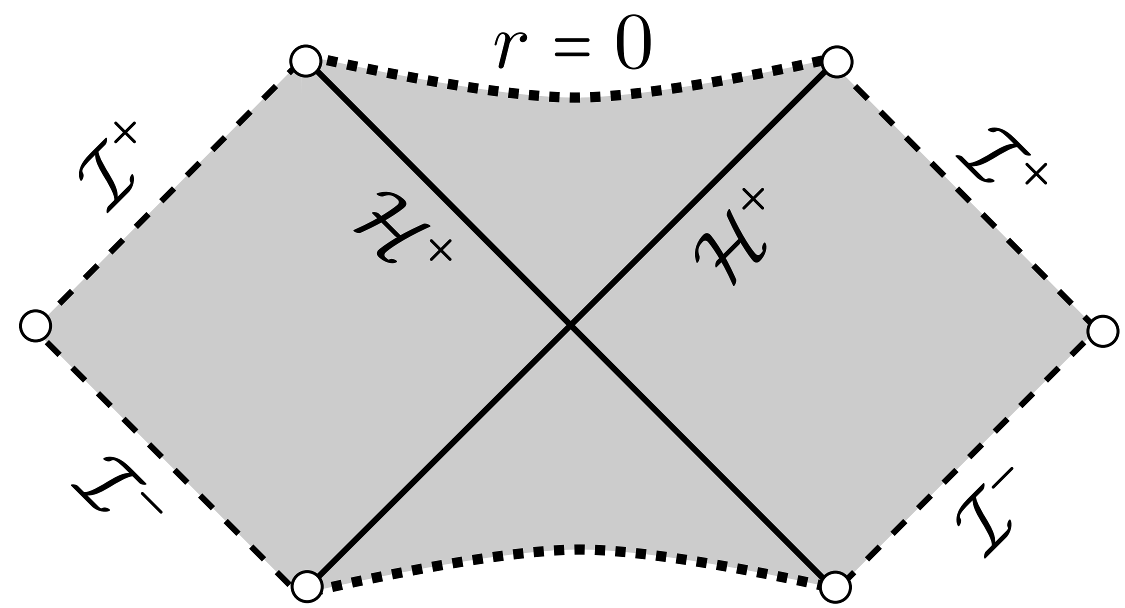
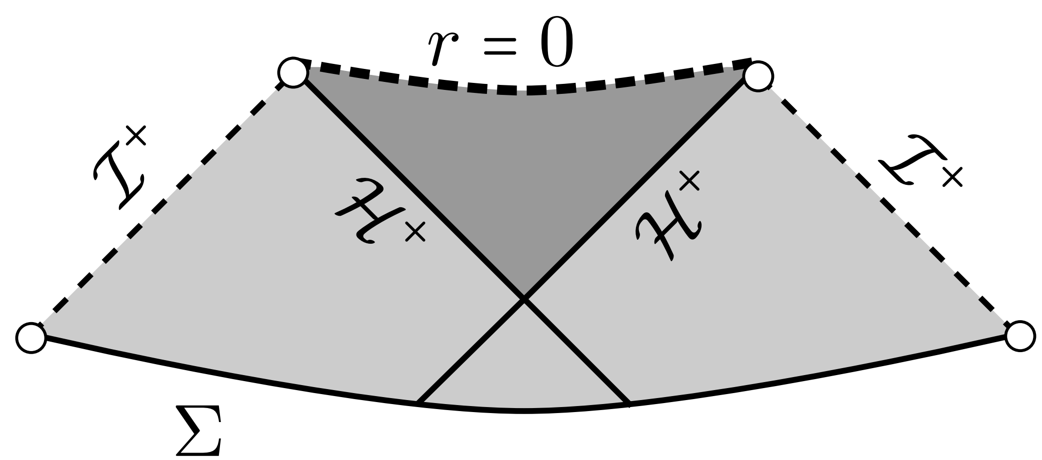
Doubling Schwarzschild



Synge 1950, Kruskal 1960

Schwarzschild as a maximal globally hyperbolic future Cauchy development



$(\Sigma, \bar{g}_M, K_M)$ asymptotically flat with two ends

Penrose's incompleteness theorem

Theorem. (Penrose 1965) *Let* $(\mathscr{M}, g)$ *satisfy the following:*

- $(\mathscr{M}, g)$ *is globally hyperbolic with a non-compact Cauchy hypersurface* Σ.
- $\mathrm{Ric}(V, V) \geq 0$ *for all null vectors* V.
- $\mathscr{M}$ *contains a closed trapped surface* S.

Then $(\mathscr{M}, g)$ *is future causally* <u>*geodesically incomplete*</u>.

Corollary of Penrose's incompleteness theorem

Corollary. *For initial data near Schwarzschild*

$$(\Sigma, \bar{g}, K) \approx (\Sigma, \bar{g}_M, K_M)$$

the maximal future globally hyperbolic Cauchy development $(\mathscr{M}, g)$ *is again future causally geodesically incomplete.*

References for Lecture 2

- M.D. “The geometry and analysis of black hole spacetimes in general relativity” https://www.dpmms.cam.ac.uk/~md384/ETH-Nachdiplom-temp.pdf (under construction)

- M.D. and I. Rodnianski “Lectures on black holes and linear waves”, arXiv:0811.0354

- S. Hawking and G. Ellis “The large scale structure of space-time”, (Cambridge Monographs on Mathematical Physics), CUP

- R. Penrose “Gravitational collapse and space-time singularities”, Phys. Rev. Lett., 14:57–59, Jan 1965.

- J. Sbierski “The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry”, J. Differential Geom., Volume 108, Number 2 (2018), 319–378

- R. M. Wald “General Relativity”, University of Chicago Press

Plan of the lectures

Lecture 1. *General Relativity and Lorentzian geometry*

Lecture 2. *The geometry of Schwarzschild black holes*

Lecture 3. *The analysis of waves on Schwarzschild exteriors*

Lecture 4. *The geometry of Kerr black holes and the strong cosmic censorship conjecture*

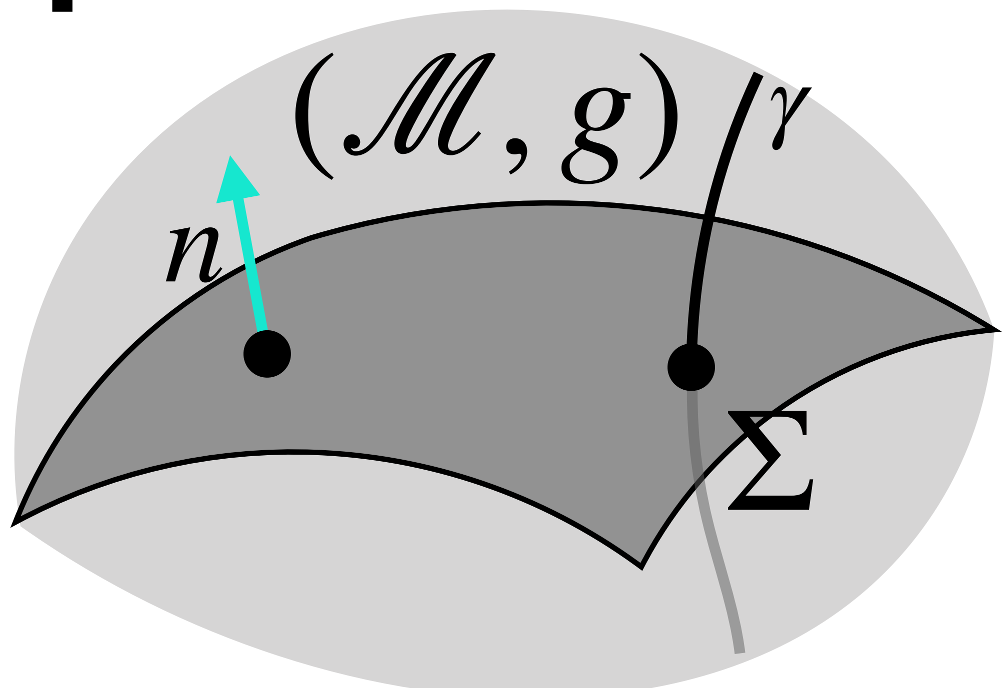
Lecture 5. *The analysis of waves on Kerr black hole interiors*

Lecture 6. *Nonlinear C^0 stability of the Kerr Cauchy horizon*

Lecture 3

The analysis of waves on Schwarzschild exteriors

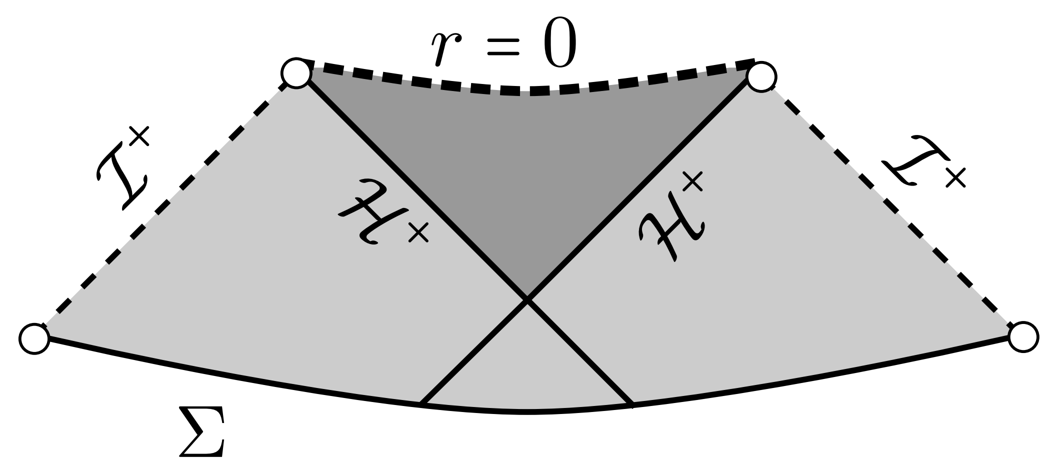
Recall: general global well posedness



- Let $(\mathscr{M}, g)$ be globally hyperbolic with Cauchy hypersurface Σ
- Given smooth initial data ϕ_0, ϕ_1 on Σ, there exists a unique smooth **global** solution ϕ of $\Box_g \phi = 0$ s.t.

$$\phi\,|_{\Sigma} = \phi_0, \qquad n^{\alpha}\partial_{\alpha}\phi\,|_{\Sigma} = \phi_1$$

The wave equation on Schwarzschild



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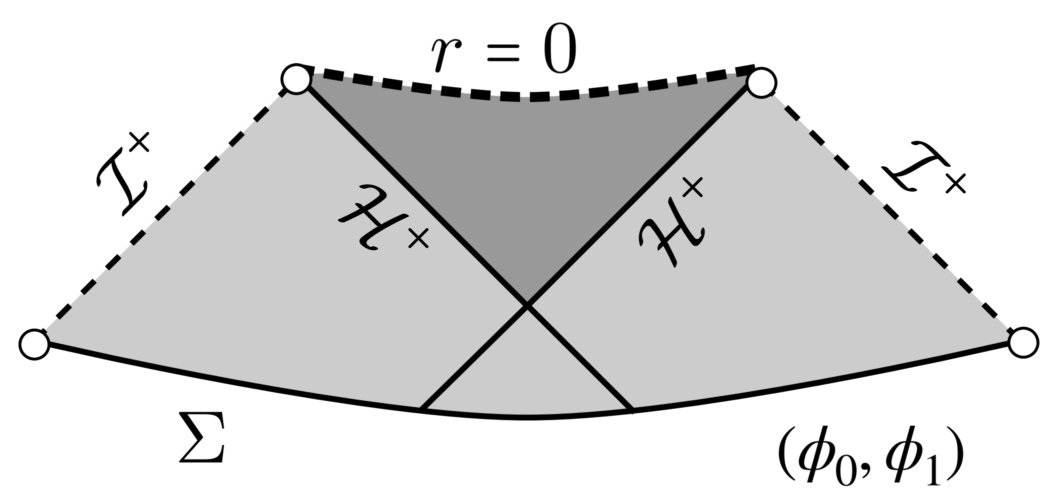
Main problem

$$\Box_g \phi = 0 \qquad \phi|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi|_\Sigma = \phi_1$$



- Does ϕ remain uniformly bounded in the lighter shaded region $J^{-}(\mathscr{I}^{+})$ (the black hole **exterior**)?
- Does ϕ decay to 0 …

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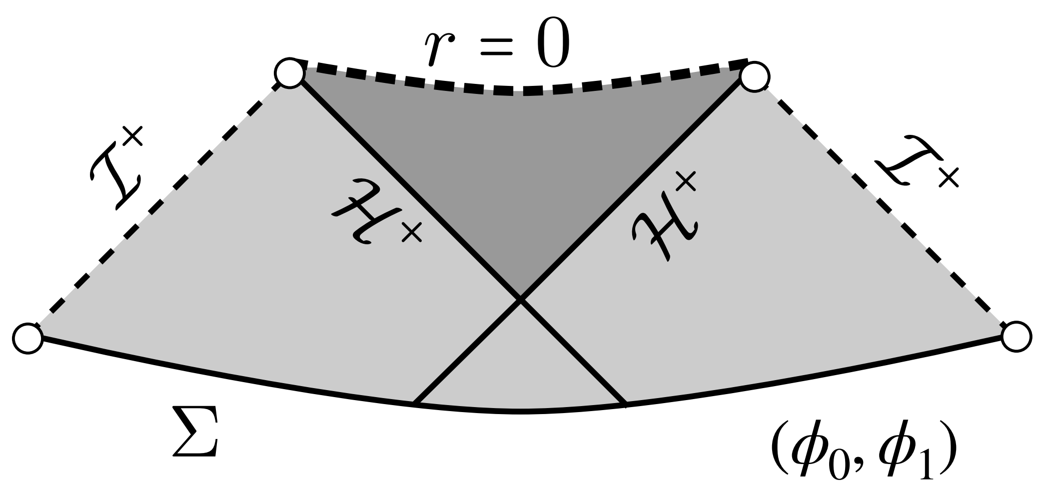

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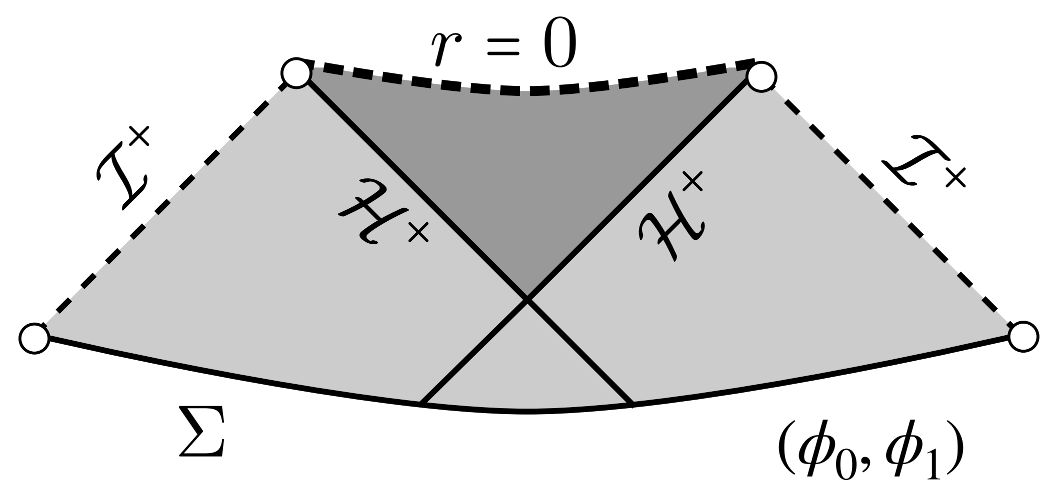

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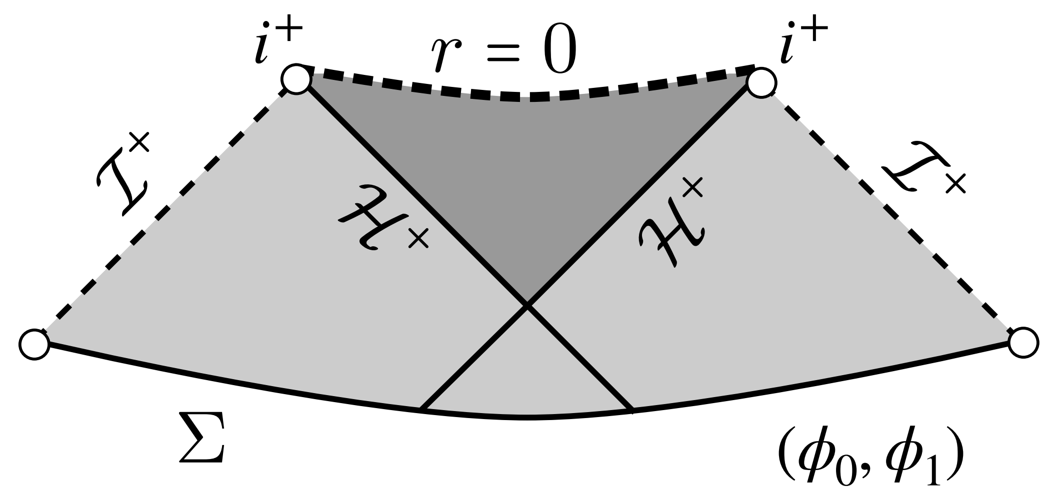
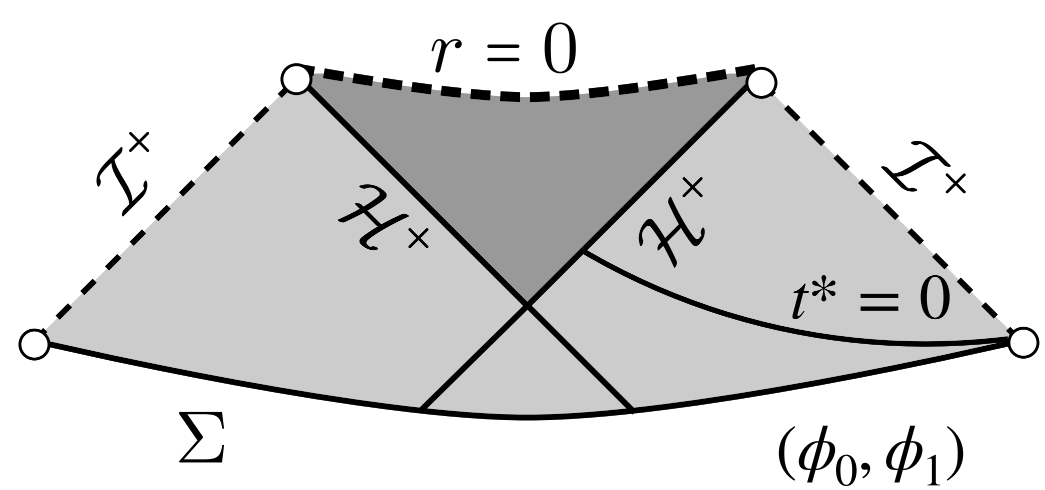


- Does ϕ remain uniformly bounded in the lighter shaded region $J^-(\mathscr{I}^+)$ (the black hole **exterior**)?
- Does ϕ decay to 0 as one approaches i^+?

Reduction

$$\Box_g \phi = 0 \qquad \phi|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi|_\Sigma = \phi_1$$



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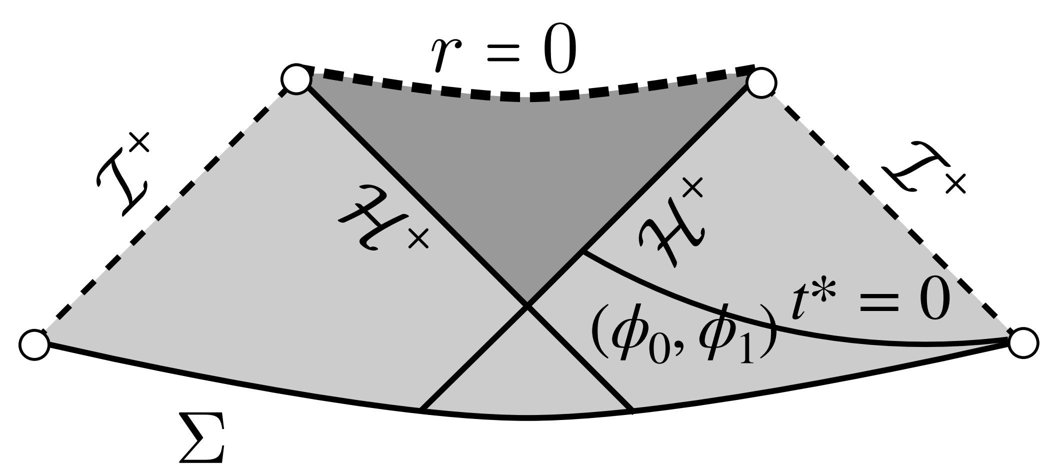

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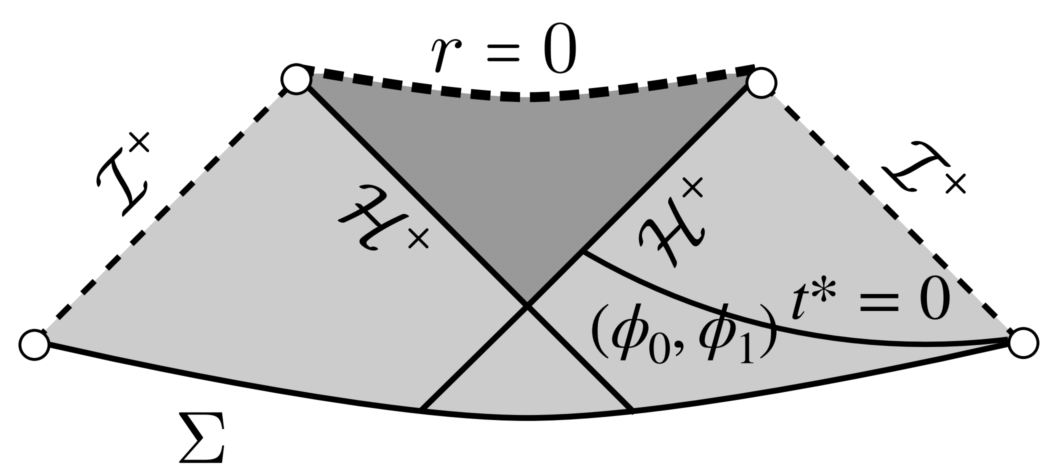

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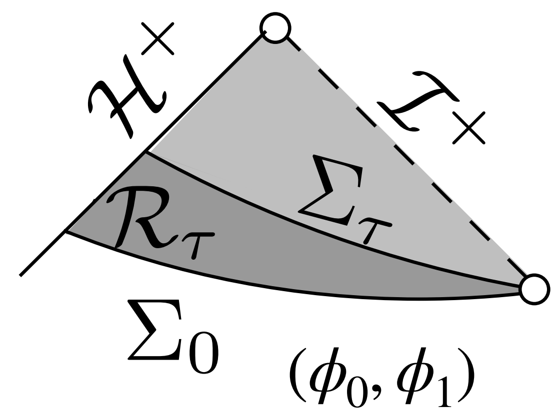

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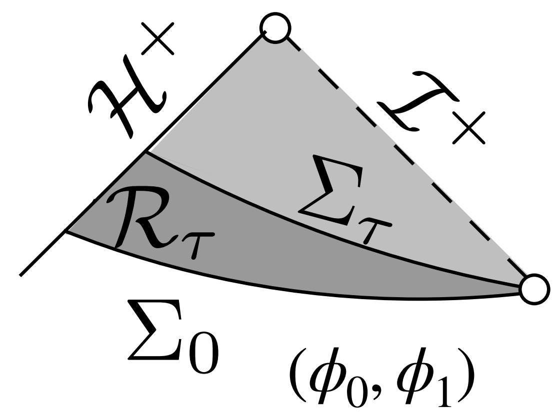

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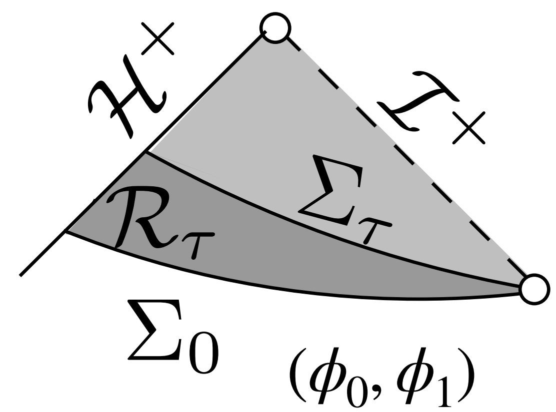

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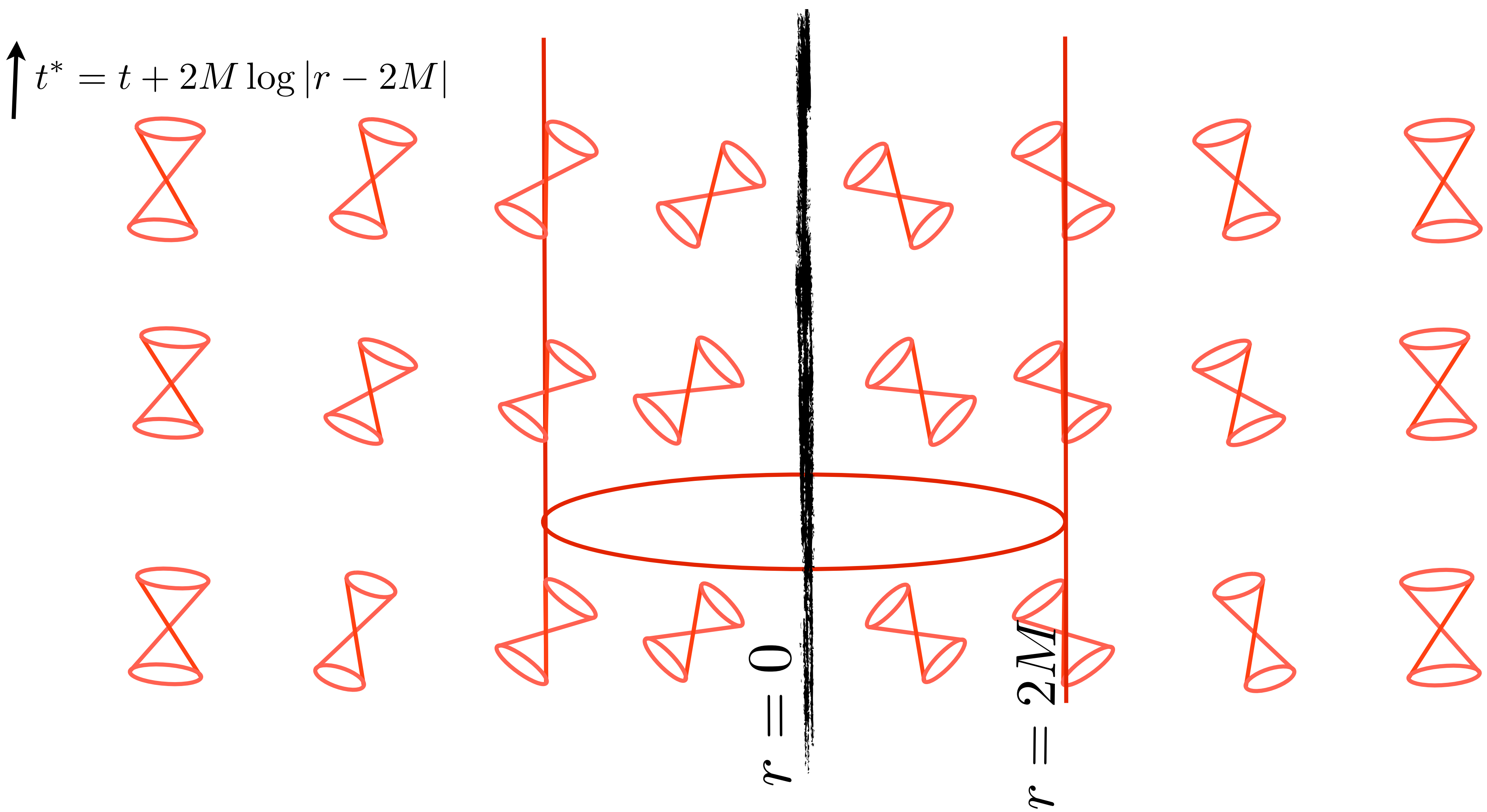
The (t^*, r) picture



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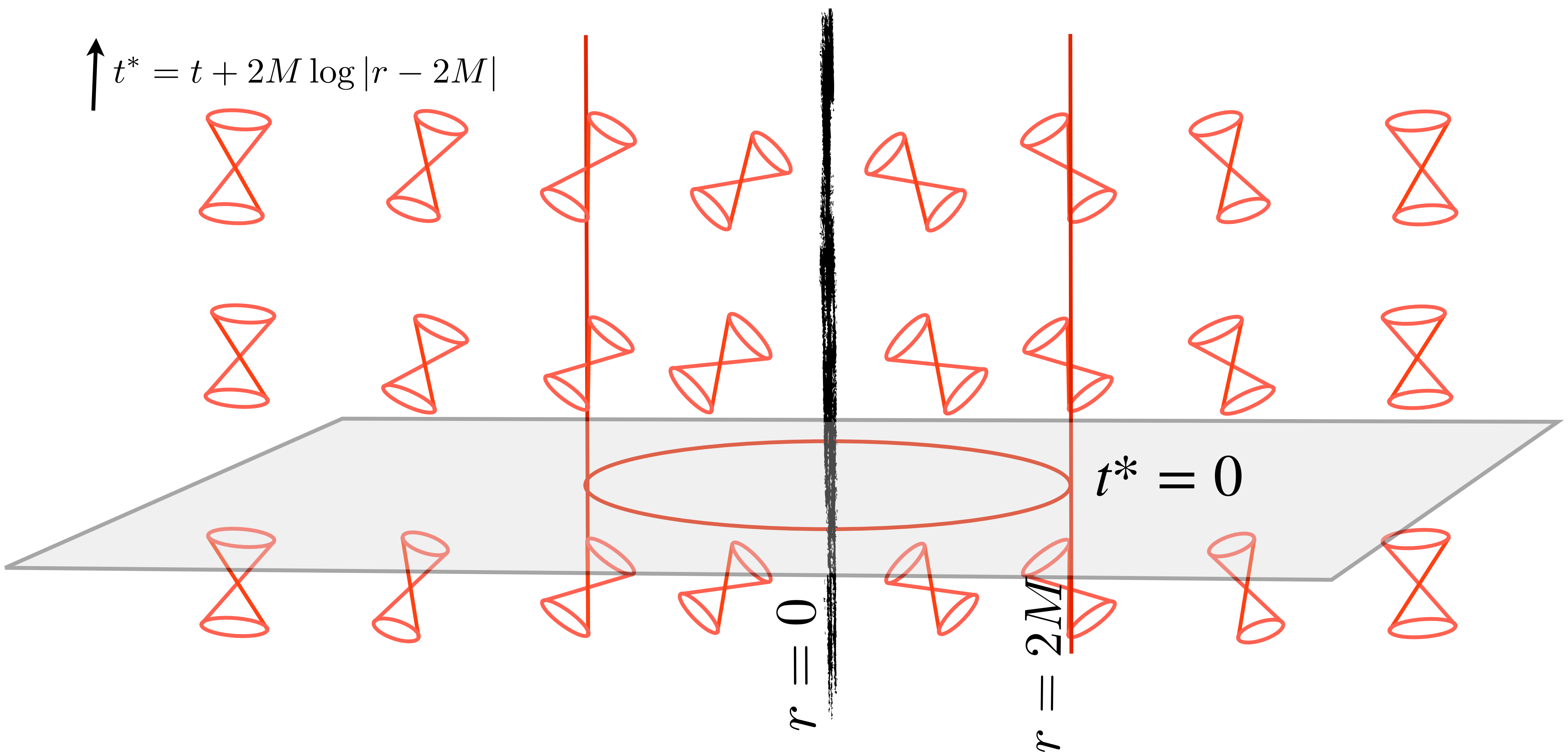

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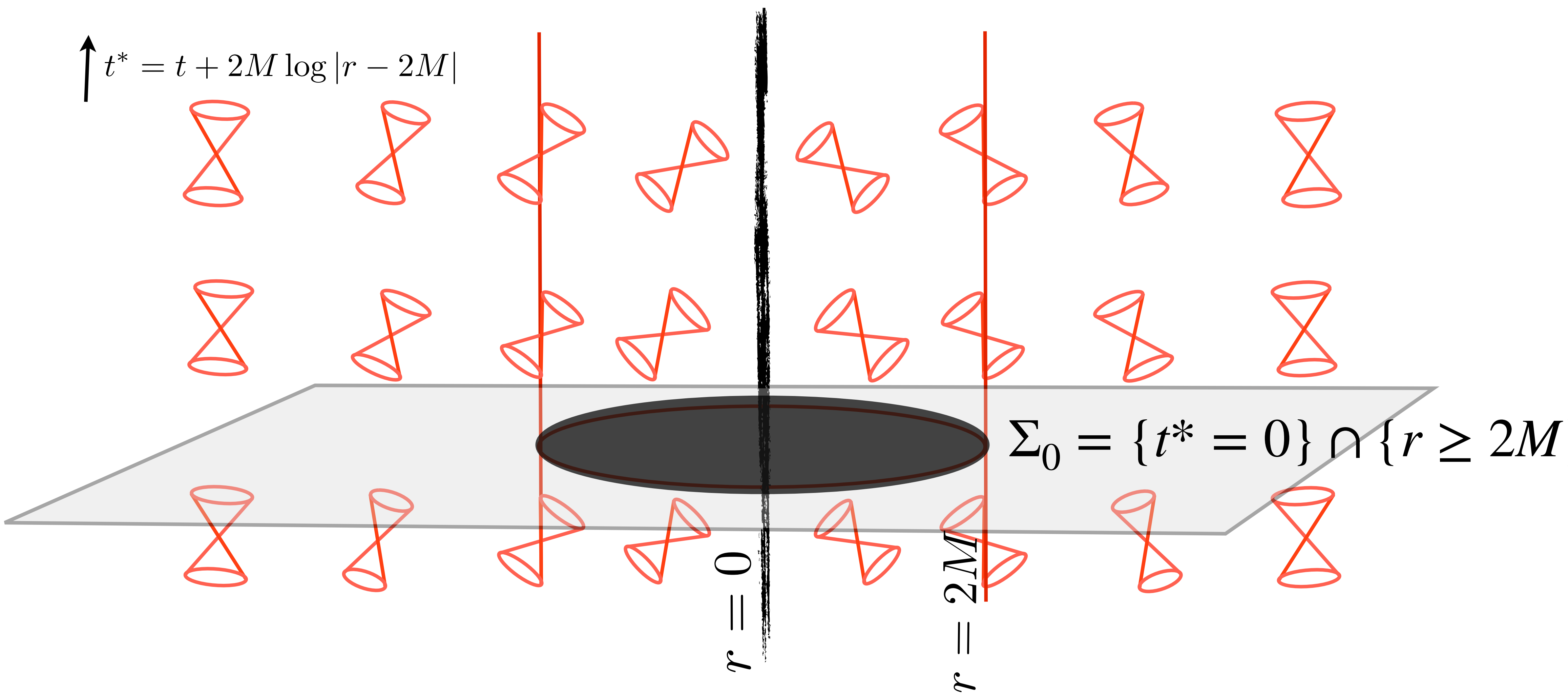

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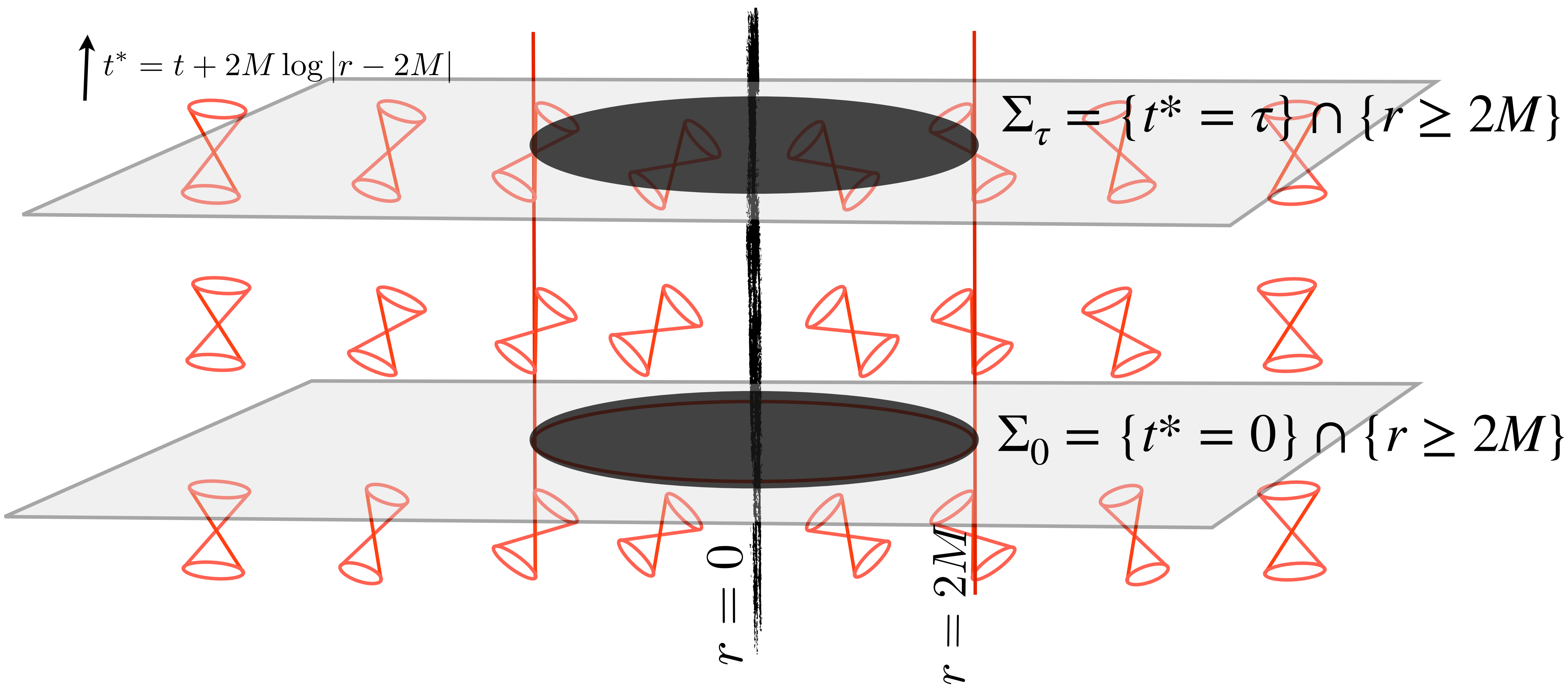

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$$T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi$$

$$\Box_g\psi = 0 \implies \nabla^\mu T_{\mu\nu}[\psi] = 0$$

X vector field $\longrightarrow$

$$J^X_\mu[\psi] \doteq T_{\mu\nu}[\psi]X^\nu$$

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$$T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi$$

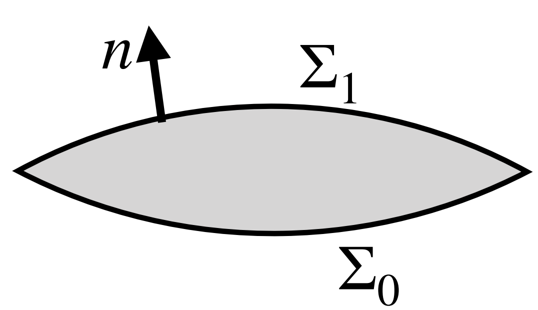
$$\Box_g\psi = 0 \implies \nabla^\mu T_{\mu\nu}[\psi] = 0$$

X vector field $\longrightarrow$

$$J^X_\mu[\psi] \doteq T_{\mu\nu}[\psi]X^\nu$$

$$K^X[\psi] \doteq T_{\mu\nu}[\psi]^{(X)}\pi^{\mu\nu} = 0 \text{ if } K \text{ is Killing}$$

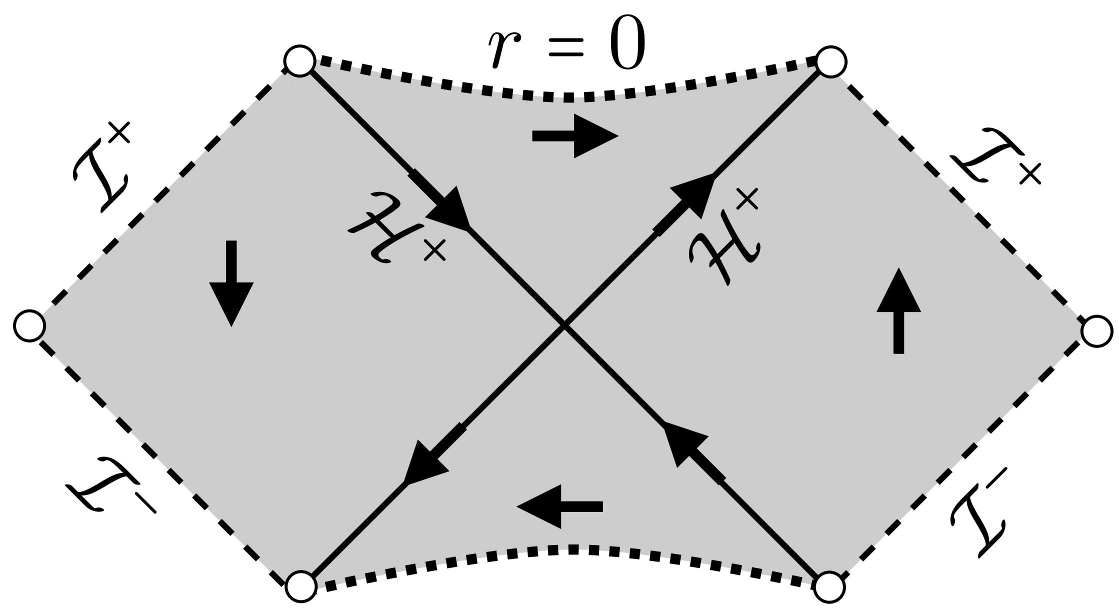
$$\nabla^\mu J^X_\mu[\psi] = K^X[\psi]$$



Σ_t spacelike, X future timelike

$$\implies \int_{\Sigma_1} J^X_\mu[\psi]n^\mu dV_{\Sigma_1} \quad \text{coercive}$$

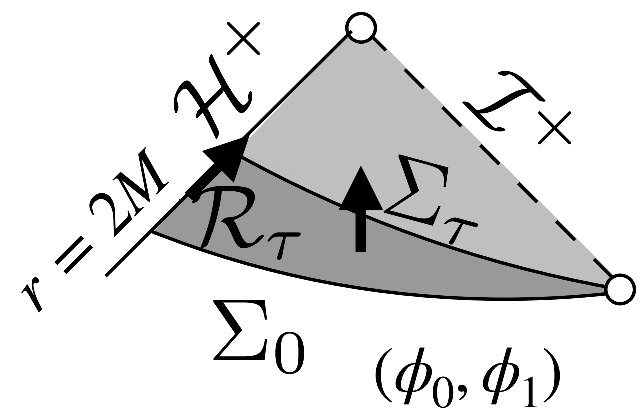
The T vector field



The coordinate vector field ∂_{t^*} extends to a globally defined Killing field T

The T identity

$$\Box_g \phi = 0 \qquad \phi|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi|_\Sigma = \phi_1$$



$$\nabla^\mu J^T_\mu[\phi] = K^T[\phi] = 0 \quad \Longrightarrow$$

$$\int_{\Sigma_\tau} J^T_\mu[\phi] n^\mu_{\Sigma_\tau} + \int_{\mathscr{H}^+_{(0,\tau)}} J^T_\mu[\phi] n^\mu_{\mathscr{H}^+} = \int_{\Sigma_0} J^T_\mu[\phi] n^\mu_{\Sigma_0}$$

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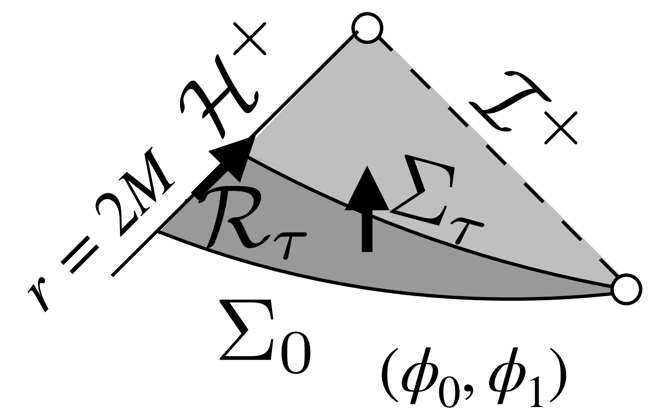

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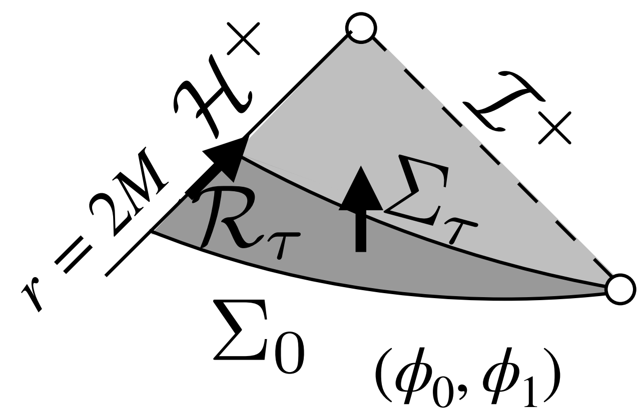

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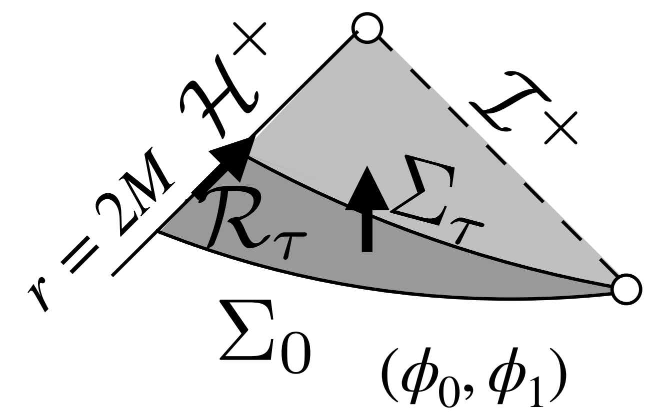

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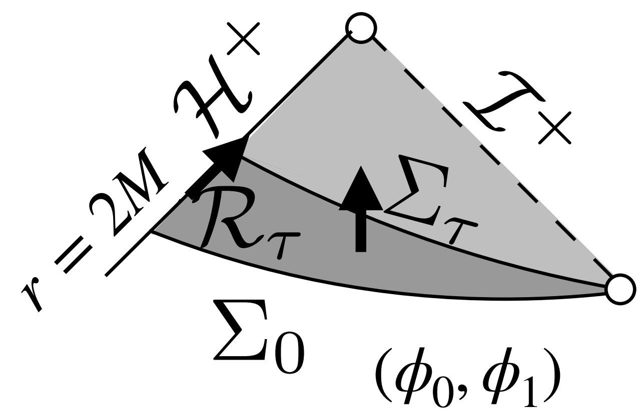

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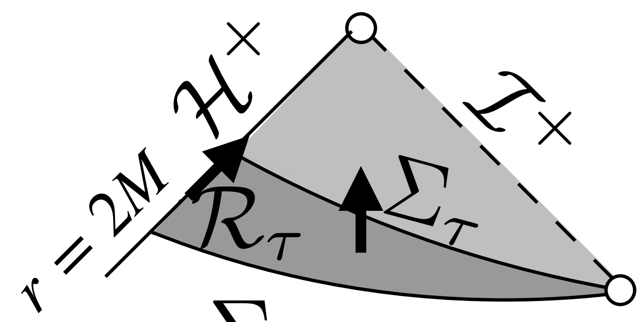

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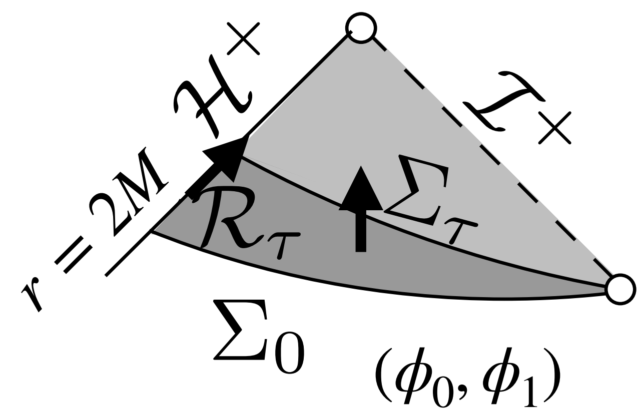
$$\Box_g \phi = 0 \qquad\qquad \phi|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi|_\Sigma = \phi_1$$



$$T \text{ Killing} \implies [\Box_g, T] = 0 \implies \Box_g (T\phi) = 0$$

$$\int_{\Sigma_\tau} (\partial_{t^*}\partial_{t^*}\phi)^2 \lesssim \int_{\Sigma_0} (\partial_{t^*}\partial_{t^*}\phi)^2 + (1 - 2M/r)(\partial_r \partial_t^* \phi)^2 + |\nabla\!\!\!/ (\partial_{t^*}\phi)|^2$$

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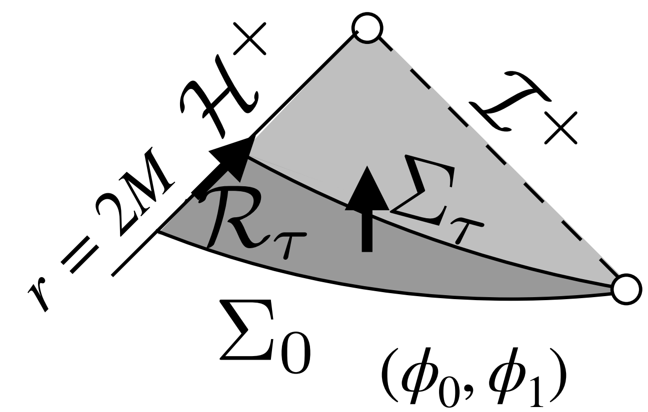

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$$\|\nabla_x \phi\|^2_{H^1(\Sigma_\tau \cap \{r \geq 2M+\epsilon\})} \lesssim_\epsilon \int_{\Sigma_\tau \cap \{r \geq 2M+\epsilon\}} (\partial_{t^*}\partial_{t^*}\phi)^2 + (1-2M/r)(\partial_r \partial_{t^*}\phi)^2 + (\partial_{t^*}\phi)^2 + (1-2M/r)(\partial_r \phi)^2 + |\nabla\!\!\!/ \phi|^2$$

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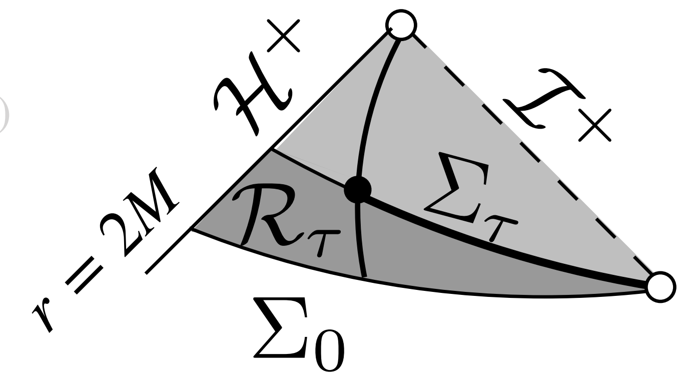

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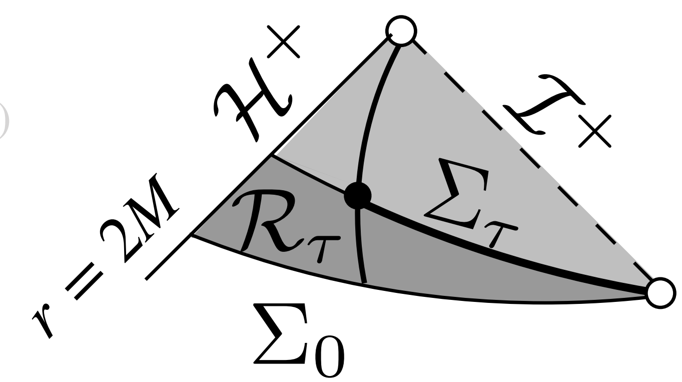

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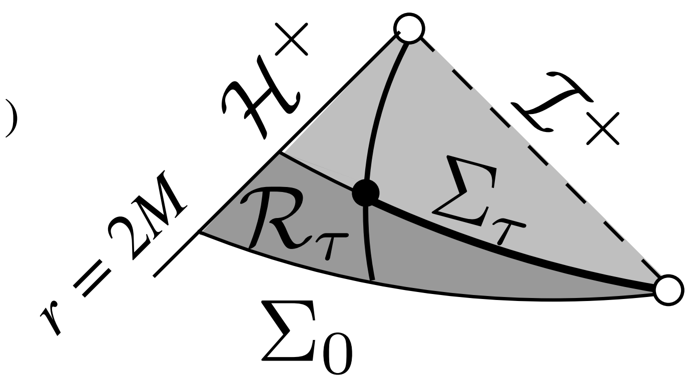

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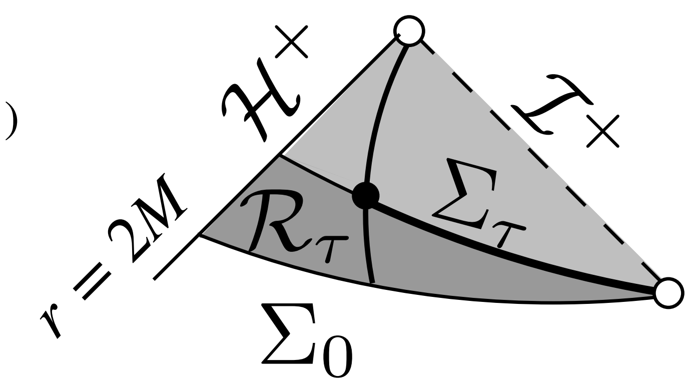

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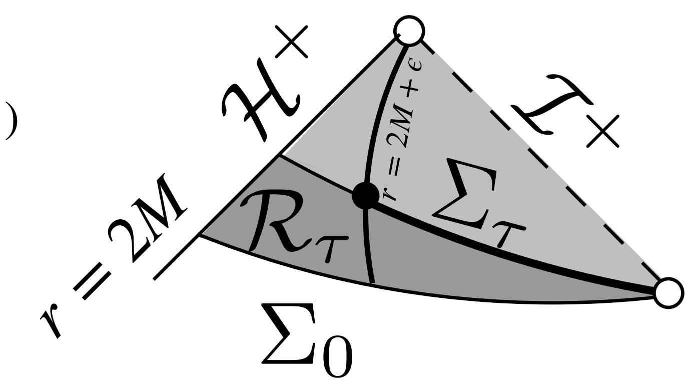

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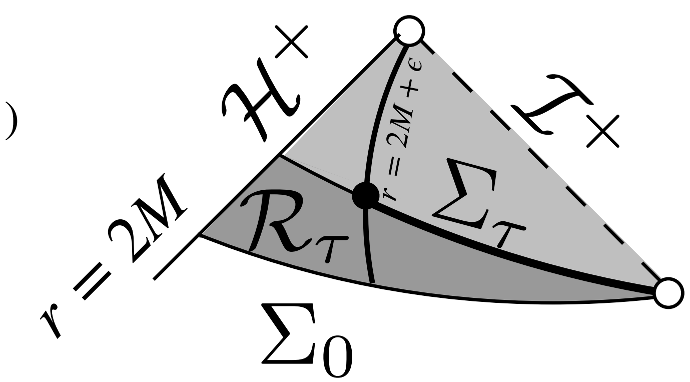

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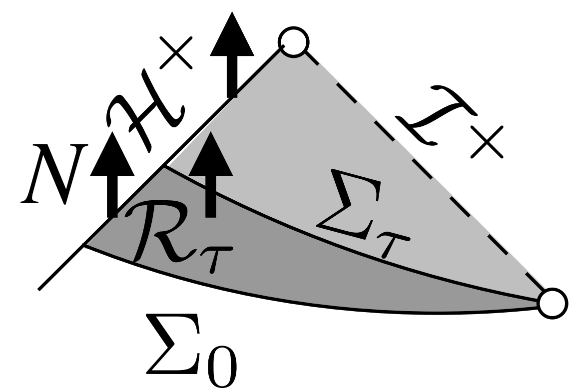
The N identity



Proposition. There exists a timelike T-invariant vectorfield N

such that $\nabla^\mu J^N_\mu[\phi] = K^N[\phi] \geq 0 \;\text{ in }\; r \leq 2M + \epsilon_0$

in fact $K^N[\phi] \gtrsim J^N_\mu[\phi] n^\mu_{\Sigma_\tau} \;\text{ in }\; r \leq 2M + \epsilon_0\,.$

The vector field N captures the celebrated red-shift.

Exercise: N-identity + T-estimate $\implies \int_{\Sigma_\tau} J^N_\mu[\phi] n^\mu_{\Sigma_\tau} \lesssim \int_{\Sigma_0} J^N_\mu[\phi] n^\mu_{\Sigma_0}$

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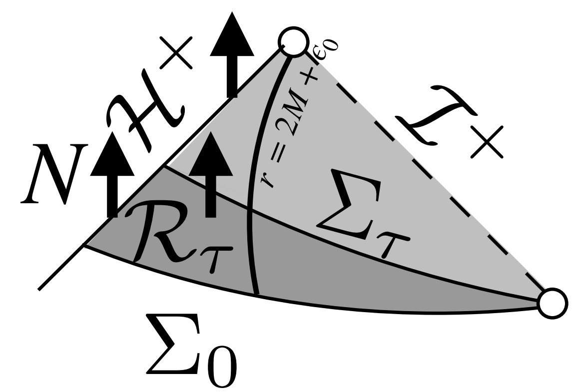

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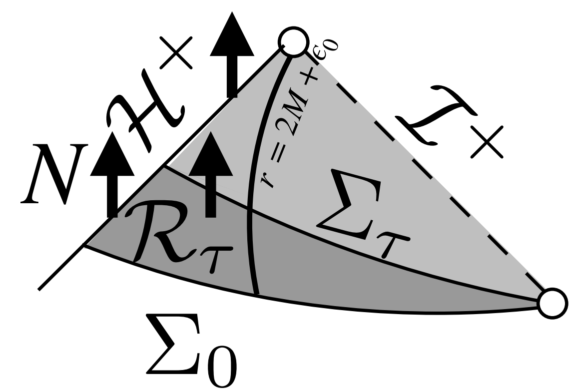

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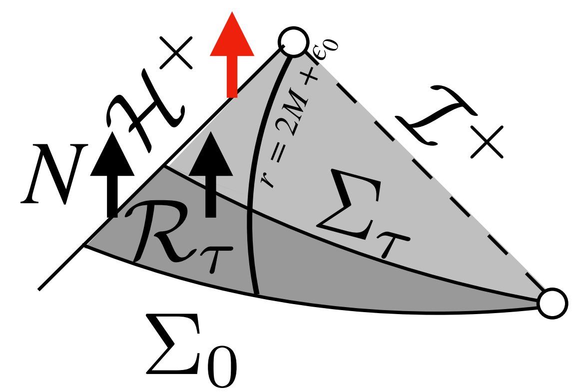

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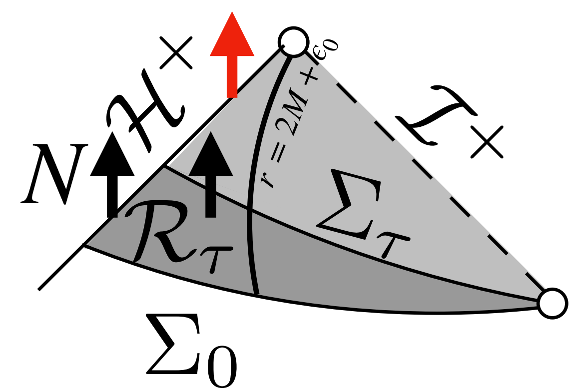

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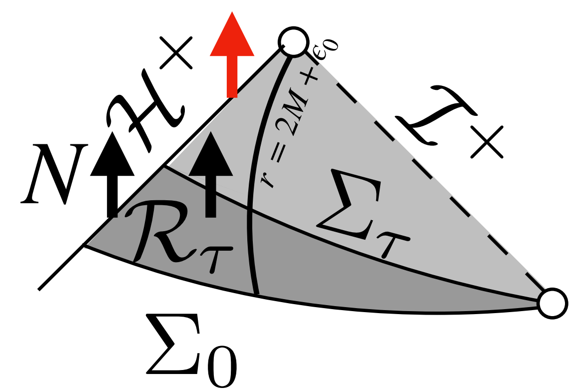

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$$[\Box_g, N] \neq 0$$

Nonetheless the "worst terms" have a good sign and are in fact coercive near $\mathscr{H}^+$

Exercise:

$$\implies \int_{\Sigma_\tau} (\partial_{t^*}(N\phi))^2 + (\partial_r(N\phi))^2 + |\nabla\!\!\!/ (N\phi)|^2 \lesssim \int_{\Sigma_0} (\partial_{t^*}(N\phi))^2 + (\partial_r\phi)^2 + |\nabla\!\!\!/ (N\phi)|^2 + \dots$$

This and the previous now controls $\|\nabla_x \phi\|^2_{H^1(\Sigma_\tau)}$

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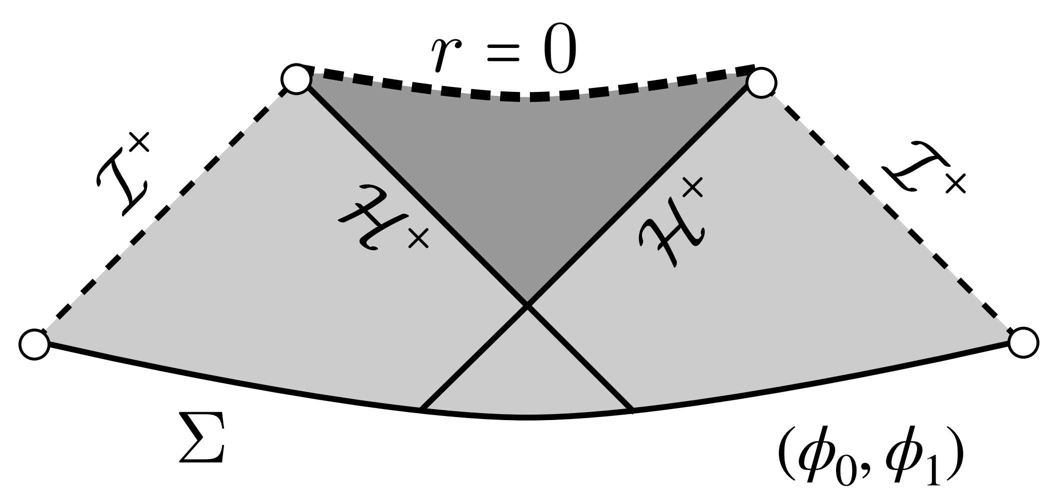
This and the previous now controls $\|\nabla_x\phi\|^2_{H^1(\Sigma_\tau)}$

The boundedness theorem

$$\Box_g \phi = 0 \qquad\qquad \phi|_\Sigma = \phi_0, \qquad n^\alpha \partial_\alpha \phi|_\Sigma = \phi_1$$



Theorem. We have the uniform bound:

$$|\phi| \lesssim D$$

in the black hole <u>exterior</u> region $J^-(\mathscr{I}^+)$, where D is a suitable norm on initial data (ϕ_0, ϕ_1).

References for Lecture 3

- D. Christodoulou “The action principle and partial differential equations”, Ann. Math. Studies No. 146, 1999

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- S. Klainerman “Brief history of the vector-field method”, https://web.math.princeton.edu/~seri/homepage/papers/John2010.pdf

Plan of the lectures

Lecture 1. *General Relativity and Lorentzian geometry*

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Lecture 3. *The analysis of waves on Schwarzschild exteriors*

Lecture 4. *The geometry of Kerr black holes and the strong cosmic censorship conjecture*

Lecture 5. *The analysis of waves on Kerr black hole interiors*

Lecture 6. *Nonlinear C^0 stability of the Kerr Cauchy horizon*

Lecture 4

The geometry of Kerr black holes
and the strong cosmic censorship conjecture

Kerr spacetime

GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*

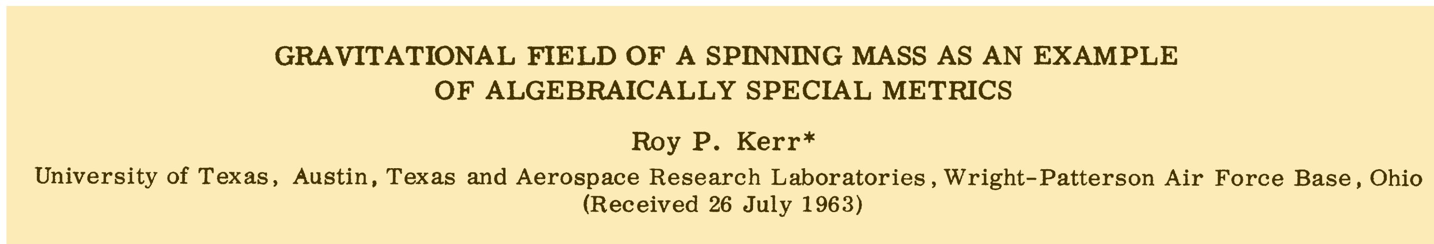
University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

(Received 26 July 1963)

- discovered by Roy Kerr in 1963
- $g = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(adt - (r^2+a^2)\,d\phi)^2$

  $r_\pm = M \pm \sqrt{M^2 - a^2}, \quad \Delta = (r - r_+)(r - r_-), \quad \rho^2 = r^2 + a^2\cos^2\theta$
- explicit solution of the vacuum equations $\mathrm{Ric}(g) = 0$
- conjectured to be unique family of stationary black holes
- conjectured to be asymptotically stable in the exterior

Kerr spacetime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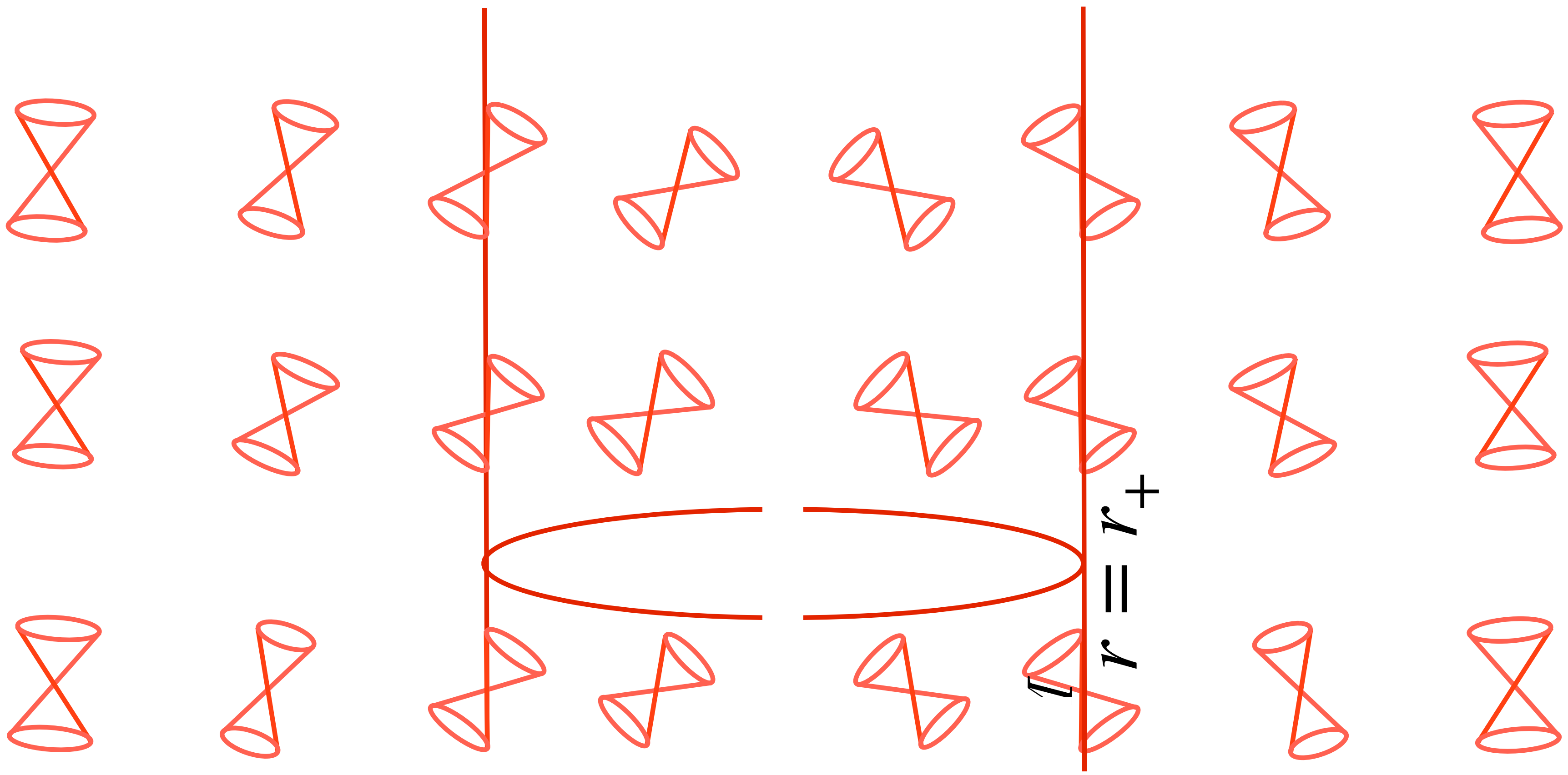

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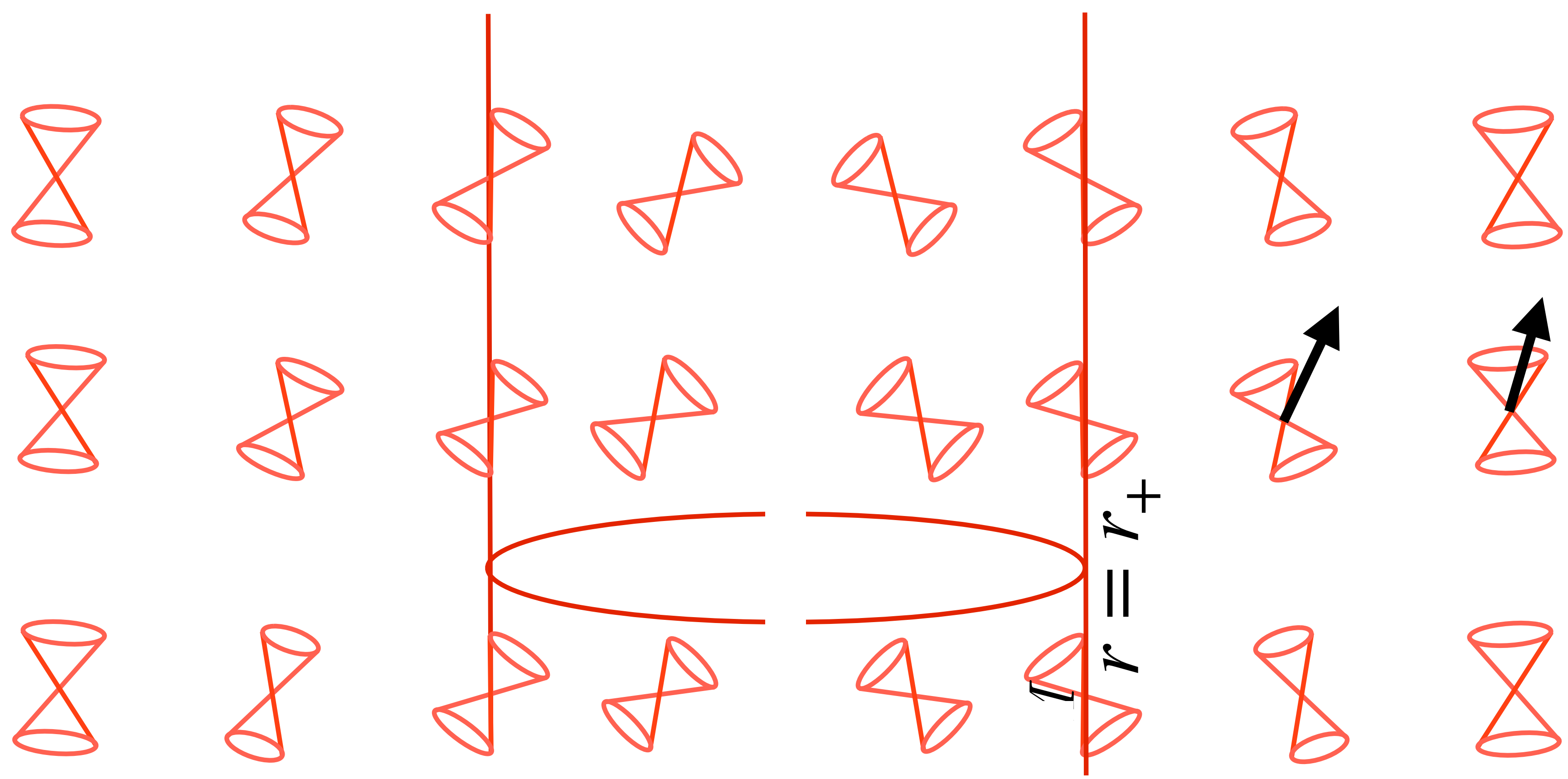
There is a region outside the horizon where ∂_t is spacelike!

The ergoregion



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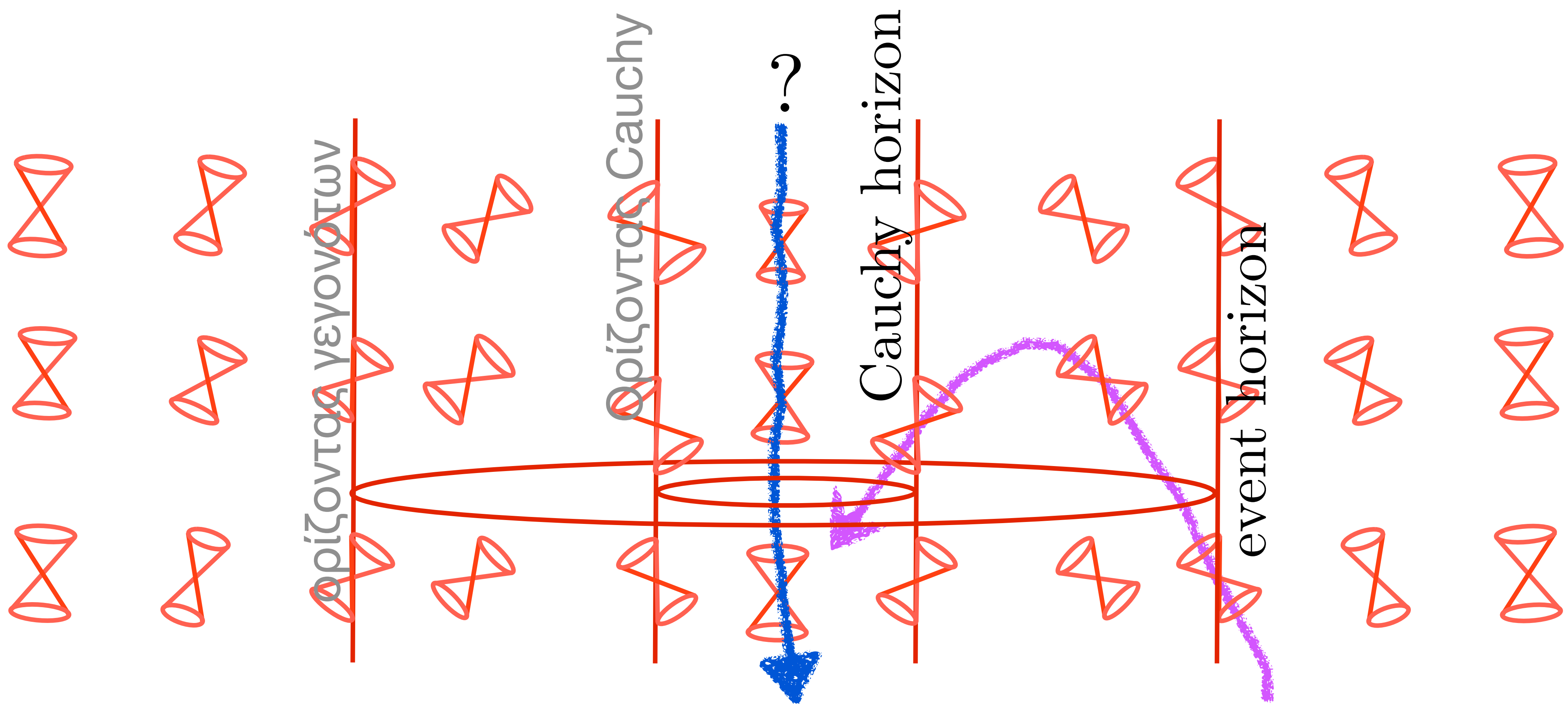
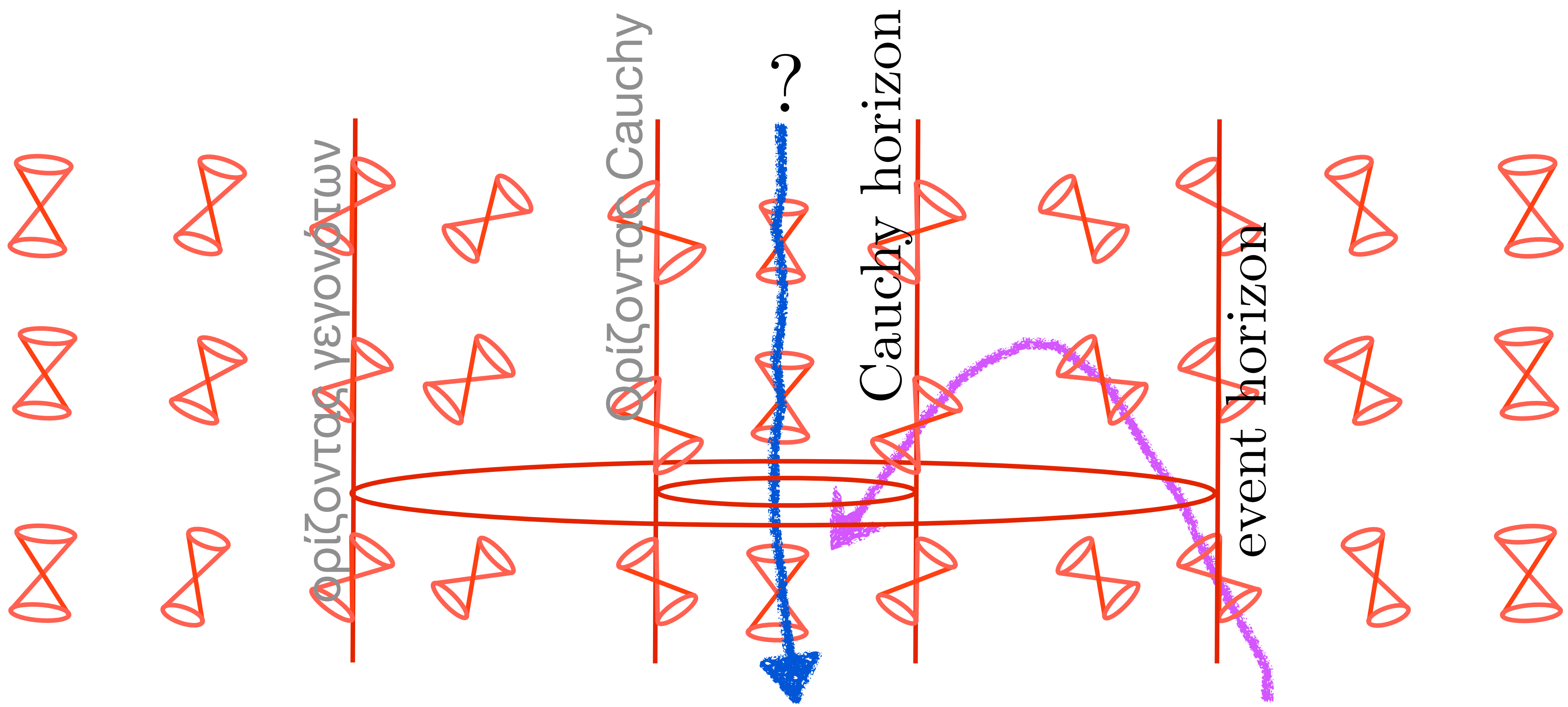
The Cauchy horizon



Before encountering any singularity, the observer falling in the black hole reaches a second horizon, the so-called Cauchy horizon (Hawking 1966).

The Cauchy horizon



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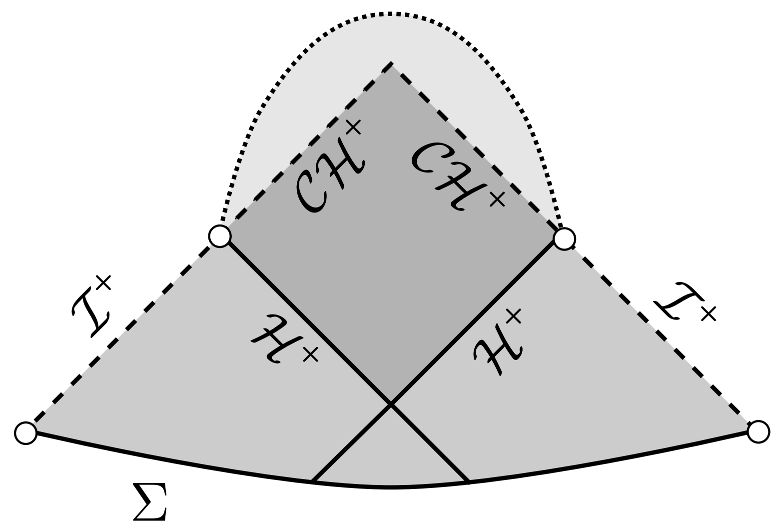
The Penrose diagram



- From the point of view of the Cauchy problem, the maximal future Cauchy development of Kerr initial data posed on Σ is incomplete but smoothly extendible beyond a bifurcate Cauchy horizon $\mathscr{CH}^+$. In fact, all incomplete geodesics can pass into the extension.

- This is analogous to the phenomenon that arises when one tries to solve the Cauchy problem for the classical wave equation $\Box\psi = 0$ with data posed only on the set $\{t = 0\} \cap \{r \leq 1\}$.

- Thus, applied to Kerr, Penrose's incompleteness theorem does not signify singularity formulation, but rather loss of *determinism*.

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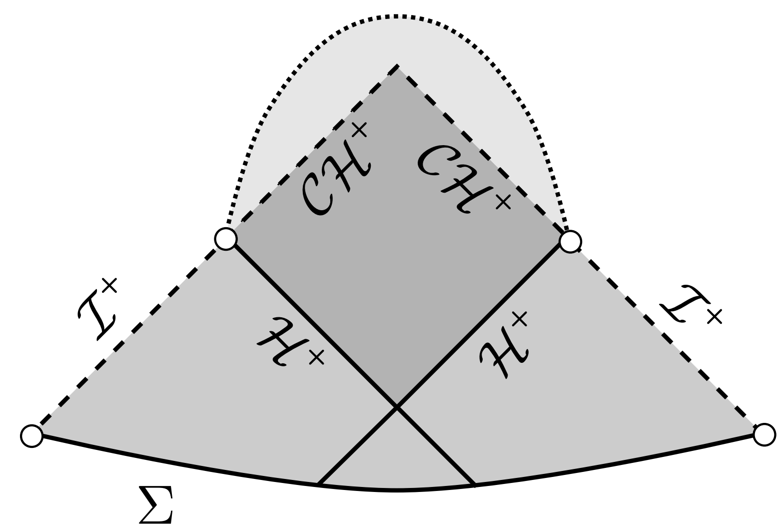

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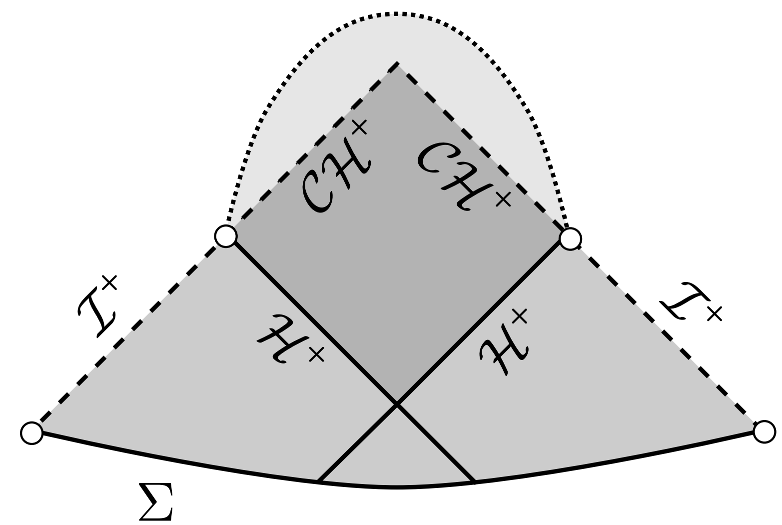

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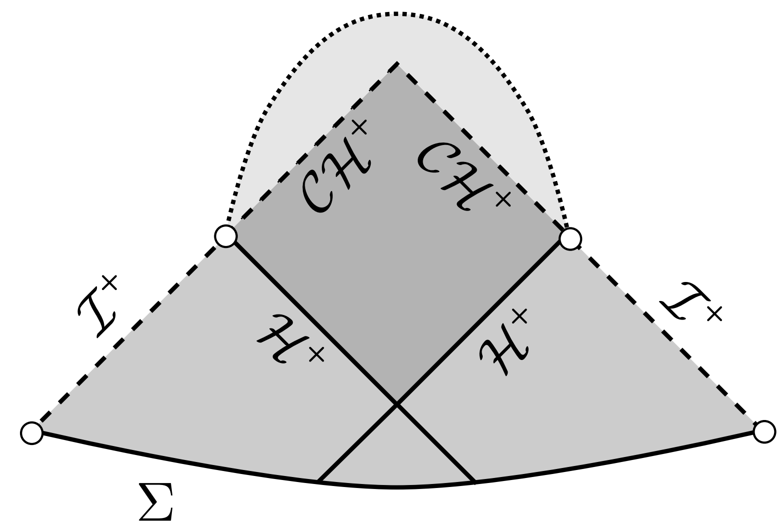

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Which behaviour is preferable, Schwarzschild or Kerr?

Strong cosmic censorship

Conjecture. (R. Penrose, 1973) *The Kerr Cauchy horizon is a fluke! For generic asymptotically flat initial data* $(\Sigma, \bar{g}, K)$ *for the vacuum equations, the maximal future Cauchy development* $(\mathscr{M}, g)$ *is inextendible…*

Strong cosmic censorship
(C^0 formulation)

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This formulation ensures the strong form of determinism that we saw holds for the Schwarzschild solution.

Strong cosmic censorship (C^0 formulation)

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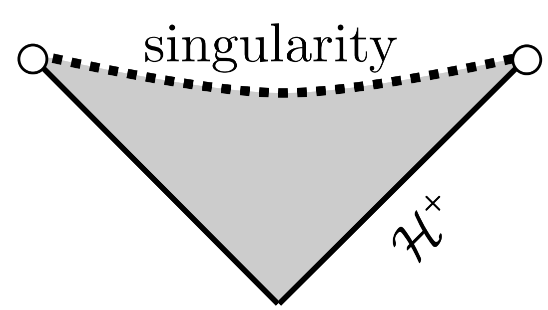
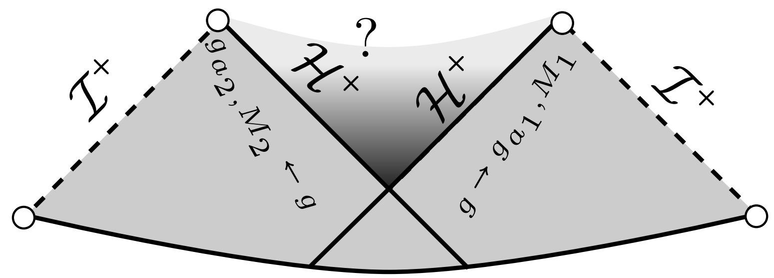
Spacelike singularity conjecture

Conjecture. (R. Penrose) *For generic asymptotically flat initial data for the vacuum equations, the "finite future boundary" of the maximal future Cauchy development is spacelike.*



holds for Schwarzschild

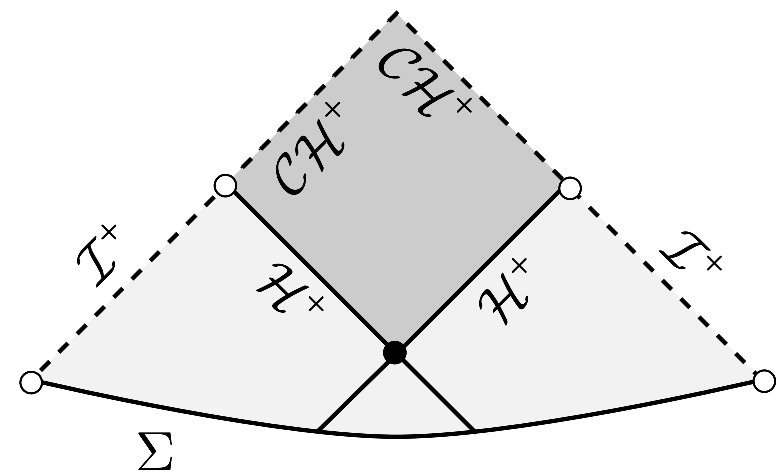
Stability of the Kerr exterior



$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$

Conjecture. *The Kerr family is stable <u>in the exterior</u> as solutions to the vacuum Einstein equations: Small perturbations of (two-ended) Kerr initial data lead to a maximal future Cauchy development with complete null infinity* $\mathscr{I}^+$ *such that in* $J^-(\mathscr{I}^+)$*, in particular on* $\mathscr{H}^+$*, the induced geometry approaches—inverse polynomially—two nearby Kerr solutions.*

Poor man's linear stability of Kerr



Theorem. (M.D.–Rodnianski–Shlapentokh-Rothman 2014)
Consider smooth localised initial data (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on sub-extremal Kerr. Then* ψ *remains uniformly bounded in the exterior region* $J^-(\mathscr{I}^+)$*. Moreover, one has sufficiently fast inverse polynomial decay for* ψ *towards* i^+*, in particular, along the event horizon* $\mathscr{H}^+$*.*

….Andersson–Blue, Blue–Soffer, Blue–Sterbenz, Kay–Wald, M.D.–Rodnianski, Tataru–Tohaneanu, Luk, Moschidis, Shlapentokh-Rothman, Whiting….

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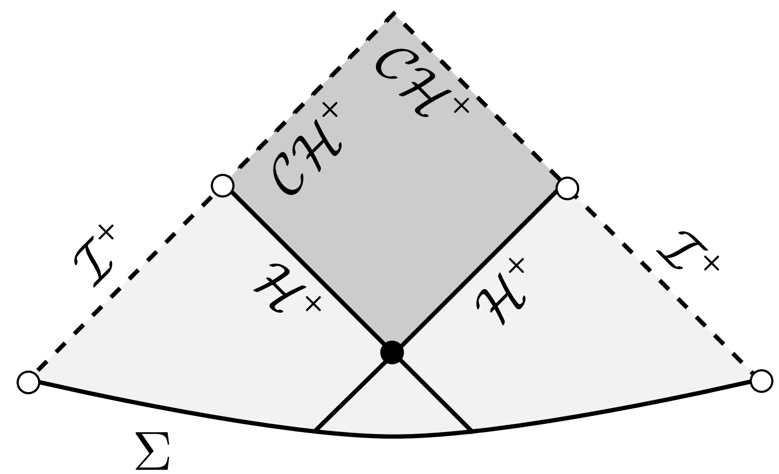

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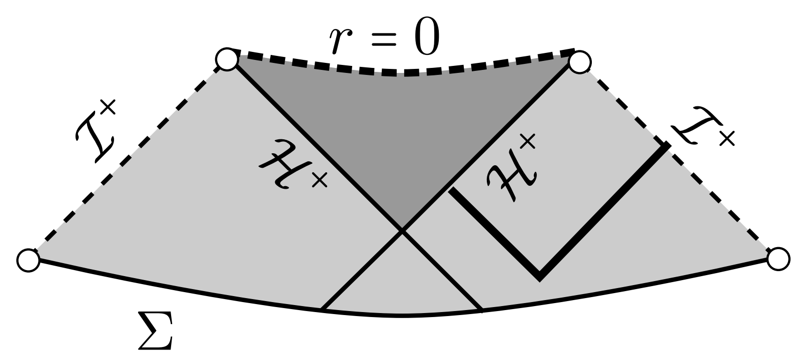
Full linear stability of Schwarzschild



Theorem. (M.D.–Holzegel–Rodnianski 2016) *Consider smooth characteristic initial data for the* ***linearised Einstein equations*** *around Schwarzschild, expressed in double null gauge. Then the arising solution remains uniformly bounded in the exterior region $J^-(\mathscr{I}^+)$ in terms of its initial data. Moreover, after adding a pure gauge solution, which itself is quantitatively controlled by the data, the solution approaches inverse polynomially a standard linearised Kerr metric as i^+ is approached, in particular along $\mathscr{H}^+$.*

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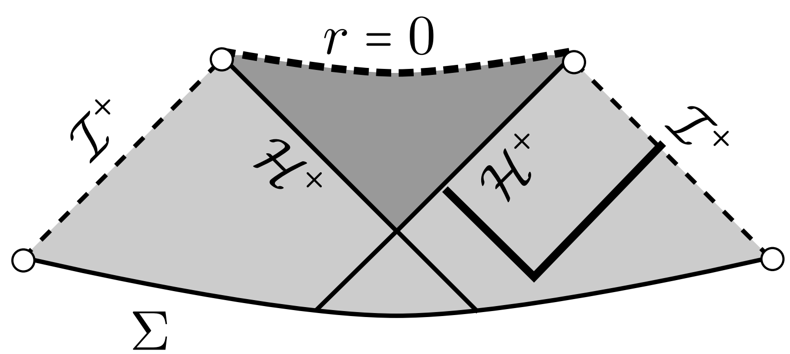

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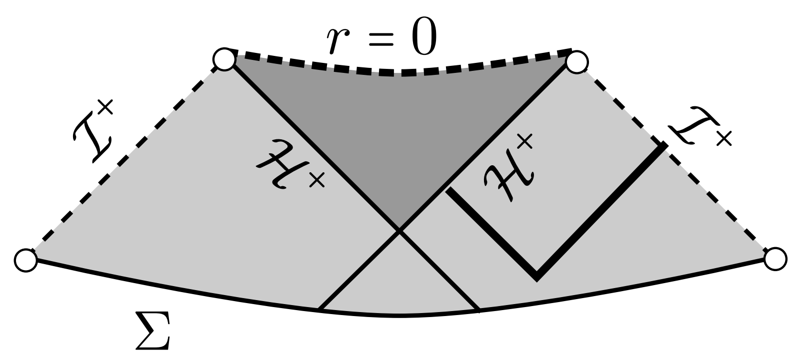

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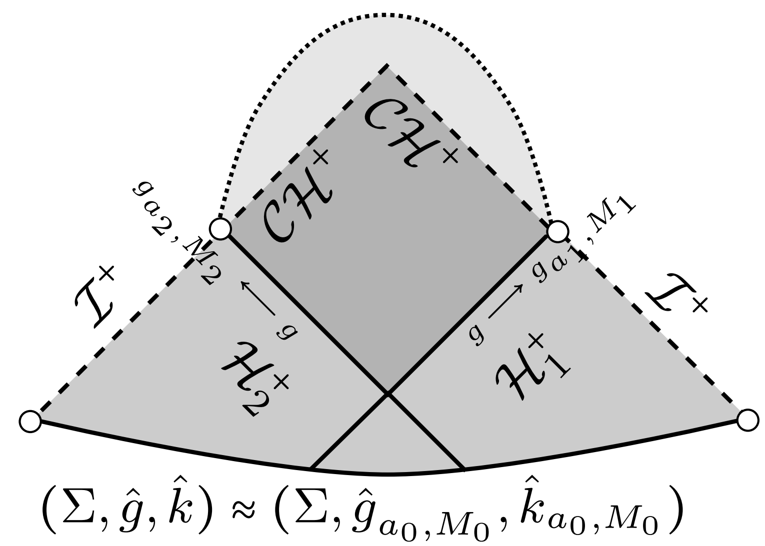
C^0 stability of the Kerr Cauchy horizon



Theorem. (M.D.—J. Luk, 2017) *If Kerr is nonlinearly stable in the black hole exterior (as conjectured), then its Penrose diagram is globally stable, and the metric again extends, at least C^0, across a Cauchy horizon $\mathscr{CH}^+$.*

The C^0 formulation of strong cosmic censorship is false

Corollary. *If Kerr is nonlinearly stable in the black hole exterior (as conjectured), then the C^0 formulation of the strong cosmic censorship conjecture, as well as the spacelike singularity conjecture, are both* ***false***.

References for Lecture 4

- M.D., G. Holzegel and I. Rodnianski “The linear stability of the Schwarzschild solution to gravitational perturbations”, arXiv:1601.06467

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- M.D. and J. Luk “The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon”, arXiv:1710.01772

- B. O’Neil “The Geometry of Kerr Black Holes”, Dover Books on Physics

- R. Penrose “Gravitational collapse” In C. Dewitt-Morette, editor, Gravitational Radiation and Gravitational Collapse, volume 64 of IAU Symposium, pages 82–91. Springer, 1974.

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Lecture 5

The analysis of waves on Kerr black hole interiors

Review of Kerr spacetime

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Review of Kerr spacetime

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OF ALGEBRAICALLY SPECIAL METRICS

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University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio
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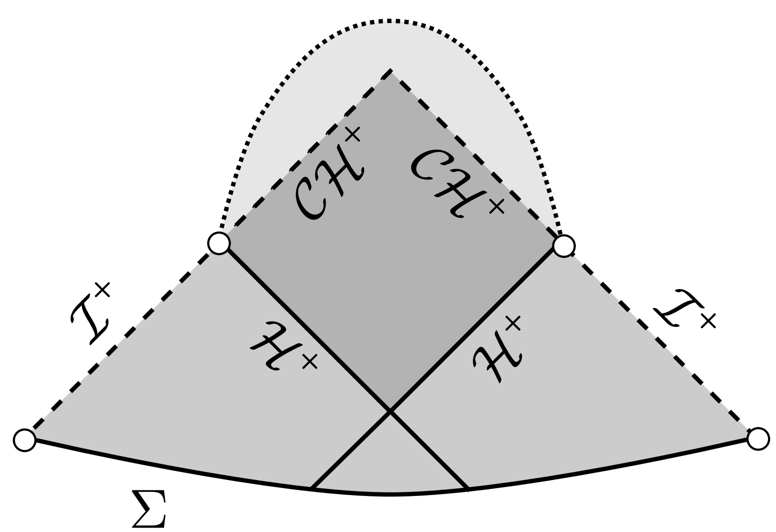
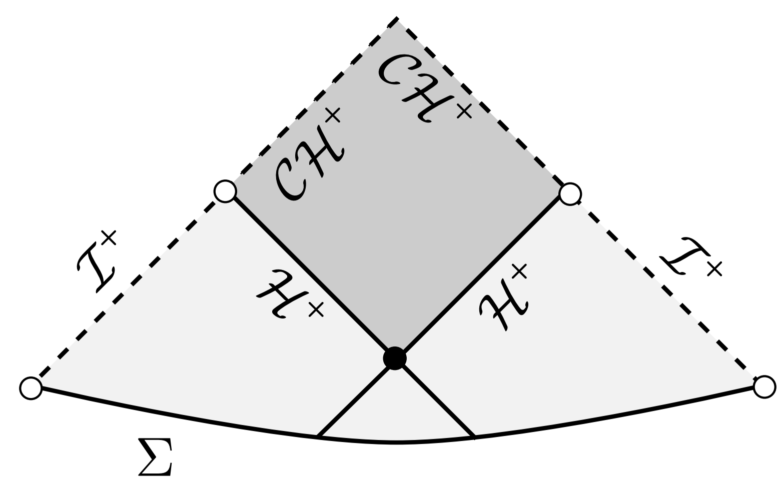


The main problem



Main problem. *Consider smooth localised initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on subextremal rotating Kerr. Understand the global properties of* ψ *in the darker shaded black hole interior region, in particular, the behaviour at* $\mathscr{CH}^+$.

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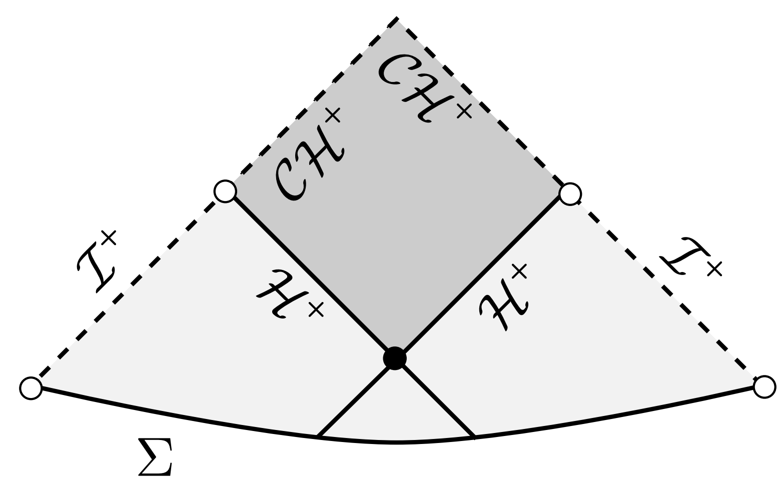

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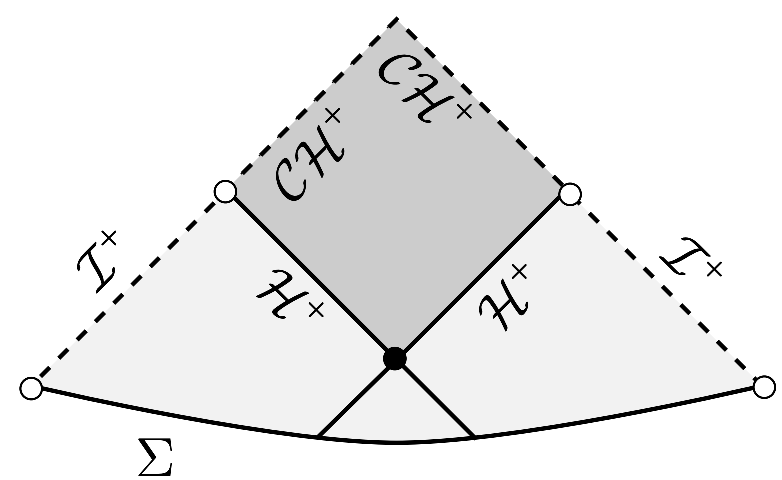

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….Andersson–Blue, Blue–Soffer, Blue–Sterbenz, Kay–Wald, M.D.–Rodnianski, Tataru–Tohaneanu, Luk, Moschidis, Shlapentokh-Rothman, Whiting….

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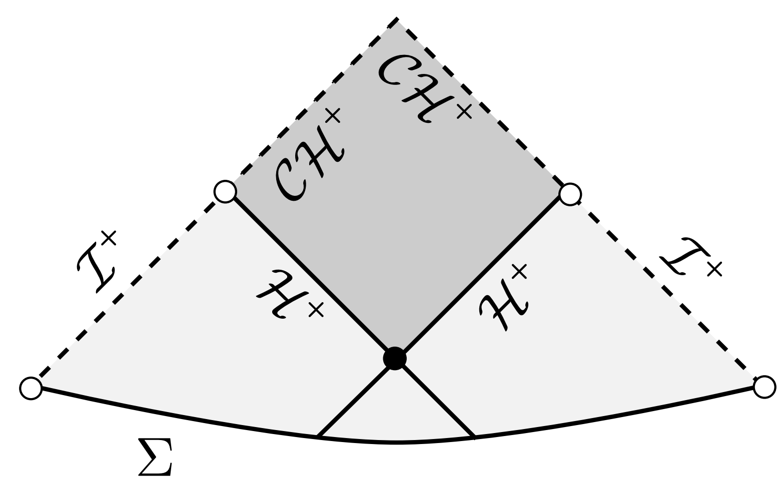

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Reissner–Nordström

- discovered independently by Reissner, Nordström 1916/8
- $g = -(1 - 2M/r + e^2/r^2)dt^2 + (1 - 2M/r + e^2/r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$
- explicit solution of the Einstein–Maxwell equations
- spherically symmetric, static: Killing fields $\Omega_1, \Omega_2, \Omega_3, T = \partial_t$
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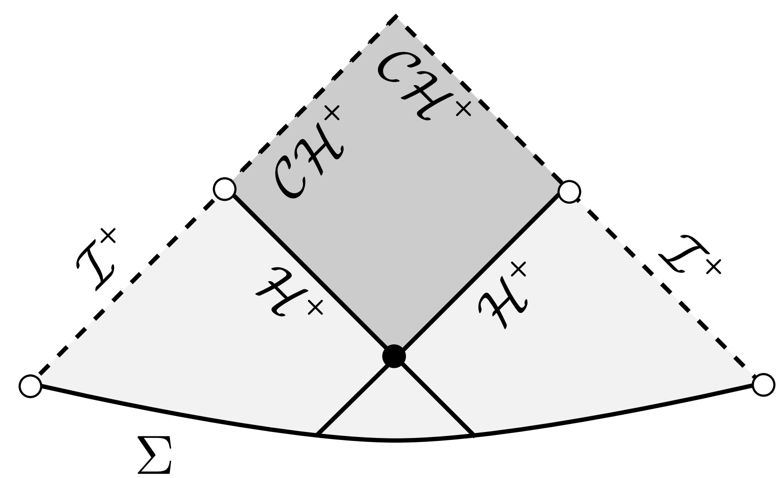

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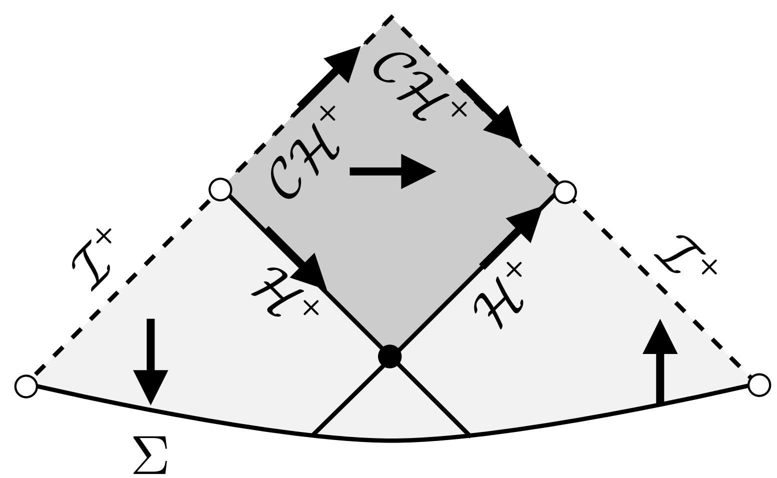
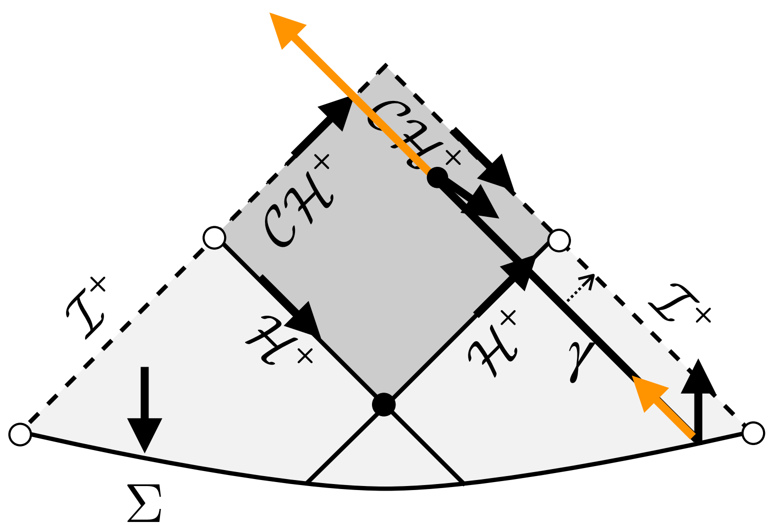
The T vector field



The blue-shift instability (Penrose)



Normalising $g(\dot\gamma, T) = -1$ at Σ, since $\dot\gamma(g(\dot\gamma, T)) = 0$, it follows that *as the geodesic γ is moved to the right*, then $\dot\gamma \to -\infty T$ at its future endpoint.

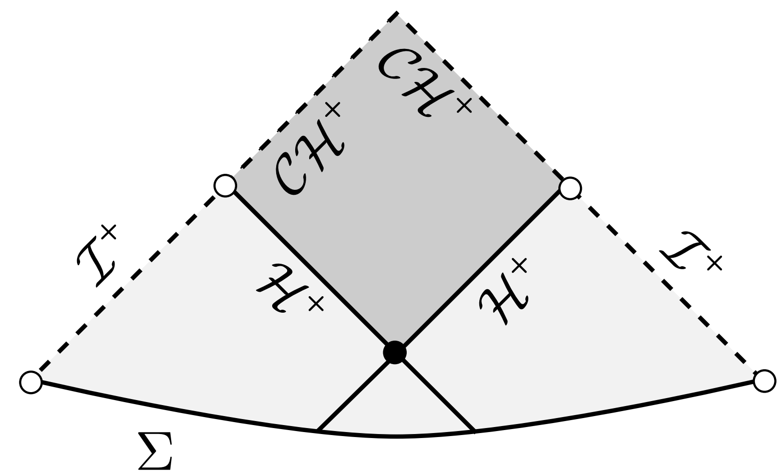
Gaussian beam approximation

Corollary. (Sbierski) *Consider generic finite energy initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on subextremal Kerr or Reissner–Nordström. Then the local energy of* ψ *blows up at* $\mathcal{CH}^+$*, i.e.* ψ *is inextendible in* H^1_{loc} *across* $\mathcal{CH}^+$*.*

Generic blow up for smooth localised data

Theorem. (Luk–Oh, M.D.–Shlapentokh-Rothman, Luk–Sbierski) *Consider* *<u>generic</u>* *smooth localised* *initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on subextremal Kerr or Reissner–Nordström. Then the local energy of* ψ *blows up at* $\mathcal{CH}^+$*, i.e.* ψ *is inextendible in* H^1_{loc} *across* $\mathcal{CH}^+$*.*

C^0 stability



Theorem. (Franzen) *Consider smooth localised initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on rotating, sub-extremal Reissner–Nordström or Kerr. Then the solution* ψ *remains uniformly bounded on the black hole interior and extends continuously* C^0 *to the bifurcate Cauchy horizon* $\mathcal{CH}^+$.

See also Luk–Sbierski, Hintz

Review: vector field multipliers

$$T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi$$

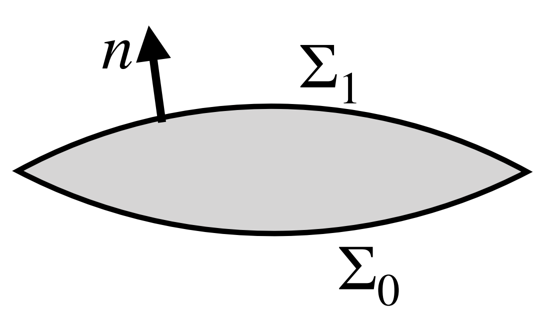
$$\Box_g \psi = 0 \implies \nabla^\mu T_{\mu\nu}[\psi] = 0$$

X vector field $\longrightarrow$

$$J^X_\mu[\psi] \doteq T_{\mu\nu}[\psi]X^\nu$$

$$K^X[\psi] \doteq T_{\mu\nu}[\psi]^{(X)}\pi^{\mu\nu} = 0 \text{ if } K \text{ is Killing}$$

$$\nabla^\mu J^X_\mu[\psi] = K^X[\psi]$$



Σ_t spacelike, X future timelike

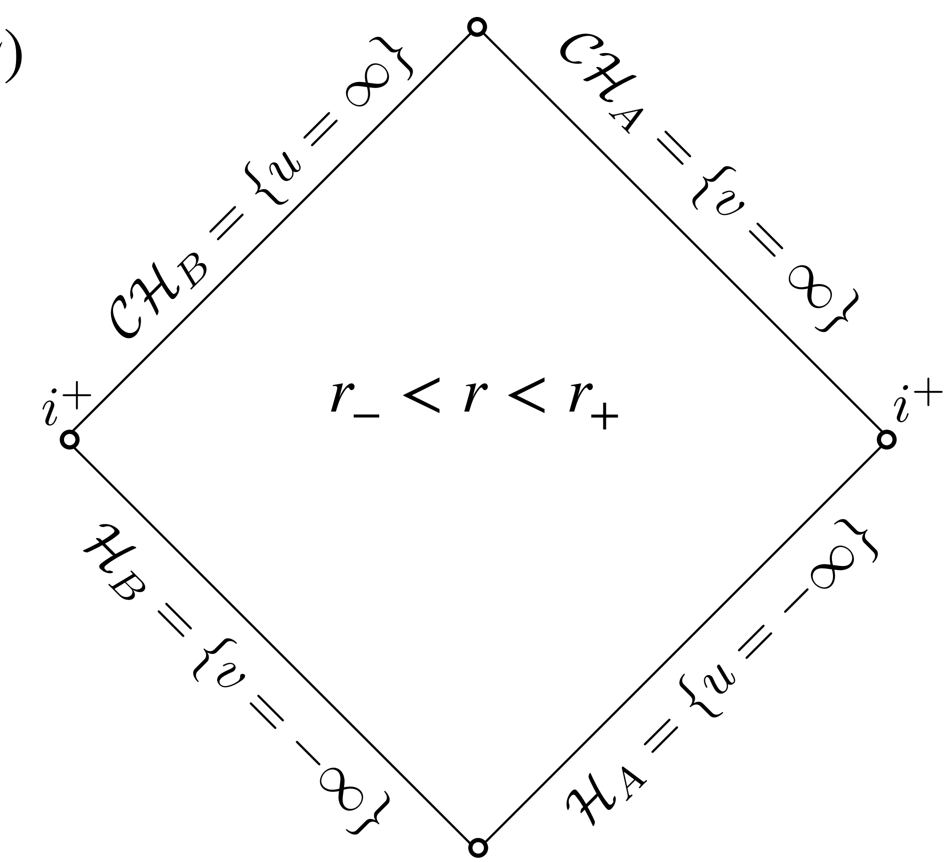
$\implies \int_{\Sigma_1} J^X_\mu[\psi]n^\mu dV_{\Sigma_1}$ coercive

Eddington–Finkelstein normalised null coordinates

$$g = (1 - 2M/r + e^2/r^2)dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$r_\pm = M^2 \pm \sqrt{M^2 - e^2}$$

$$\kappa_\pm = (r_\pm)^{-2}(r_+ - r_-)$$ surface gravities

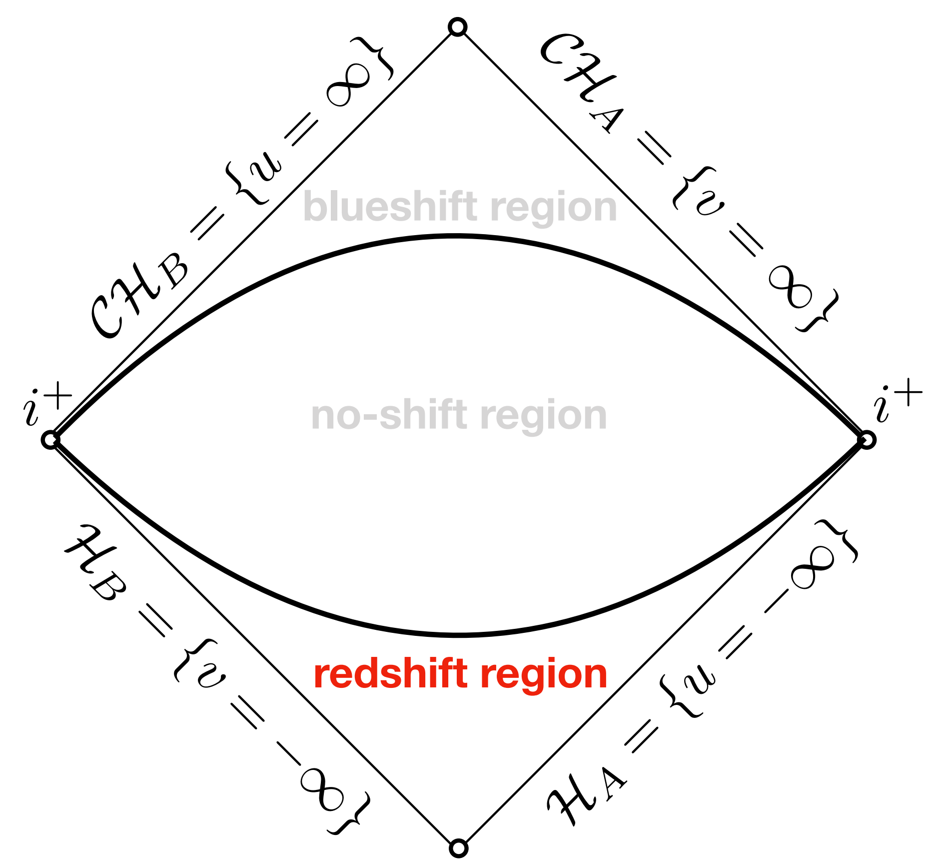


From previous theorem we know initially:

$$\int_{\mathscr{H}_A \cap \{v_* \le v \le v_* + 1\}} (\partial_v \psi)^2 dv \sin\theta d\theta d\phi \le v_*^{-2-2\delta}$$

Three regions

- red-shift region
- “no-shift” region
- blue-shift region



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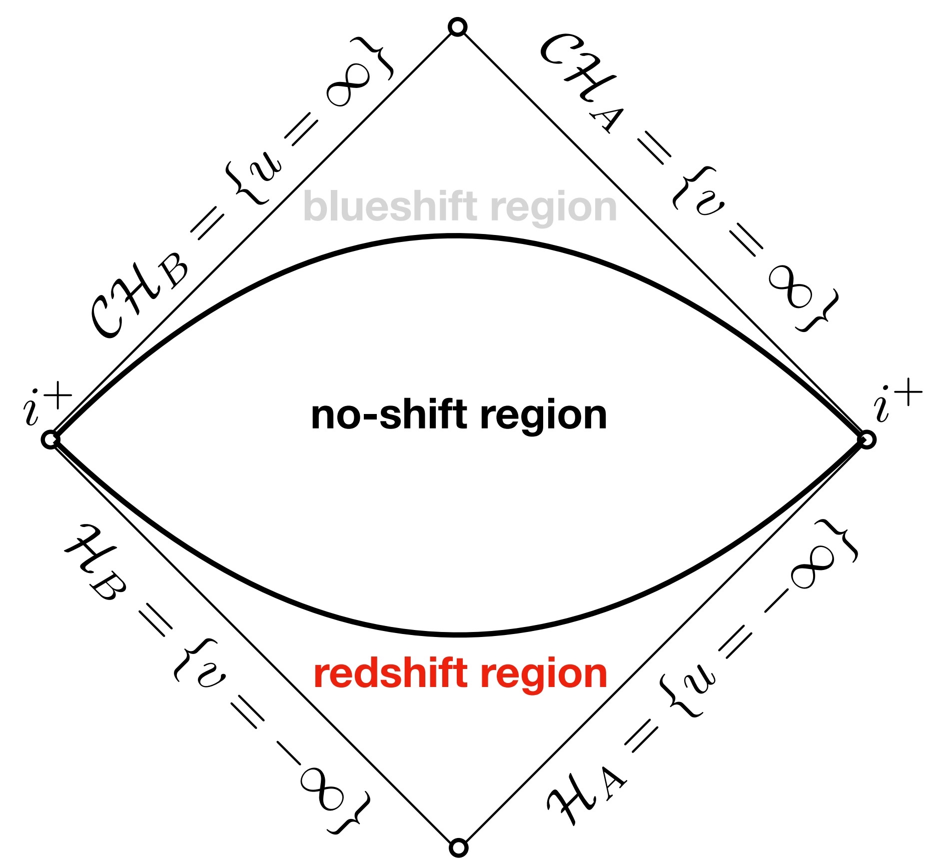

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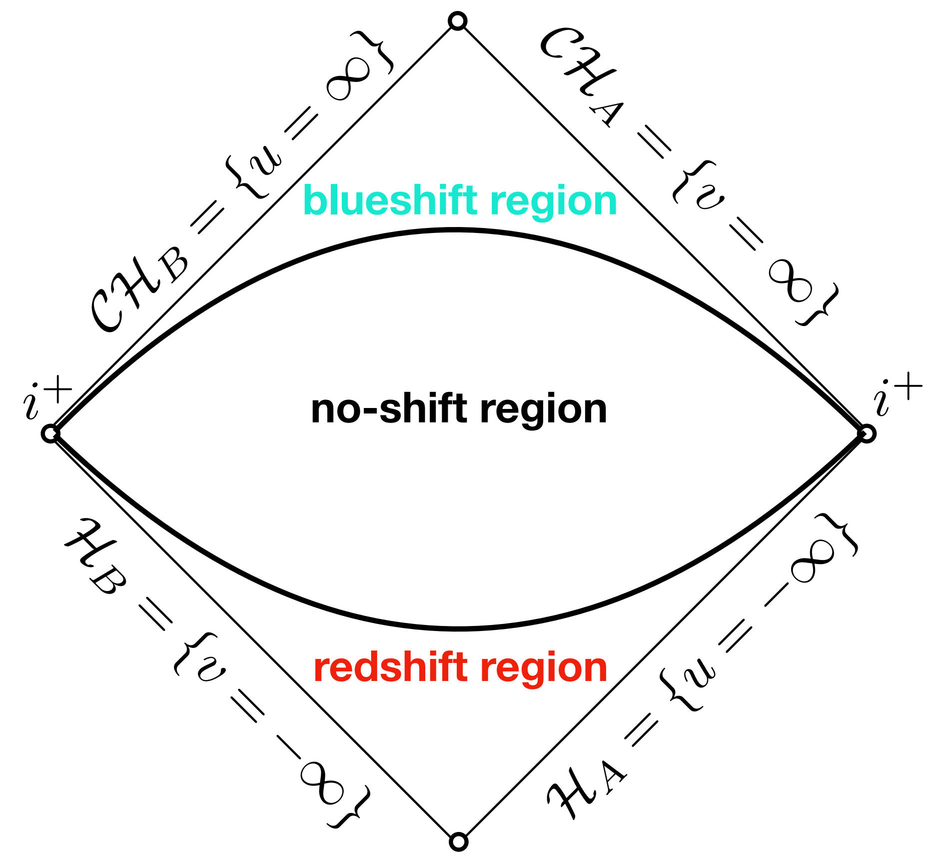
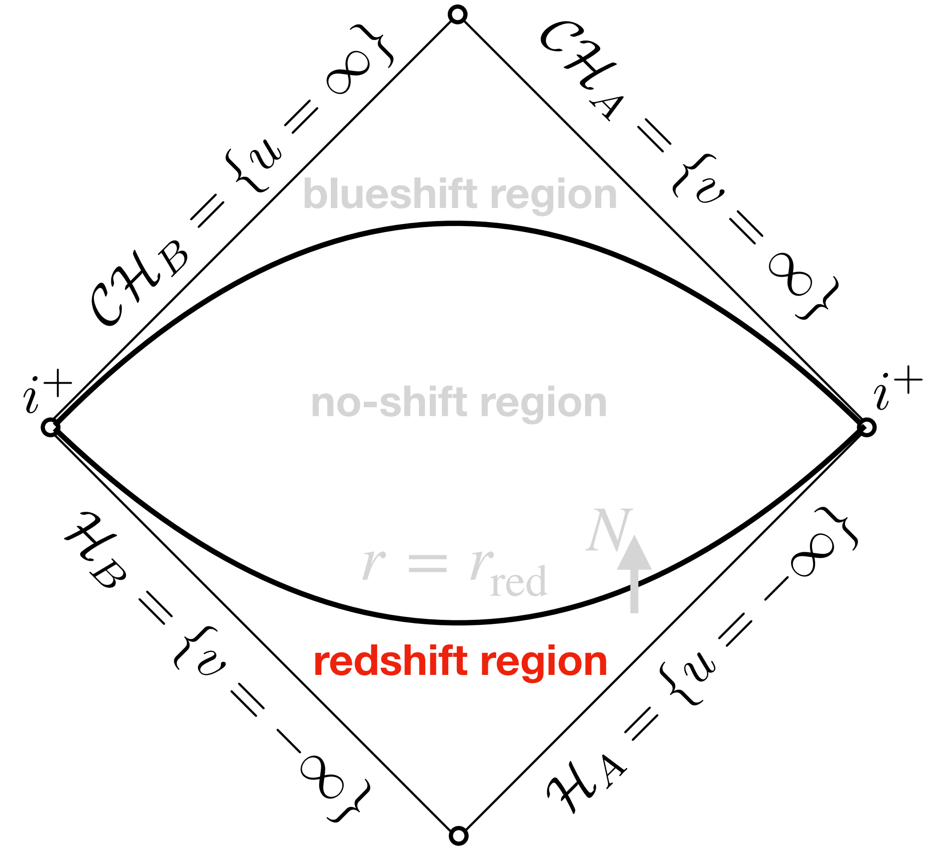


The red-shift region

- well chosen T-invariant timelike vector field N
- bulk coercivity $K^N[\psi] \gtrsim J^N_\mu[\psi]N^\mu$ for $r \leq r_{\rm red}$
- $\implies$ polynomial decay propagates to $r = r_{\rm red}$
- only uses positivity of surface gravity κ_+
- on $\mathscr{H}^+$; $\nabla_T T = \kappa_+ T$



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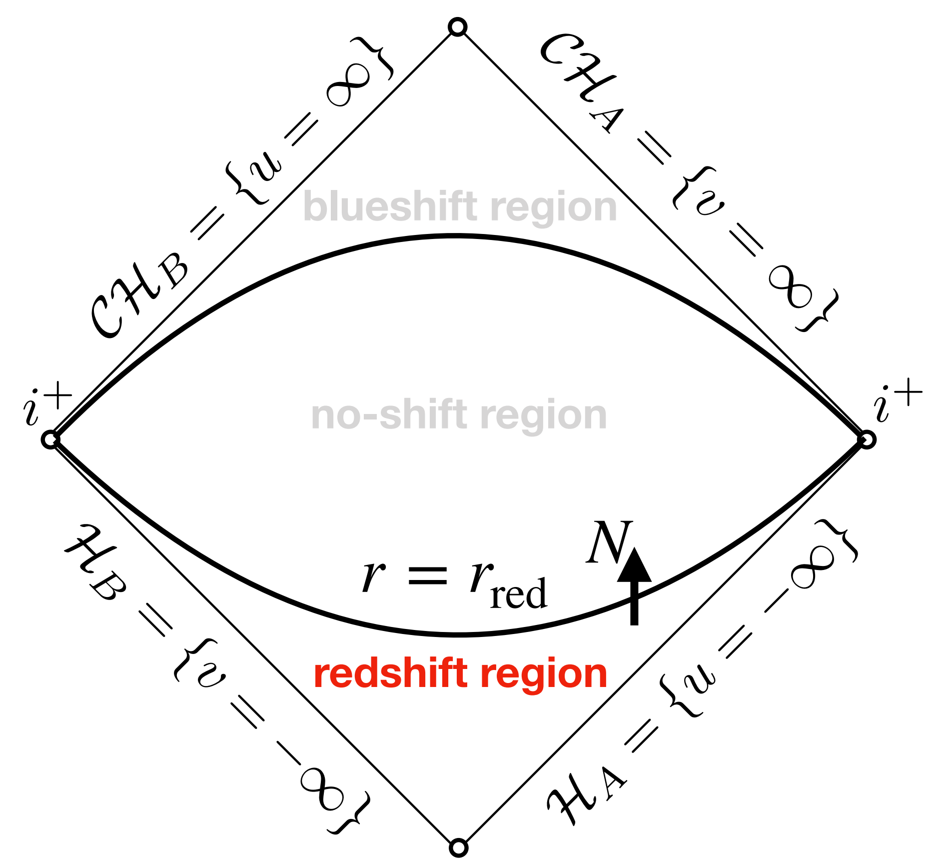

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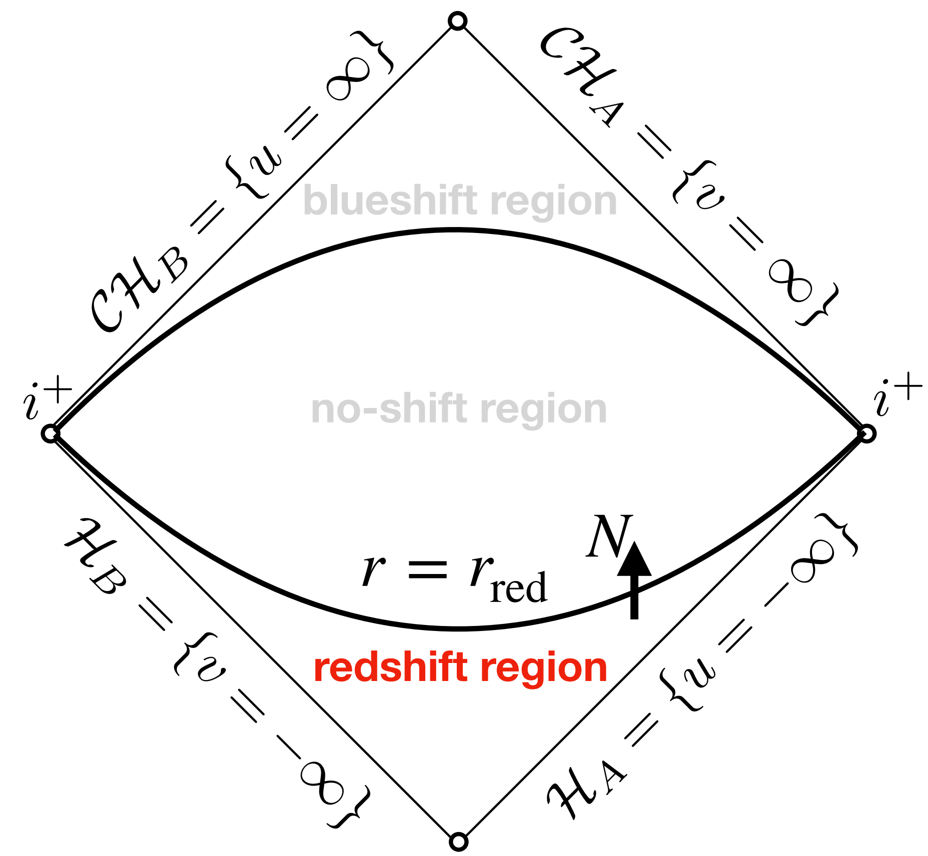

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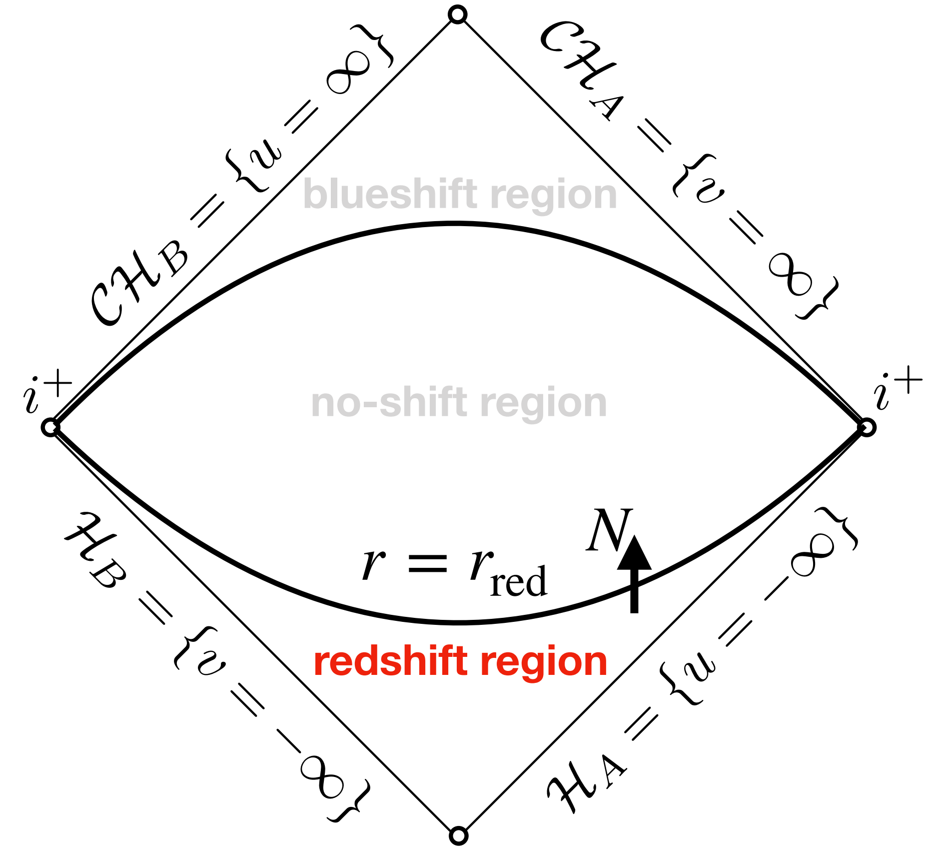

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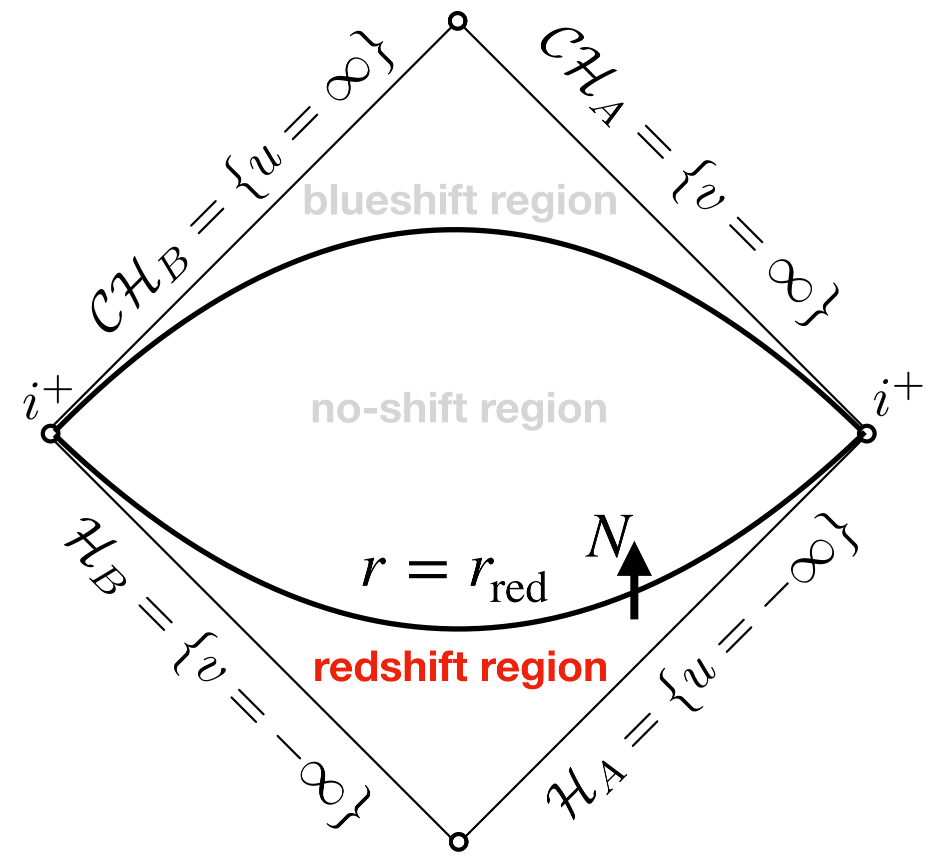

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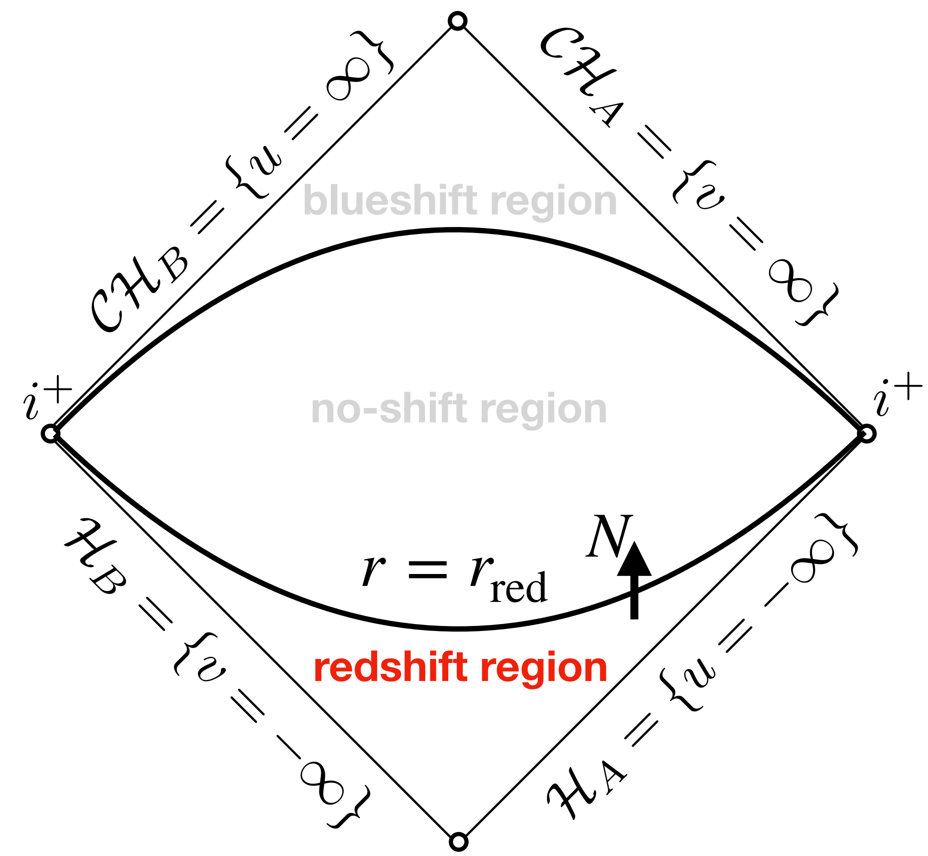

The no-shift region

- use $X = r^{2N}(\partial_u + \partial_v)$ with $N \gg 1$
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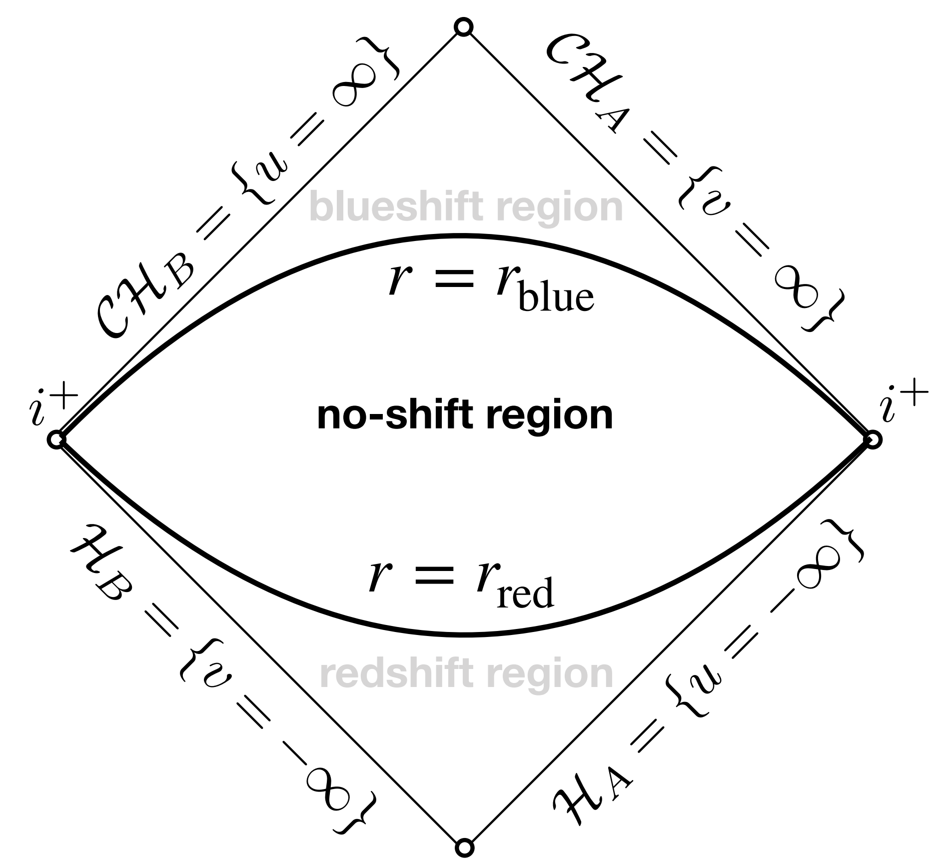

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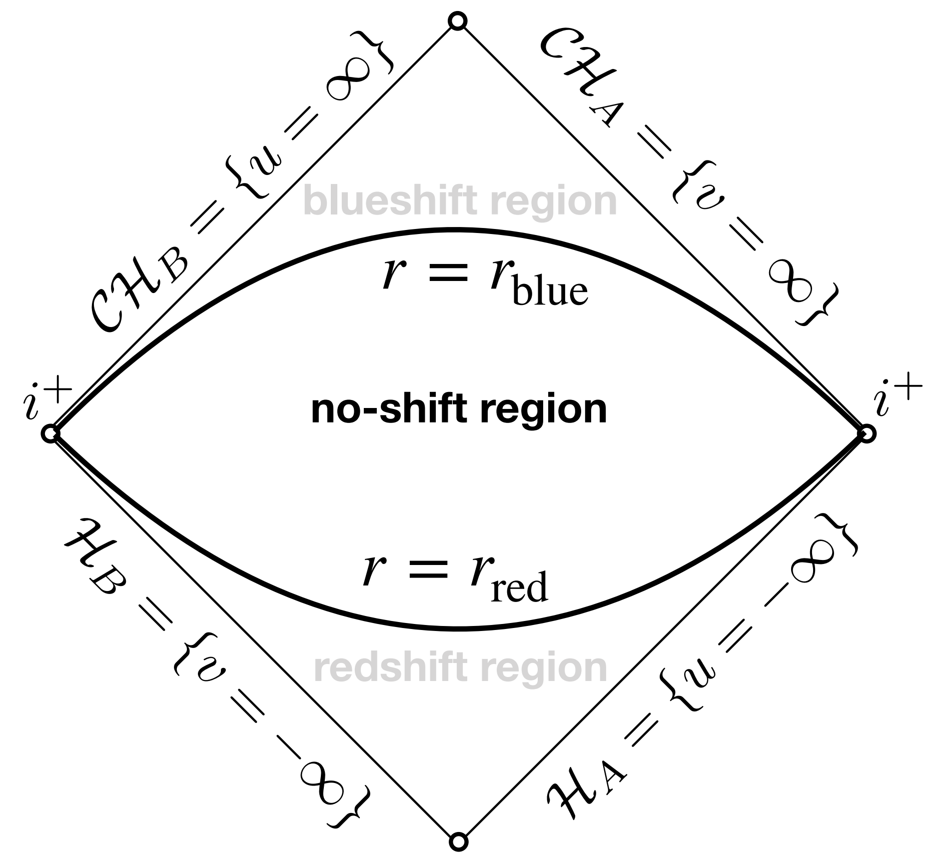

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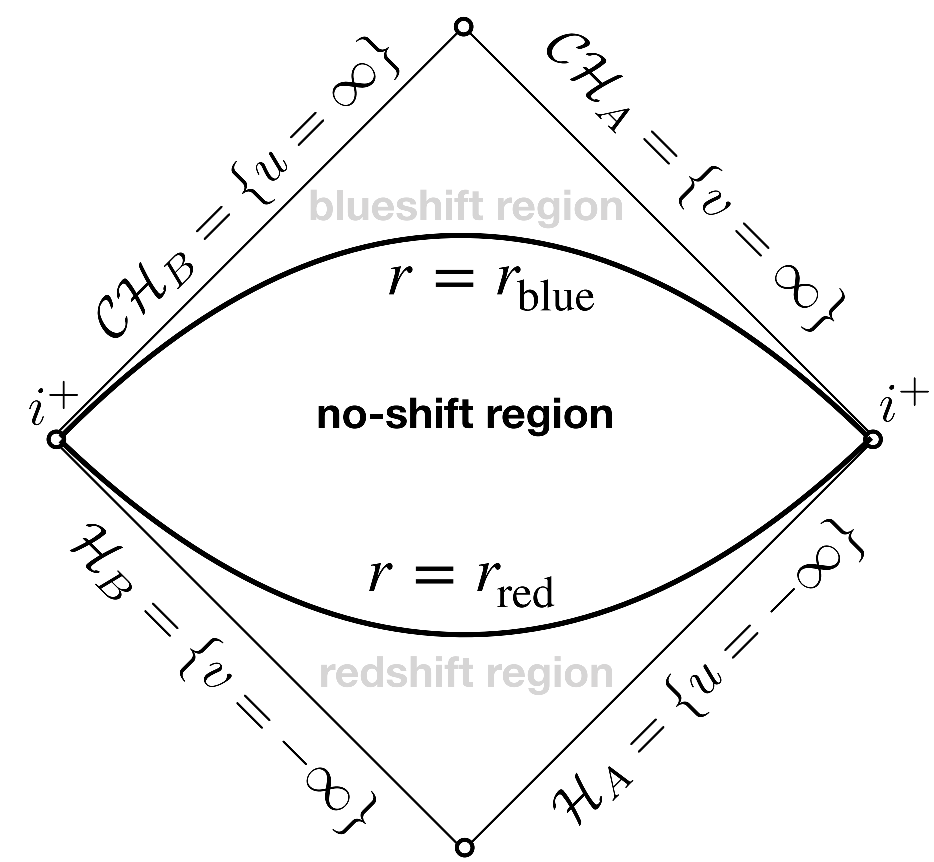

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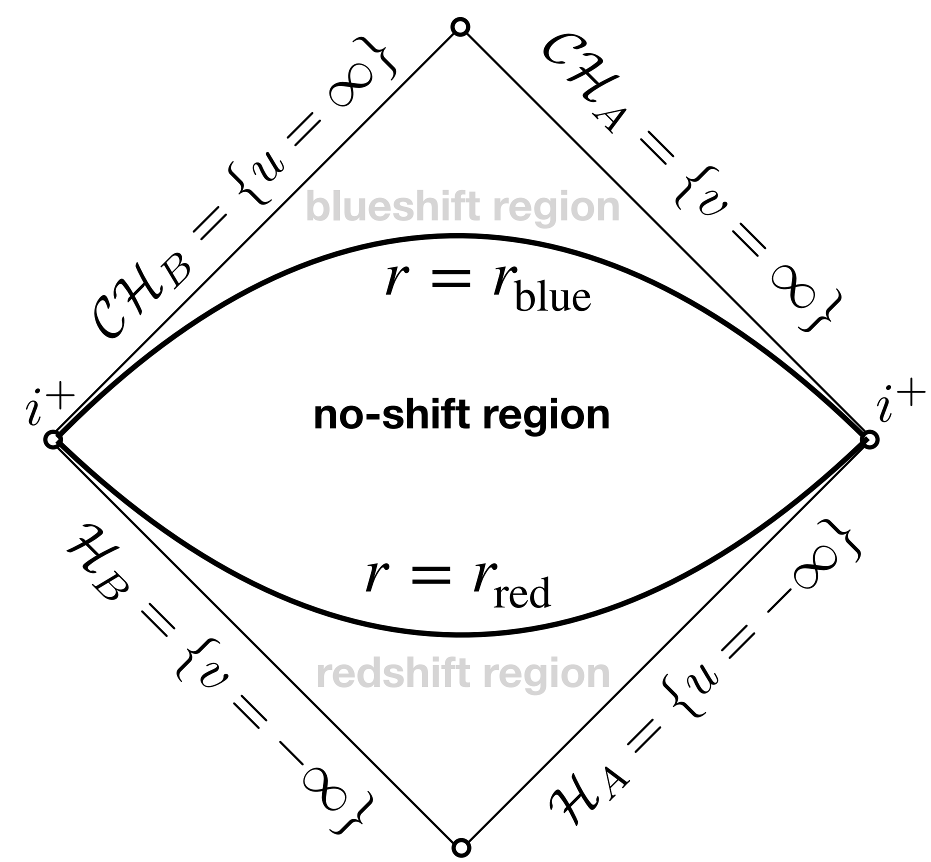
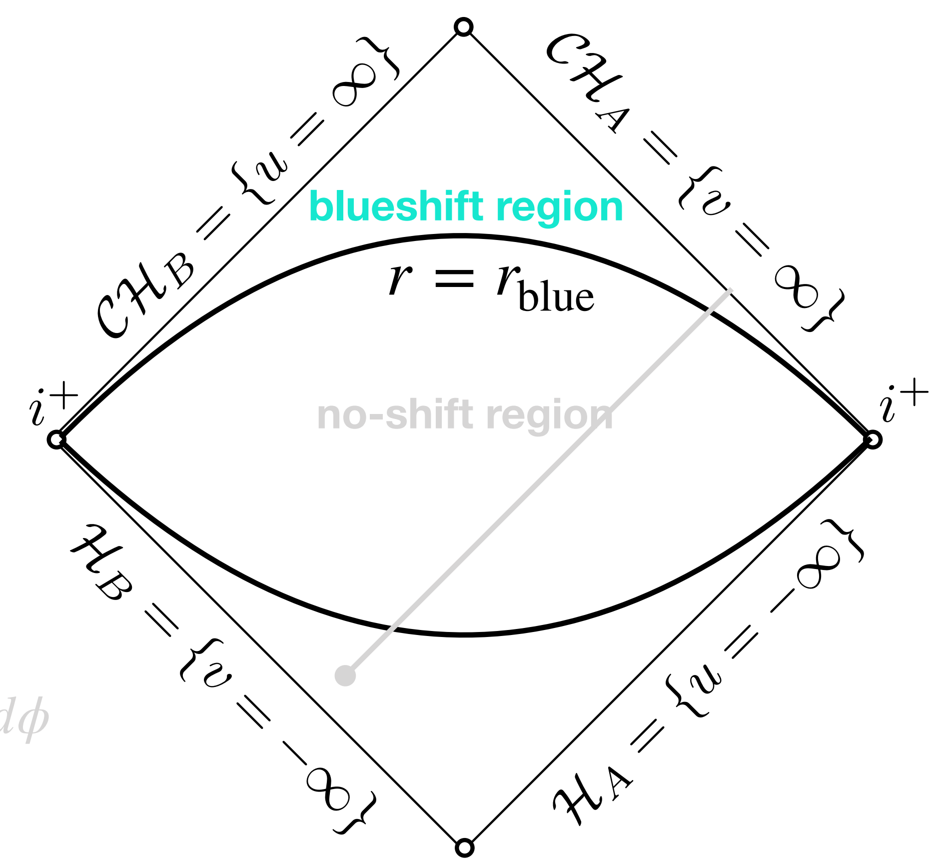


The blue-shift region

- use $X = r^{2N}(u^p \partial_u + v^p \partial_v)$ with $p > 1$
- bulk non-negativity of the worst terms
- initial flux on $r = r_{\text{blue}}$ bounded from previous
- obtain finally bounds for $\int_{u=const} v^p (\partial_v \psi)^2 dv \sin\theta d\theta d\phi$



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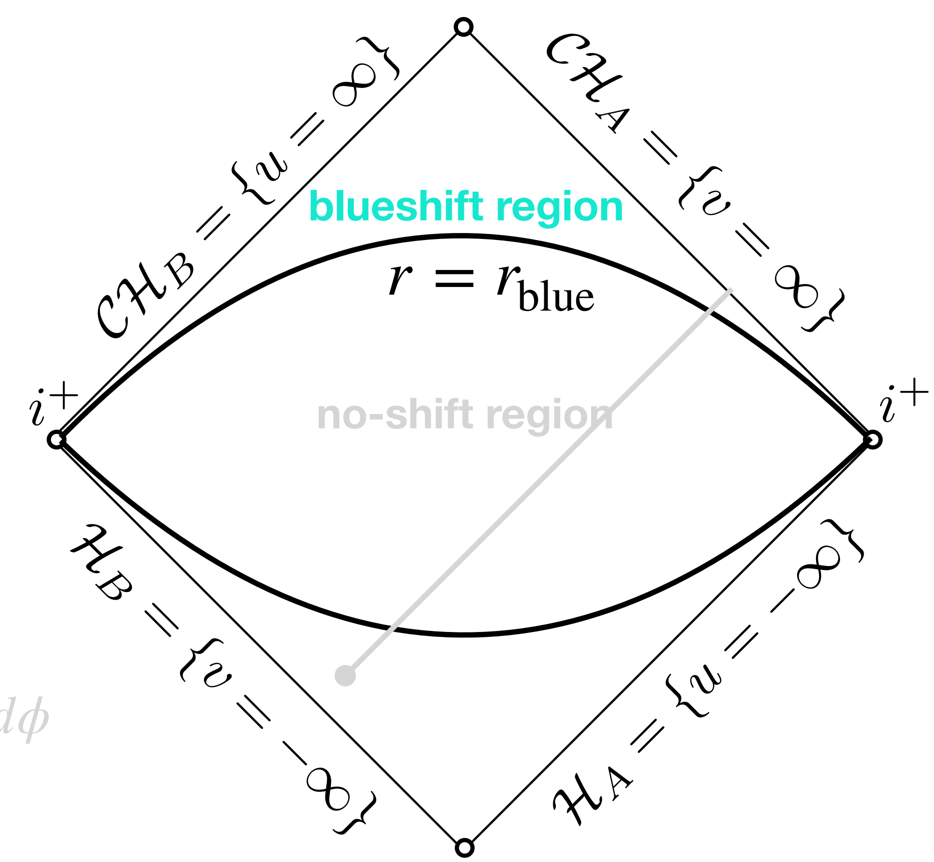

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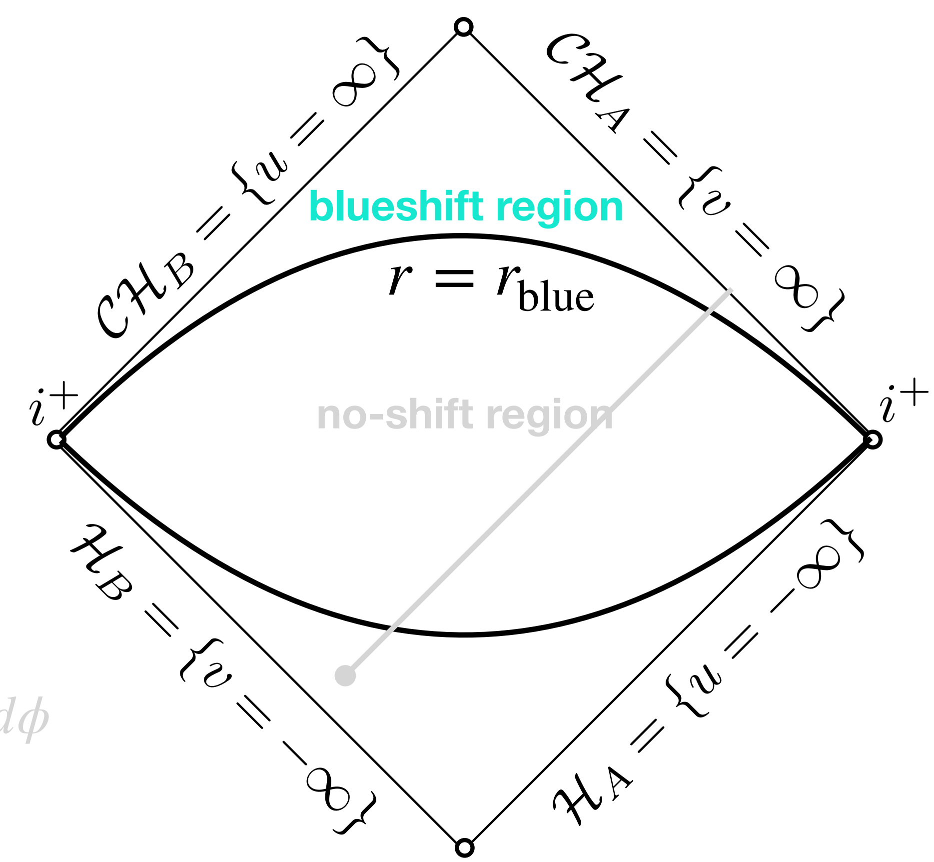

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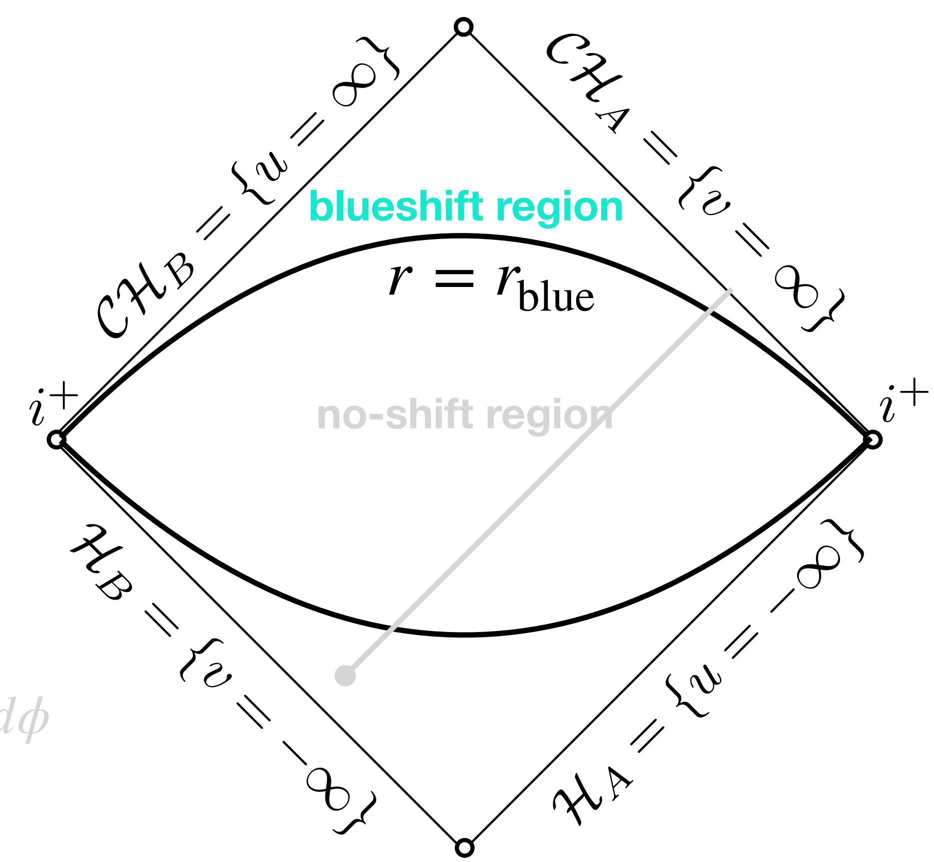

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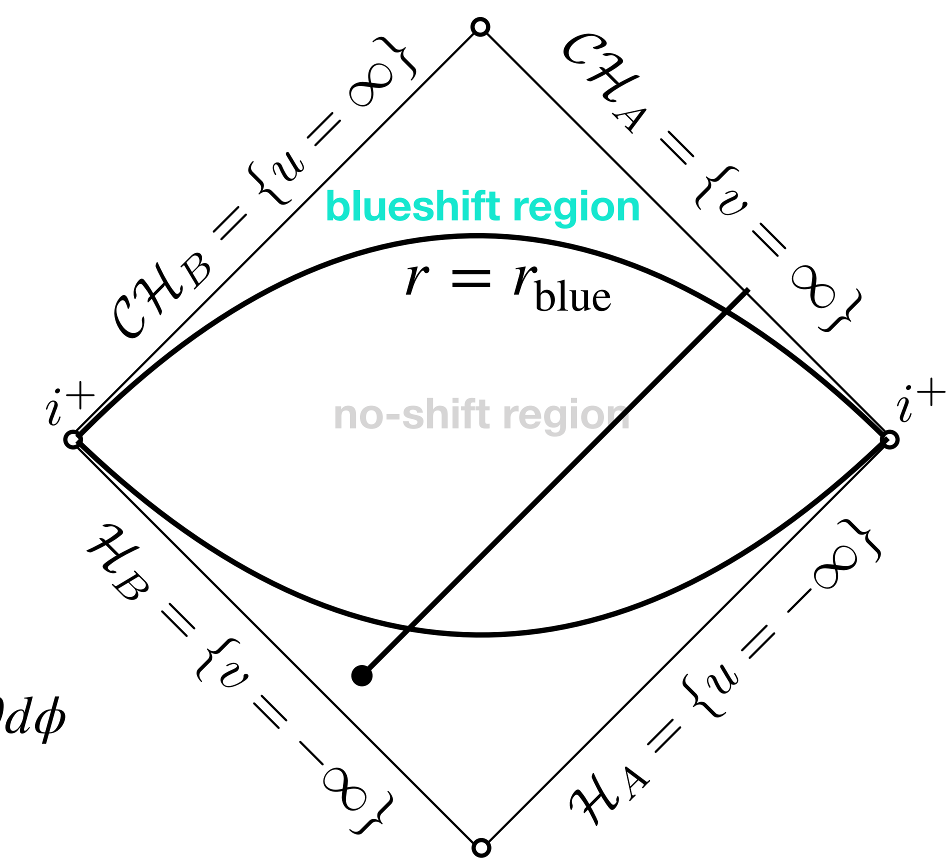
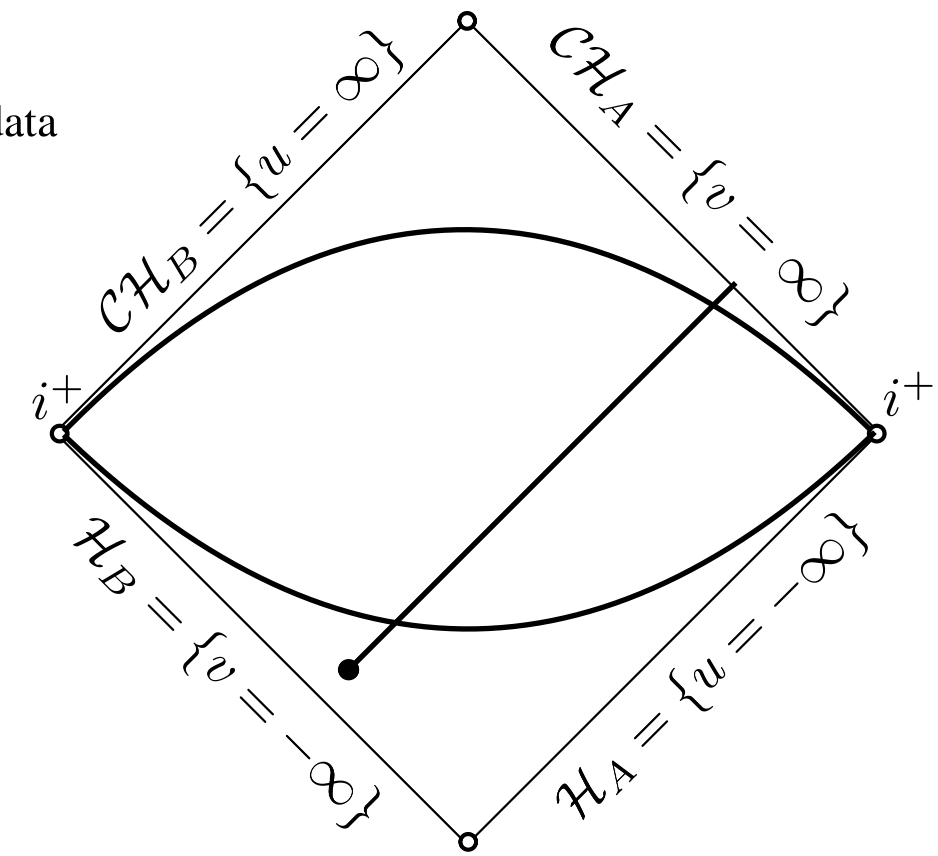


Deducing C^0 bounds

$$|\psi|^2 \leq \left(\int_1^\infty |\partial_v \psi|\right)^2 + \text{data} \leq \left(\int_1^\infty v^p |\partial_v \psi|^2 dv\right)\left(\int_1^\infty v^{-p} dv\right) + \text{data}$$

- now integrate the above over the spheres
- Since $p > 1$, the second factor is finite
- commute with $\mathbf{\Omega}_i$
- and apply Sobolev on the spheres



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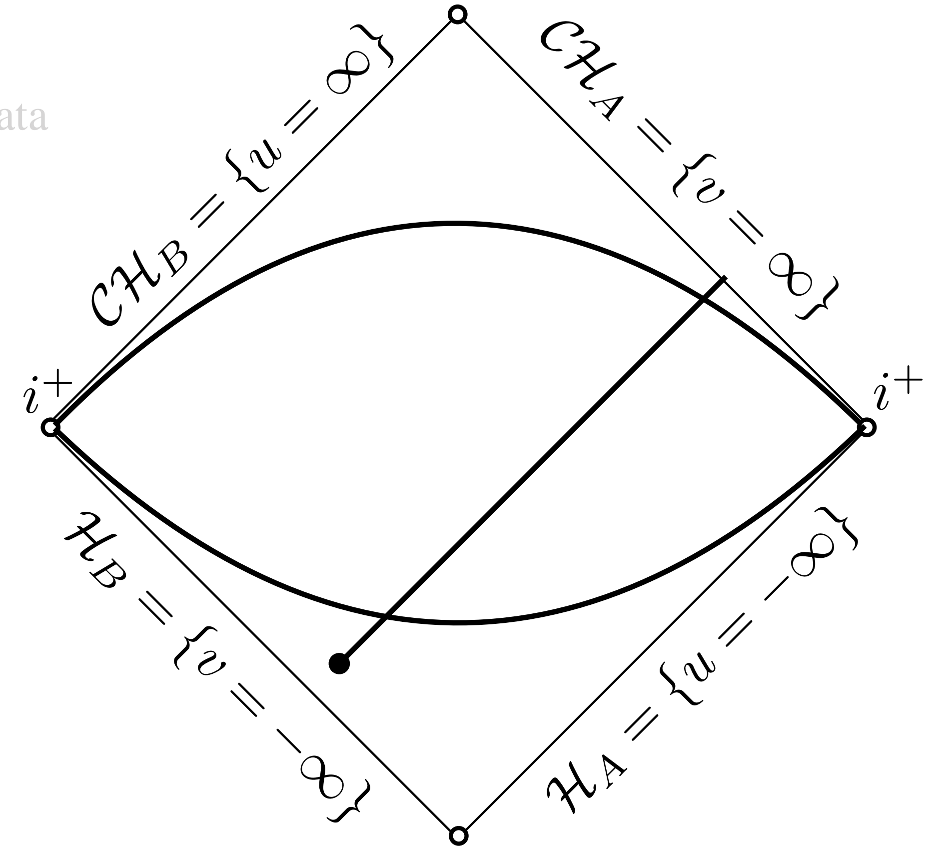
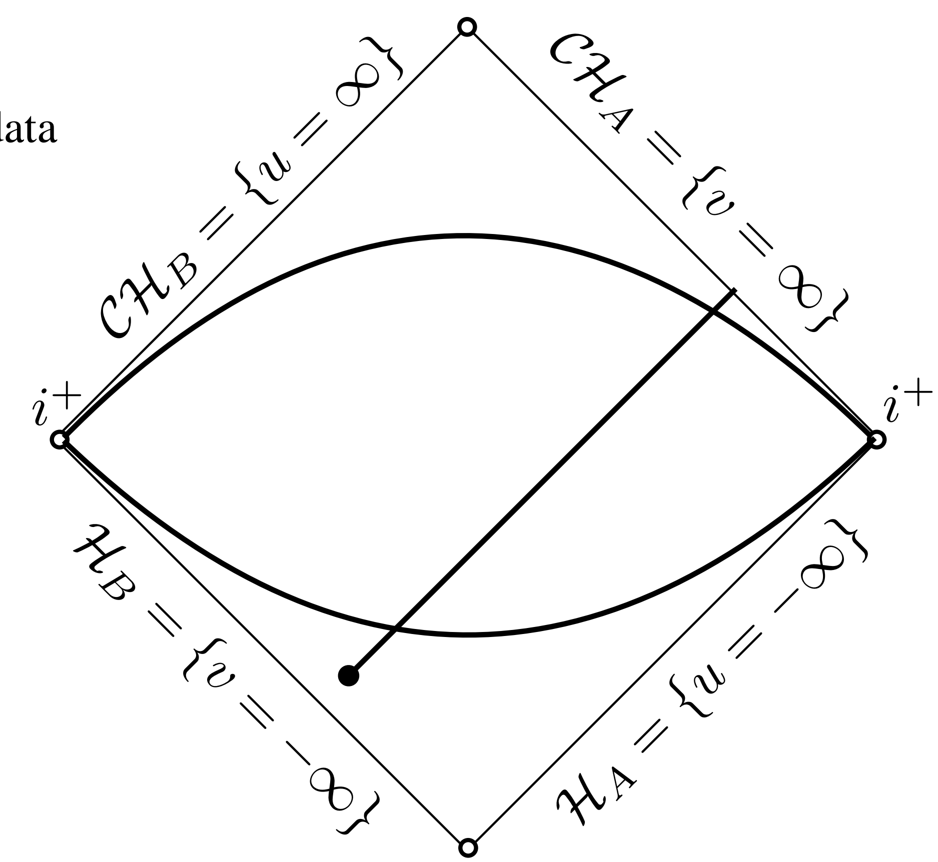


Deducing C^0 bounds

$$|\psi|^2 \leq \left(\int_1^\infty |\partial_v \psi|\right)^2 + \text{data} \leq \left(\int_1^\infty v^p |\partial_v \psi|^2 dv\right)\left(\int_1^\infty v^{-p} dv\right) + \text{data}$$



- now integrate the above over the spheres
- Since $p > 1$, the second factor is finite
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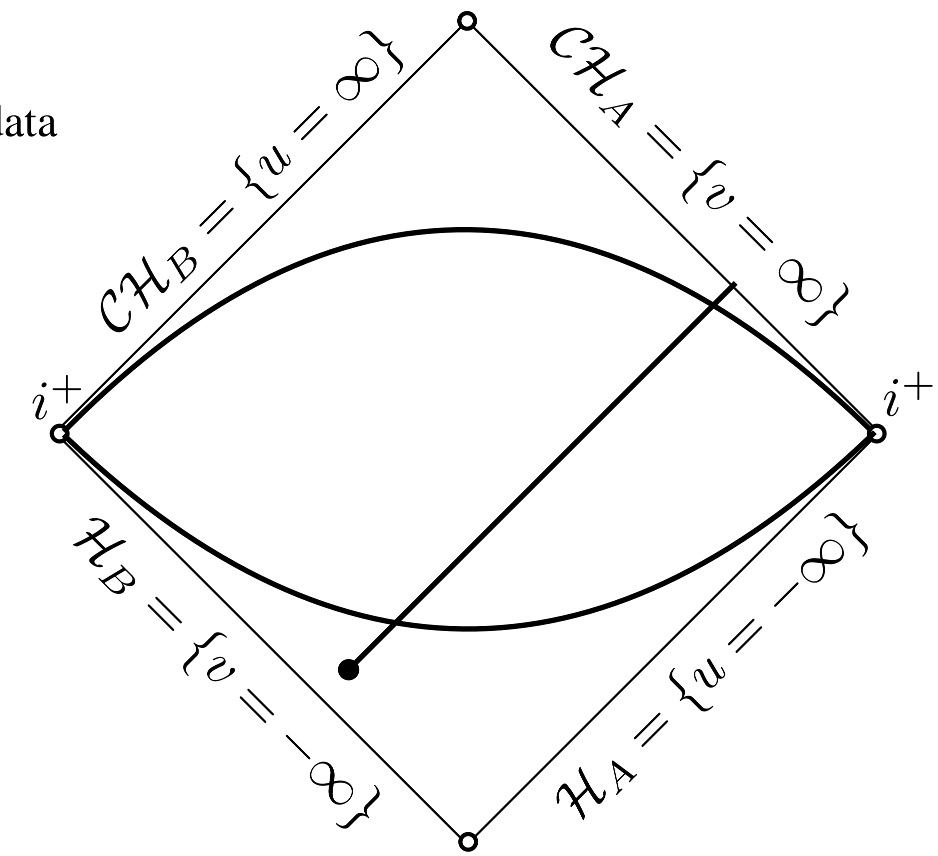

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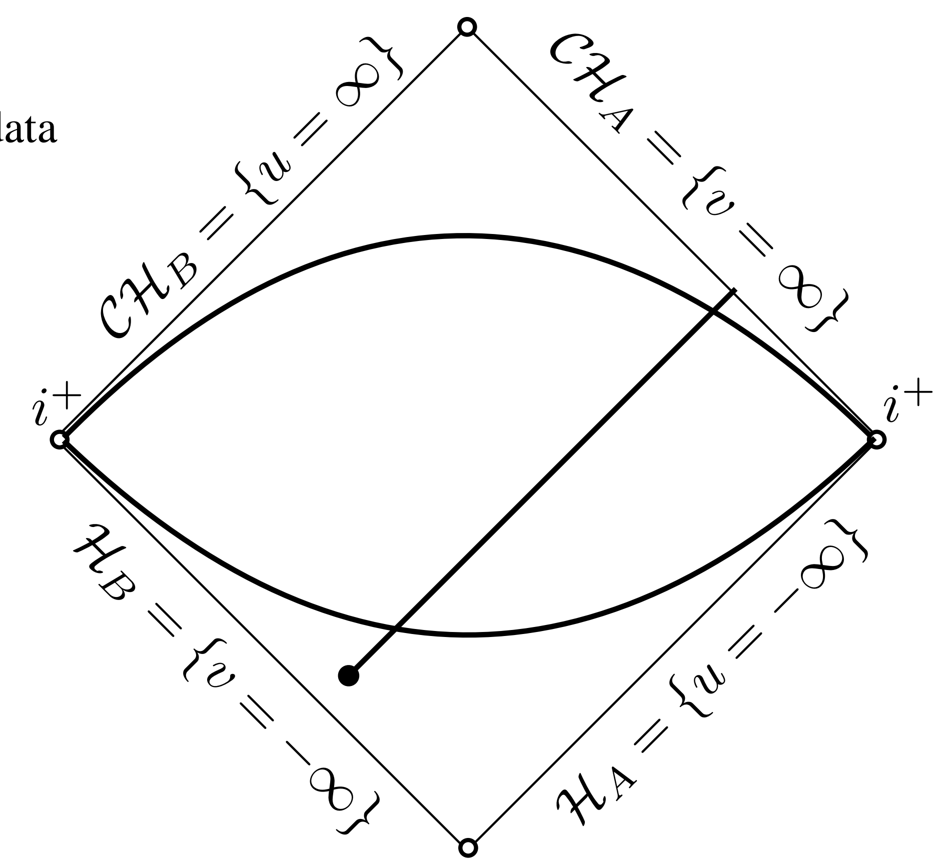

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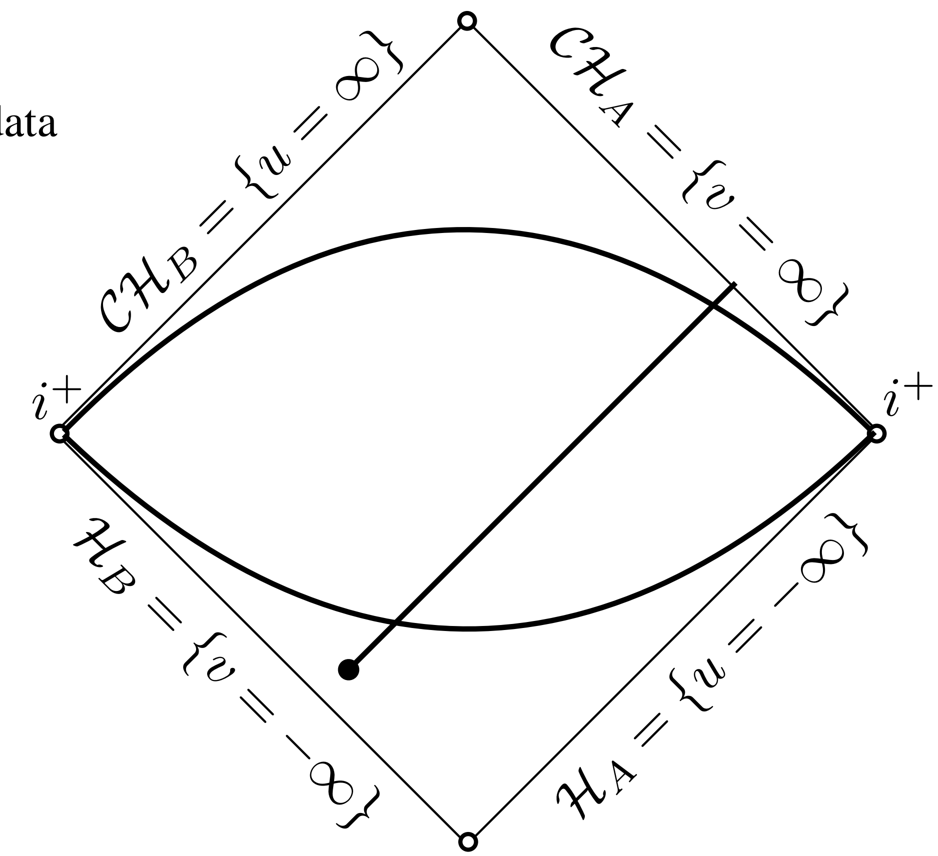

The Kerr case

- use double null foliation
- operators tangential to the sphere no longer commute
- additional error terms in the bulk can be absorbed by Gronwall

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The main theorem quoted in Lecture 4 can be thought of as a fully nonlinear analogue of the previous result.

Strong cosmic censorship (Christodoulou formulation)

Conjecture. (R. Penrose, 1973) *The Kerr Cauchy horizon is a fluke! For generic asymptotically flat initial data* $(\Sigma, \bar{g}, K)$ *for the vacuum equations, the maximal future Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a* C^0 *Lorentzian manifold with locally square integrable Christoffel symbols.*

This formulation is sufficiently strong to assure that there is no extension even as a weak solution.

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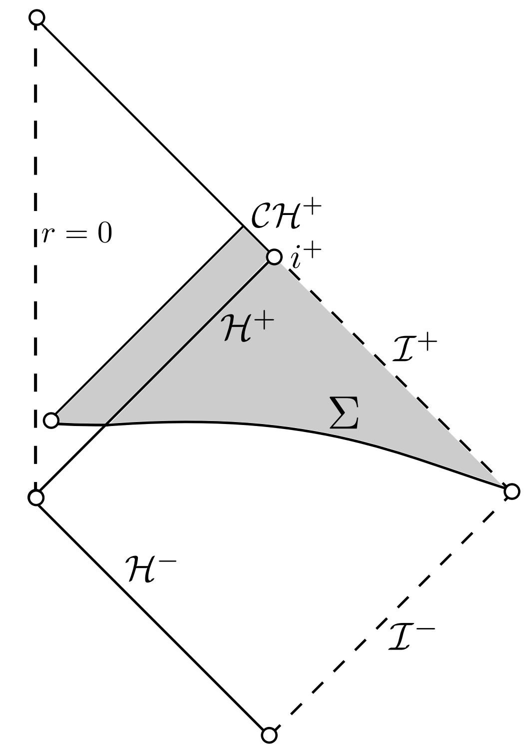
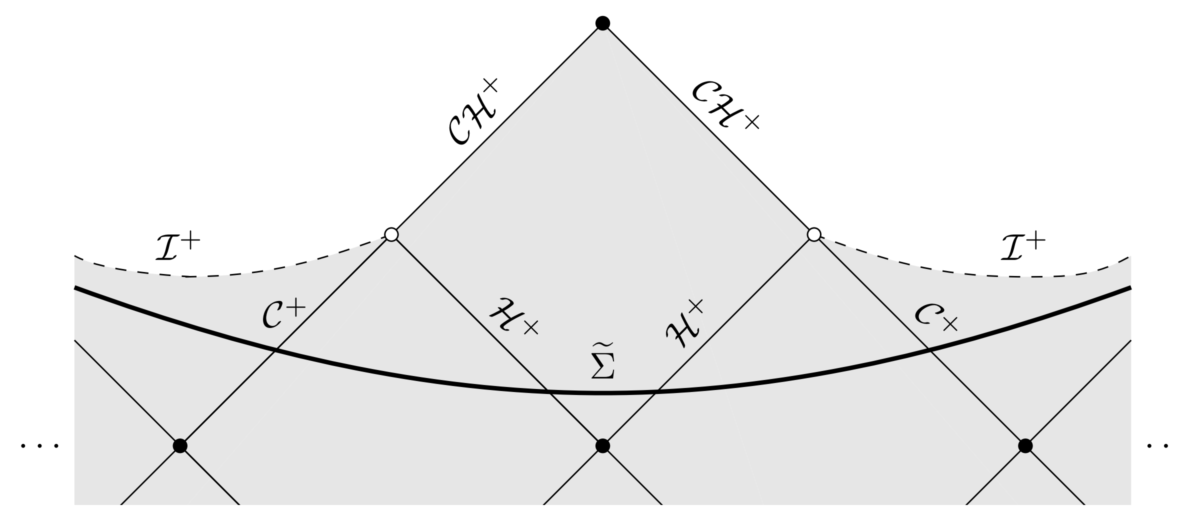
Addendum I: the extremal case



Theorem. (Gajic, Angelopoulos–Aretakis–Gajic) *Consider smooth compactly supported initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on extremal Reissner–Nordstrom. Then* ψ *is extendible across the black hole inner horizon in* H^1_{loc}.

Addendum II: the $\Lambda > 0$ case



Conjecture. (Moss, M.D., Cardoso et al) *Consider smooth initial data* (ψ_0, ψ_1) *on* $\widetilde{\Sigma}$ *for the wave equation* $\Box_g \psi = 0$ *on subextremal Reissner-Nordström–de Sitter. Then* ψ *extends in* H^1_{loc} *across the Cauchy horizon* $\mathscr{CH}^+$.

See also discussion in Dias–Reall–Santos.

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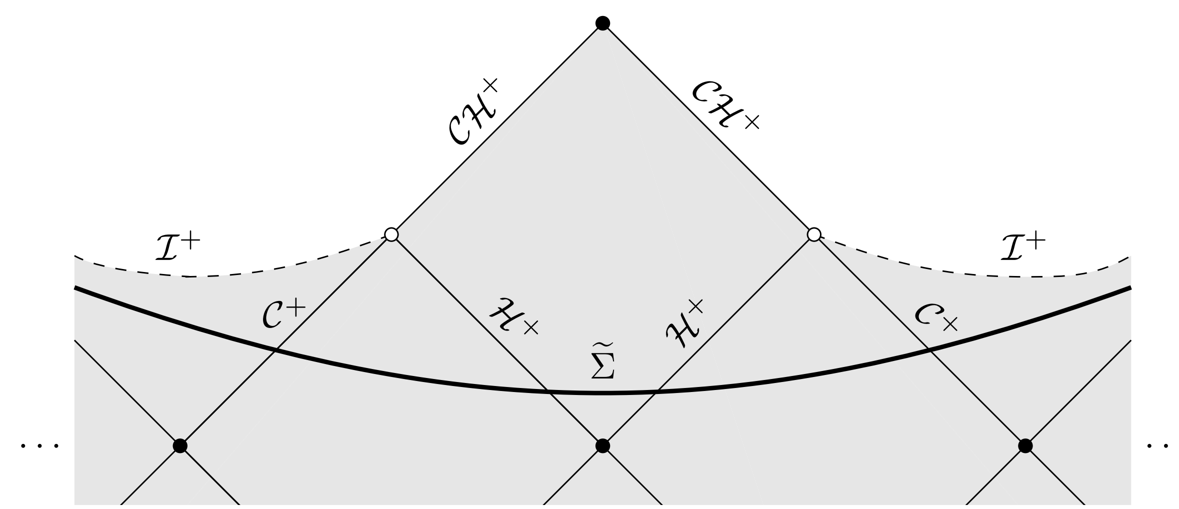

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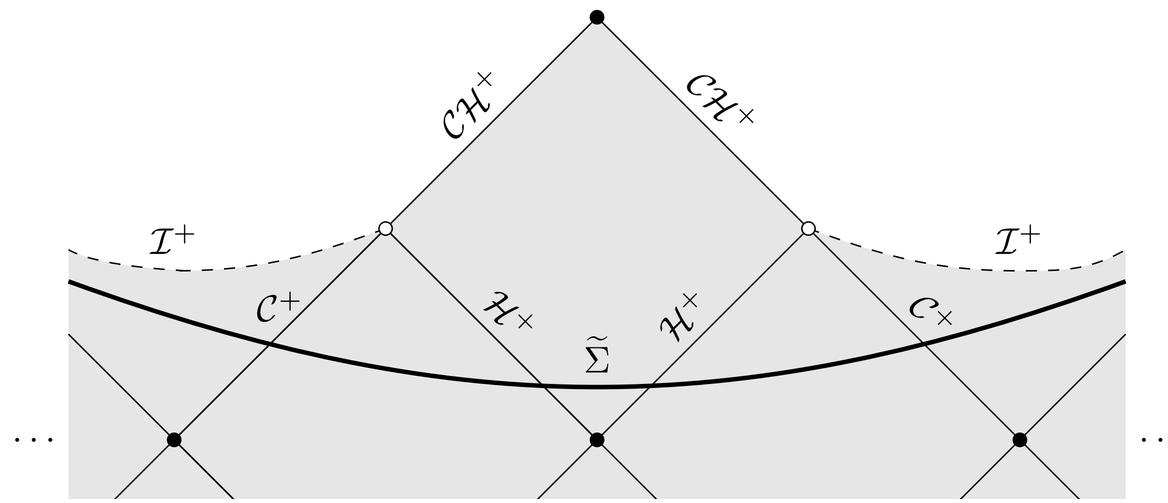

Theorem. (M.D.–Shlapentokh-Rothman 2018) *Consider* *generic* $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$ *initial data* (ψ_0, ψ_1) *on* $\widetilde{\Sigma}$ *for the wave equation* $\Box_g \psi = 0$ *on subextremal Reissner-Nordström (or Kerr)–de Sitter. Then* ψ *is* *inextendible* *in* H^1_{loc} *across the Cauchy horizon* $\mathscr{CH}^+$*.*

Addendum III: the $\Lambda < 0$ case

Theorem. (Holzegel–Smulevici 2014) *Consider smooth initial data* (ψ_0, ψ_1) *on a spacelike slice* Σ *for the wave equation* $\Box_g \psi = 0$ *with reflective boundary conditions at* $\mathscr{I}$ *on Kerr–anti de Sitter. Then solutions* ψ *decay logarithmically on the event horizon* $\mathscr{H}^+$*. Moreover, this decay bound is sharp.*

Implications for the interior? See upcoming work of Kehle!

Addendum III: the $\Lambda < 0$ case

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References for Lecture 5

- S. Aretakis, Y. Angelopoulos and D. Gajic “Late-time asymptotics for the wave equation on extremal Reissner–Nordström backgrounds”, arXiv:1807.03802
- D. Christodoulou. The formation of black holes in general relativity. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.
- M.D. and Y. Shlapentokh-Rothman, “Time-translation invariance of scattering maps and blue-shift instabilities on Kerr black hole sapcetimes”, Commun. Math. Phys. 350 (2017), 985–1016
- M.D. and Y. Shlapentokh-Rothman, “Rough initial data and the strength of the blue-shift instability on cosmological black holes with $\Lambda>0$”, Class. Quantum Grav. 35 (2018) 195010
- M.D., I. Rodnianski and Y. Shlapentokh-Rothman “Decay for solutions of the wave equation on Kerr exterior spacetimes III: the full subextremal case $|a| < M$”, Ann. of Math., 183 (2016), 787–913
- M.D. and J. Luk “The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon”, arXiv:1710.01772
- A. Franzen “Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds”, Commun. Math. Phys. (2016) 343: 601–650.
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Plan of the lectures

Lecture 1. *General Relativity and Lorentzian geometry*

Lecture 2. *The geometry of Schwarzschild black holes*

Lecture 3. *The analysis of waves on Schwarzschild exteriors*

Lecture 4. *The geometry of Kerr black holes and the strong cosmic censorship conjecture*

Lecture 5. *The analysis of waves on Kerr black hole interiors*

Lecture 6. *Nonlinear C^0 stability of the Kerr Cauchy horizon*

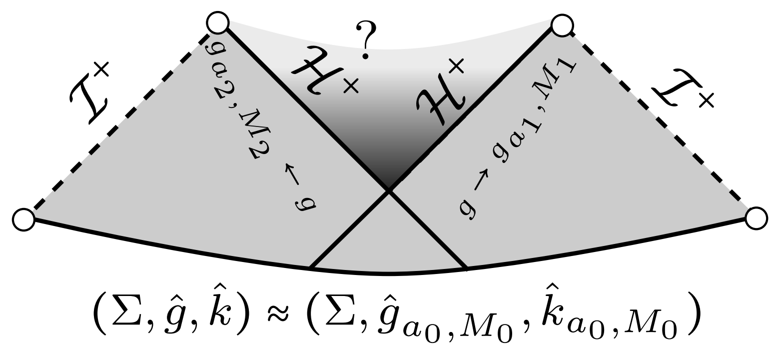
Lecture 6

The full nonlinear C^0 stability of the Kerr Cauchy horizon
(joint work with J. Luk)

Strong cosmic censorship (C^0 formulation)

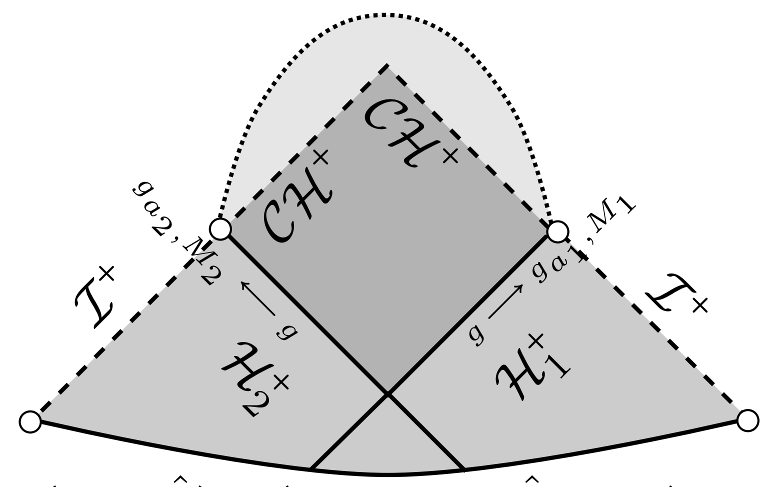
Conjecture. (R. Penrose, 1973) *The Kerr Cauchy horizon is a fluke! For generic asymptotically flat initial data* $(\Sigma, \bar{g}, K)$ *for the vacuum equations, the maximal future Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a manifold with continuous* (C^0) *Lorentzian metric.*

Review: Stability of the Kerr exterior



Conjecture. *The Kerr family is stable <u>in the exterior</u> as solutions to the vacuum Einstein equations: Small perturbations of (two-ended) Kerr initial data lead to a maximal future Cauchy development with complete null infinity* $\mathscr{I}^+$ *such that in* $J^-(\mathscr{I}^+)$ *, in particular on* $\mathscr{H}^+$ *, the induced geometry approaches—inverse polynomially—two nearby Kerr solutions.*

C^0 stability of the Kerr Cauchy horizon



$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$

Theorem. (M.D.—J. Luk, 2017) *If Kerr is stable in the black hole exterior (as conjectured), then its Penrose diagram is globally stable, and the metric again extends, at least* C^0*, across a Cauchy horizon* $\mathscr{CH}^+$.

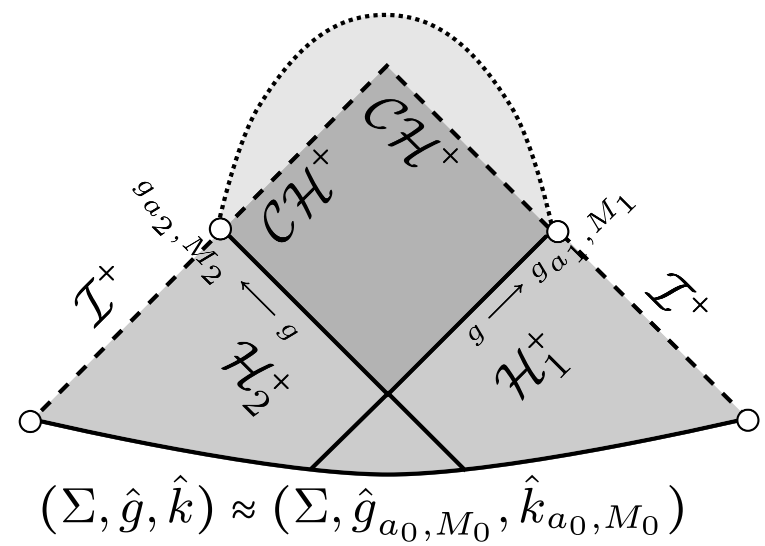
The C^0 formulation of strong cosmic censorship is false

Corollary. *If Kerr is stable in the black hole exterior (as conjectured), then the C^0 formulation of the strong cosmic censorship conjecture, as well as the spacelike singularity conjecture, are both false.*

Heuristic studies and symmetric model problems

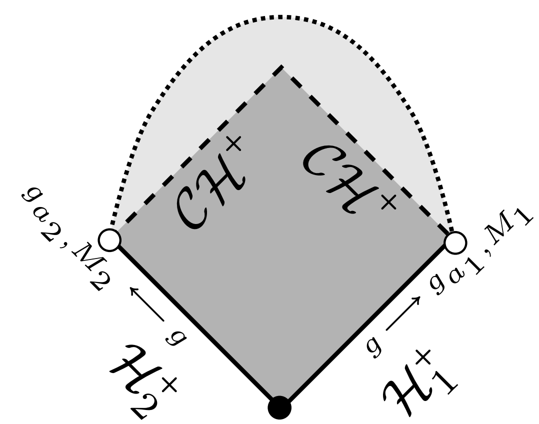
- Hiscock 1981, Poisson–Israel 1989, Ori 1991, Brady 1996, Ori 1997, etc.
- M.D., Luk–Oh, van de Moortel 2017

Reduction to characteristic initial value problem



Theorem. *Consider characteristic initial data for the Einstein vacuum equations which are both globally close to Kerr and approach two nearby Kerr solutions. Then the Cauchy evolution has Penrose diagram as depicted, and the metric extends, at least C^0, across a Cauchy horizon $\mathscr{CH}^+$.*

Reduction to characteristic initial value problem



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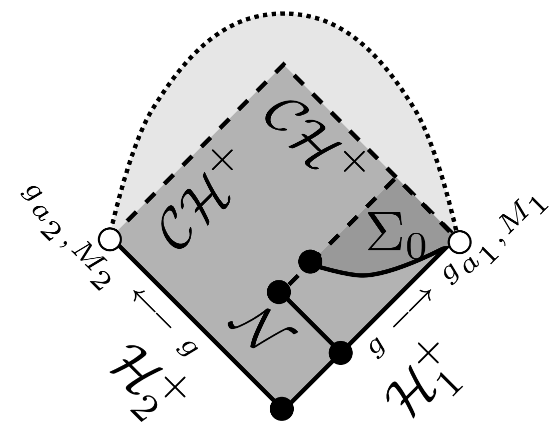
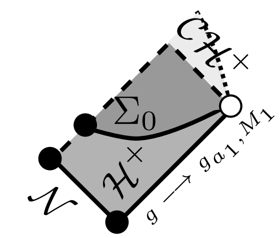
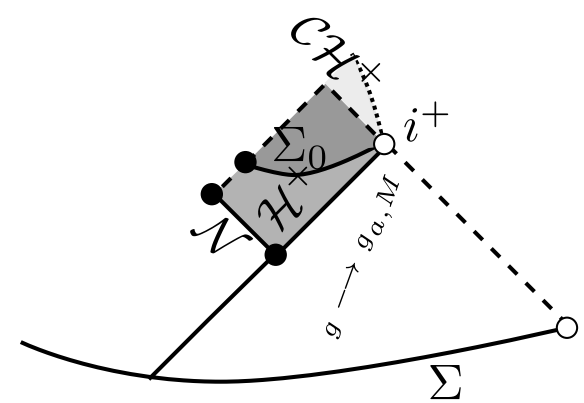
Stability of a piece of the Cauchy horizon





Theorem. *Consider characteristic initial data on $\mathcal{N} \cup \mathcal{H}^+$ for the Einstein vacuum equations which are both close to Kerr and approach a Kerr solution along $\mathcal{H}^+$. Then the Cauchy evolution has Penrose diagram as shown on the right, and the metric extends, at least C^0, across a Cauchy horizon $\mathcal{CH}^+$.*

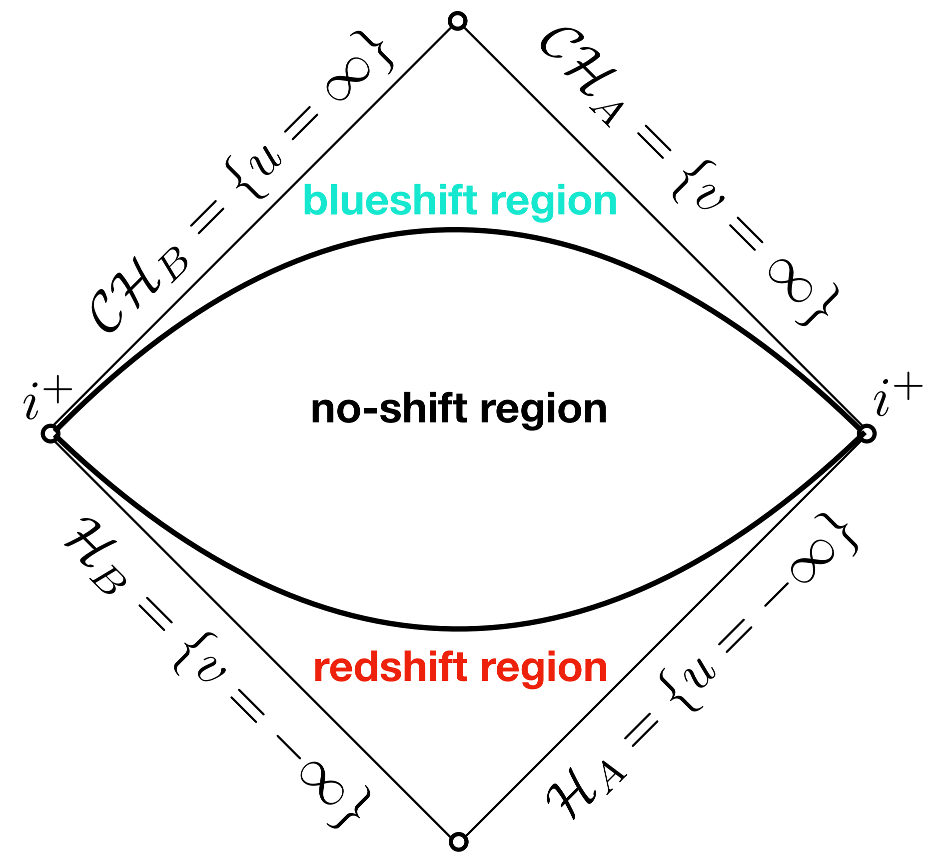
Application to gravitational collapse



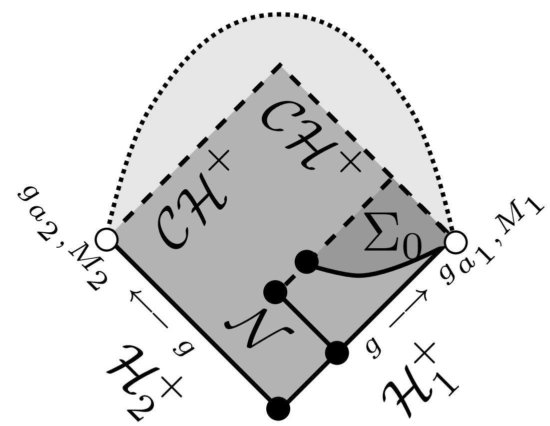
Corollary. *All dynamic vacuum black holes appropriately settling down to Kerr along* $\mathscr{H}^+$ *have a piece of Cauchy horizon* $\mathscr{CH}^+$ *in their interior across which the metric extends at least* C^0.

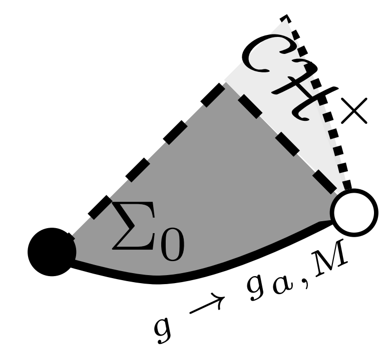
Recall from Lecture 5: three regions

- red-shift region
- "no-shift" region
- blue-shift region



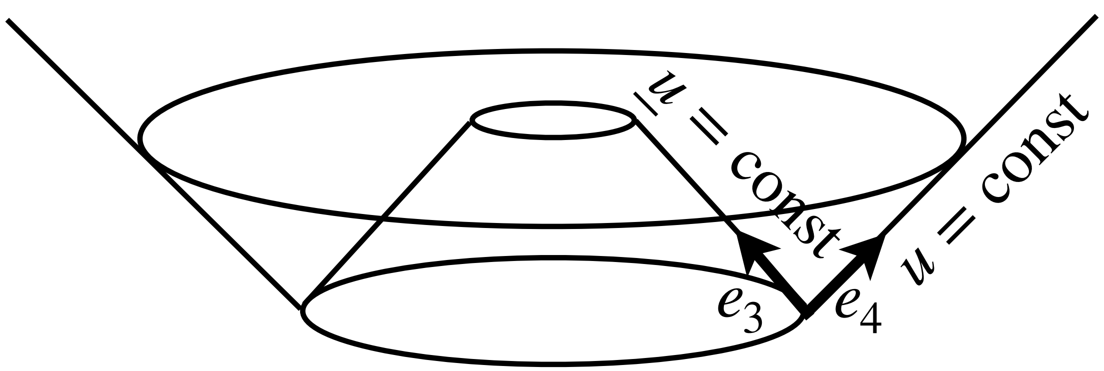
Stability of piece of Cauchy horizon from spacelike data

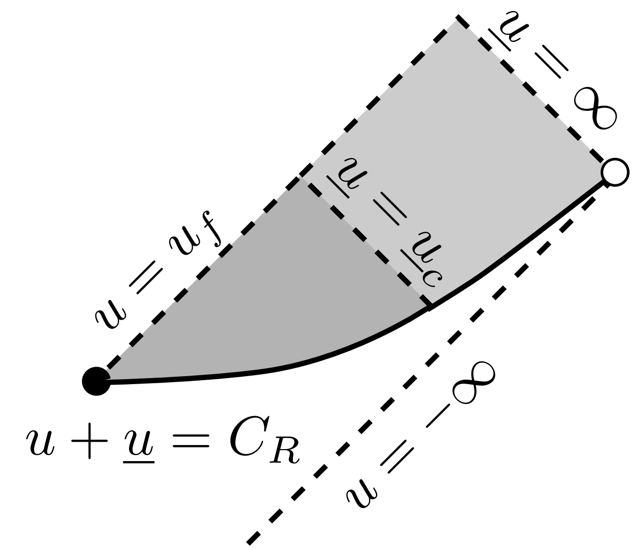




Theorem. *Consider Cauchy initial data on* Σ_0 *for the Einstein vacuum equations which are both close to Kerr and approach a Kerr solution. Then the Cauchy evolution has Penrose diagram as shown on the right, and the metric extends, at least* C^0*, across a Cauchy horizon* $\mathscr{C}\mathscr{H}^+$*.*

Setup: double null foliation





- $\underline{u}$ and u are Eddington-Finkelstein-like normalised

$$g = -2\Omega^2(du \otimes d\underline{u} + d\underline{u} \otimes du) + \gamma_{AB}(d\theta^A - b^A d\underline{u}) \otimes (d\theta^B - b^B d\underline{u}),$$

$$\chi_{AB} \doteq g(D_A e_4, e_B), \qquad \alpha_{AB} \doteq R(e_A, e_4, e_B, e_4), \quad \rho \doteq \frac{1}{4} R(e_4, e_3, e_4, e_3), \quad \sigma \doteq \frac{1}{4} {}^*R(e_4, e_3, e_4, e_3)$$

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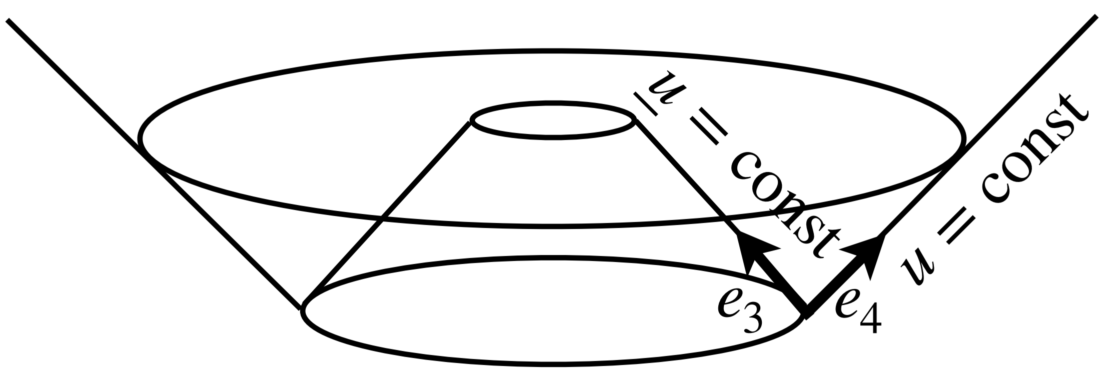

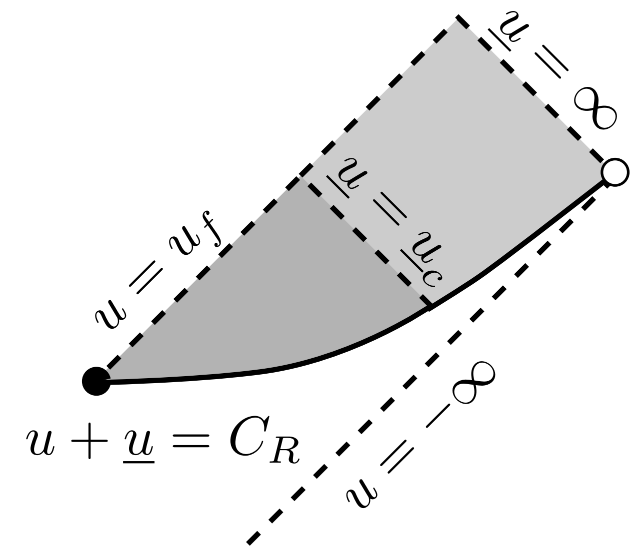

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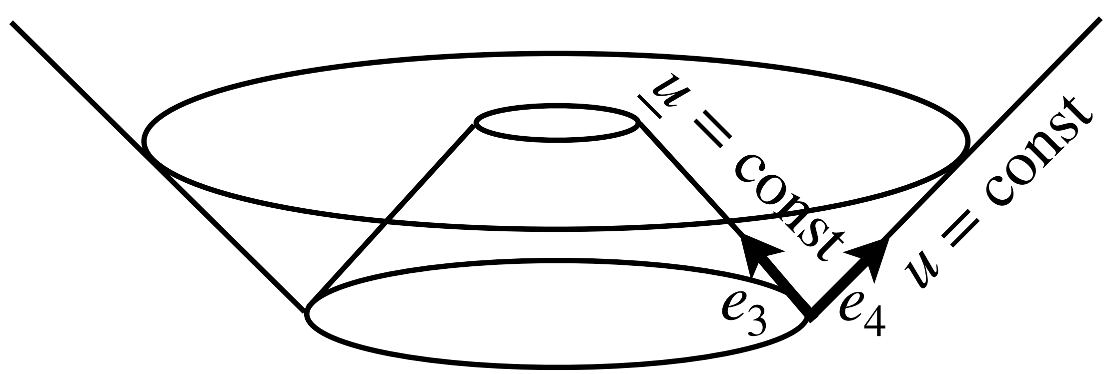

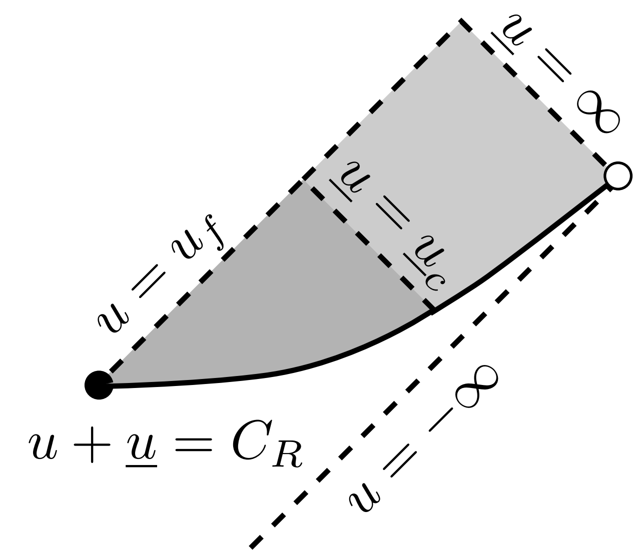

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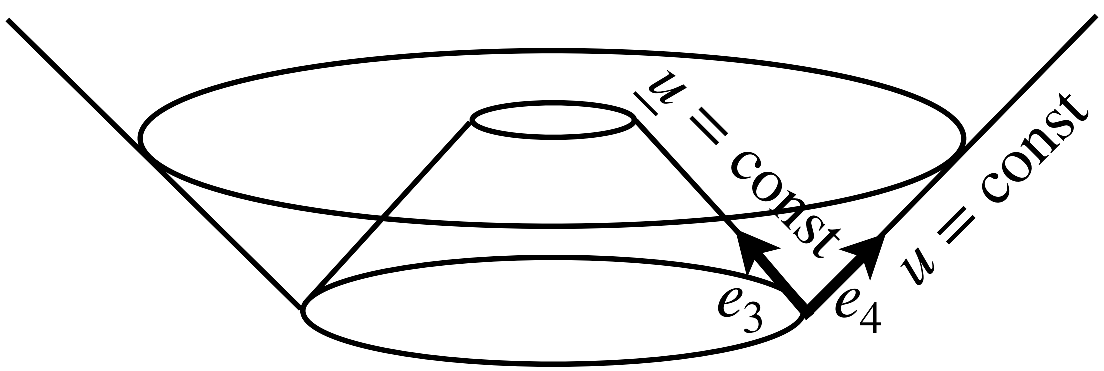

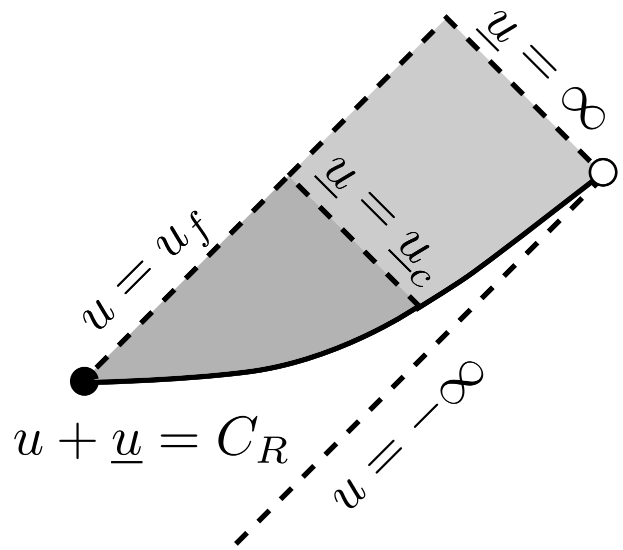

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The equations I

$$
\begin{aligned}
\nabla\!\!\!/_4 \operatorname{tr\!\!/}\chi + \frac{1}{2}(\operatorname{tr\!\!/}\chi)^2 &= -|\hat{\chi}|^2 - 2\omega \operatorname{tr\!\!/}\chi, \\
\nabla\!\!\!/_4 \hat{\chi} + \operatorname{tr\!\!/}\chi \hat{\chi} &= -2\omega\hat{\chi} - \alpha, \\
\nabla\!\!\!/_3 \operatorname{tr\!\!/}\underline{\chi} + \frac{1}{2}(\operatorname{tr\!\!/}\underline{\chi})^2 &= -2\underline{\omega} \operatorname{tr\!\!/}\underline{\chi} - |\underline{\hat{\chi}}|^2, \\
\nabla\!\!\!/_3 \underline{\hat{\chi}} + \operatorname{tr\!\!/}\underline{\chi}\, \underline{\hat{\chi}} &= -2\underline{\omega}\underline{\hat{\chi}} - \underline{\alpha}, \\
\nabla\!\!\!/_4 \operatorname{tr\!\!/}\underline{\chi} + \frac{1}{2}\operatorname{tr\!\!/}\chi \operatorname{tr\!\!/}\underline{\chi} &= 2\omega \operatorname{tr\!\!/}\underline{\chi} + 2\rho - \hat{\chi}\cdot\underline{\hat{\chi}} + 2\operatorname{di\!\!/v}\, \underline{\eta} + 2|\underline{\eta}|^2, \\
\nabla\!\!\!/_4 \underline{\hat{\chi}} + \frac{1}{2}\operatorname{tr\!\!/}\chi \underline{\hat{\chi}} &= \nabla\!\!\!/ \widehat{\otimes} \underline{\eta} + 2\omega\underline{\hat{\chi}} - \frac{1}{2}\operatorname{tr\!\!/}\underline{\chi}\hat{\chi} + \underline{\eta}\widehat{\otimes}\underline{\eta}, \\
\nabla\!\!\!/_3 \operatorname{tr\!\!/}\chi + \frac{1}{2}\operatorname{tr\!\!/}\underline{\chi} \operatorname{tr\!\!/}\chi &= 2\underline{\omega} \operatorname{tr\!\!/}\chi + 2\rho - \hat{\chi}\cdot\underline{\hat{\chi}} + 2\operatorname{di\!\!/v}\, \eta + 2|\eta|^2, \\
\nabla\!\!\!/_3 \hat{\chi} + \frac{1}{2}\operatorname{tr\!\!/}\underline{\chi} \hat{\chi} &= \nabla\!\!\!/ \widehat{\otimes} \eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\operatorname{tr\!\!/}\chi\underline{\hat{\chi}} + \eta\widehat{\otimes}\eta.
\end{aligned}
$$

The equations II

$$
\begin{aligned}
\nabla\!\!\!/_4 \eta &= -\chi \cdot (\eta - \underline{\eta}) - \beta, \\
\nabla\!\!\!/_3 \underline{\eta} &= -\underline{\chi} \cdot (\underline{\eta} - \eta) + \underline{\beta}, \\
\nabla\!\!\!/_4 \underline{\omega} &= \zeta \cdot (\eta - \underline{\eta}) - \eta \cdot \underline{\eta} + \rho, \\
\mathrm{di\!\!/v}\ \hat{\chi} &= \frac{1}{2} \nabla\!\!\!/ \mathrm{t\!\!/r}\chi - \zeta \cdot (\chi - \mathrm{t\!\!/r}\chi\gamma) - \beta, \\
\mathrm{di\!\!/v}\ \underline{\hat{\chi}} &= \frac{1}{2} \nabla\!\!\!/ \mathrm{t\!\!/r}\underline{\chi} + \zeta \cdot (\underline{\hat{\chi}} - \mathrm{t\!\!/r}\underline{\chi}\gamma) + \underline{\beta}, \\
\mathrm{cu\!\!/rl}\ \eta = -\mathrm{cu\!\!/rl}\ \underline{\eta} &= \sigma + \frac{1}{2} \underline{\hat{\chi}} \wedge \hat{\chi}, \\
K &= -\rho + \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} - \frac{1}{4} \mathrm{t\!\!/r}\chi \mathrm{t\!\!/r}\underline{\chi}.
\end{aligned}
$$

The equations III

$$
\begin{aligned}
\nabla\!\!\!/_3\alpha + \frac{1}{2}\mathrm{tr}\!\!/\,\underline{\chi}\alpha &= \nabla\!\!\!/\,\widehat{\otimes}\beta + 4\underline{\omega}\alpha - 3(\hat{\chi}\rho +^{*}\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta,\\
\nabla\!\!\!/_4\beta + 2\mathrm{tr}\!\!/\,\chi\beta &= \mathrm{div}\!\!\!/\ \alpha - 2\omega\beta + (2\zeta + \underline{\eta})\cdot\alpha,\\
\nabla\!\!\!/_3\beta + \mathrm{tr}\!\!/\,\underline{\chi}\beta &= \nabla\!\!\!/\,\rho + 2\underline{\omega}\beta +^{*}\nabla\!\!\!/\,\sigma + 2\hat{\chi}\cdot\underline{\beta} + 3(\eta\rho +^{*}\eta\sigma),\\
\nabla\!\!\!/_4\sigma + \frac{3}{2}\mathrm{tr}\!\!/\,\chi\sigma &= -\mathrm{div}\!\!\!/\ {}^{*}\beta + \frac{1}{2}\underline{\hat{\chi}}\wedge\alpha - \zeta\wedge\beta - 2\underline{\eta}\wedge\beta,\\
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\end{aligned}
$$

Recall from Lecture 2: generic blow up at Cauchy horizon

Theorem. (Luk–Oh, M.D.–Shlapentokh-Rothman, Luk–Sbierski) *Consider* <u>*generic*</u> *smooth localised initial data* (ψ_0, ψ_1) *on* Σ *for the wave equation* $\Box_g \psi = 0$ *on subextremal Kerr or Reissner–Nordström. Then the local energy of* ψ *blows up at* $\mathcal{CH}^+$*, i.e.* ψ *is inextendible in* H^1_{loc} *across* $\mathcal{CH}^+$*.*

Before trying to prove our main Theorem, must address question: **Is this consistent with non-linear evolution?**

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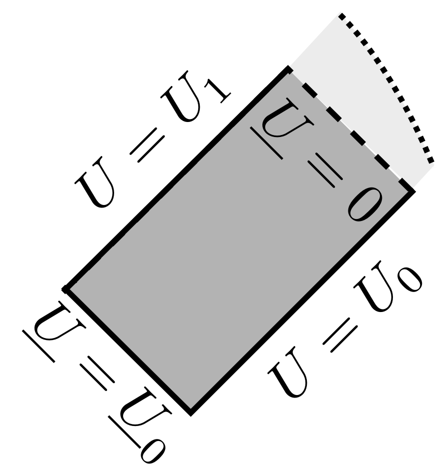
Luk’s weak null singularities



Theorem. (J. Luk 2013) *Consider vacuum characteristic initial data posed on the above hypersurfaces such that on the outgoing part* $U = U_0$ *the Christoffel symbols are bounded in a norm consistent with the singular behaviour of the previous theorem. Then the Cauchy evolution can be covered by a full rectangular domain* $[\underline{U}_0, 0) \times [U_0, U_1]$*, i.e. it has Penrose diagram as shown above, and the metric extends, at least* C^0*, across a Cauchy horizon* $\underline{U} = 0$*.*

Moreover, if the Christoffel symbols indeed have the singular profile of the previous theorem, then this profile propages. Thus $\underline{U} = 0$ *can be thought of as a weak null singularity.*

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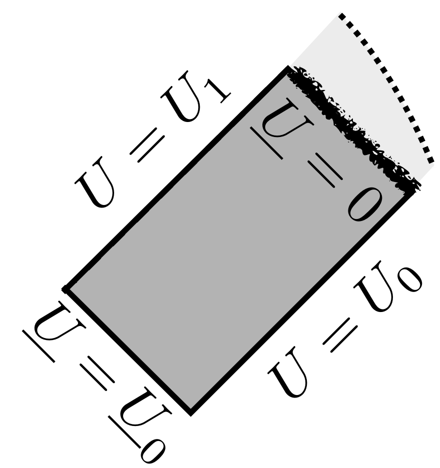

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Proof of Luk's theorem

- Best general well posedness results for the vacuum equation need curvature to be square integrable (Klainerman–Rodnianski–Szeftel)
- Here even the Christoffel symbols fail to be square integrable
- This is compensated by additional angular regularity
- Compare with impulsive gravitational waves (Penrose, Luk–Rodnianski) where Christoffel symbols were square integrable. Could do usual energy estimates but with renormalised Bianchi equations (see next slide).
- In contrast, here need in addition weighted estimates:

  i.e. estimate $\|f(\underline{U})\hat{\chi}\|_{L^2_{\underline{U}}L^2(S_{U,\underline{U}})}$ with e.g. $f(\underline{U}) = (-\underline{U})^{\frac{1}{2}}|\log(-\underline{U})|^{\frac{1}{2}+\delta}$
- Null condition ensures non-linear terms can be estimated!

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The renormalised equations

$$\rho \rightsquigarrow K \doteq -\rho + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \frac{1}{4}\slashed{\mathrm{tr}}\chi\slashed{\mathrm{tr}}\underline{\chi}, \qquad \sigma \rightsquigarrow \check{\sigma} = \sigma + \frac{1}{2}\underline{\hat{\chi}}\wedge\hat{\chi}$$

$$\begin{aligned}
\slashed{\nabla}_3\beta + \slashed{\mathrm{tr}}\underline{\chi}\beta =& -\slashed{\nabla}K +^* \slashed{\nabla}\check{\sigma} + 2\underline{\omega}\beta + 2\hat{\chi}\cdot\underline{\beta} - 3(\eta K -^* \eta\check{\sigma}) + \frac{1}{2}(\slashed{\nabla}(\hat{\chi}\cdot\underline{\hat{\chi}}) +^* \slashed{\nabla}(\hat{\chi}\wedge\underline{\hat{\chi}})) \\
&+ \frac{3}{2}(\eta\hat{\chi}\cdot\underline{\hat{\chi}} +^* \eta\hat{\chi}\wedge\underline{\hat{\chi}}) - \frac{1}{4}(\slashed{\nabla}\slashed{\mathrm{tr}}\chi\slashed{\mathrm{tr}}\underline{\chi} + \slashed{\mathrm{tr}}\chi\slashed{\nabla}\slashed{\mathrm{tr}}\underline{\chi}) - \frac{3}{4}\eta\slashed{\mathrm{tr}}\chi\slashed{\mathrm{tr}}\underline{\chi}, \\
\slashed{\nabla}_4\check{\sigma} + \frac{3}{2}\slashed{\mathrm{tr}}\chi\check{\sigma} =& -\slashed{\mathrm{div}}\,{}^*\beta - \zeta\wedge\beta - 2\underline{\eta}\wedge\beta - \frac{1}{2}\hat{\chi}\wedge(\slashed{\nabla}\widehat{\otimes}\underline{\eta}) - \frac{1}{2}\hat{\chi}\wedge(\underline{\eta}\widehat{\otimes}\underline{\eta}), \\
\slashed{\nabla}_4 K + \slashed{\mathrm{tr}}\chi K =& -\slashed{\mathrm{div}}\,\beta - \zeta\cdot\beta - 2\underline{\eta}\cdot\beta + \frac{1}{2}\hat{\chi}\cdot\slashed{\nabla}\widehat{\otimes}\underline{\eta} + \frac{1}{2}\hat{\chi}\cdot(\underline{\eta}\widehat{\otimes}\underline{\eta}) - \frac{1}{2}\slashed{\mathrm{tr}}\chi\slashed{\mathrm{div}}\,\underline{\eta} - \frac{1}{2}\slashed{\mathrm{tr}}\chi|\underline{\eta}|^2, \\
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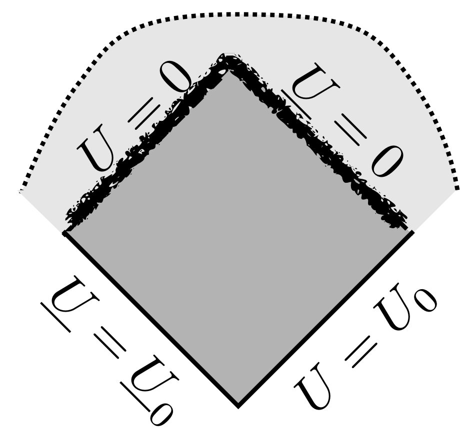
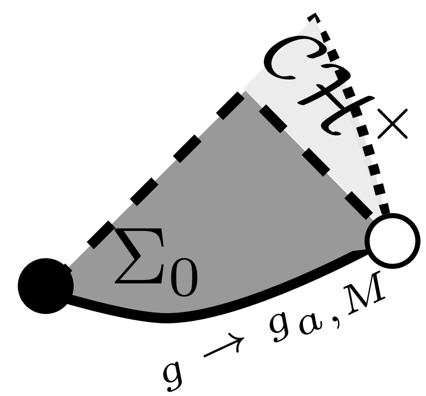
Bifurcate weak null singularities



Theorem. (J. Luk 2013) *Consider initial data such that the Christoffel symbols have the singular profile suggested by the previous on* <u>*both*</u> *ingoing and outgoing parts. Then, with an appropriate smallness condition, the maximal future development is bounded by a bifurcate null singularity, across which the metric is globally continuously extendible.*

Back to the theorem



Theorem. *Consider Cauchy initial data on* Σ_0 *for the Einstein vacuum equations which are both close to Kerr and approach a Kerr solution. Then the Cauchy evolution has Penrose diagram as shown on the right, and the metric extends, at least* C^0*, across a Cauchy horizon* $\mathscr{C}\mathscr{H}^+$*.*

Difference quantities and the reduced schematic system

$$\widetilde{\mathrm{tr}\chi} \doteq \mathrm{tr}\chi - (\mathrm{tr}\chi)_{\mathcal{K}}, \qquad \widetilde{\beta} \doteq \beta - \beta_{\mathcal{K}}$$

$$\widetilde{\psi} \in \{\widetilde{\eta}, \widetilde{\underline{\eta}}\}, \qquad \widetilde{\psi}_{\underline{H}} \in \{\widetilde{\mathrm{tr}\underline{\chi}}, \widetilde{\underline{\hat{\chi}}}\}, \qquad \widetilde{\psi}_{H} \in \{\widetilde{\mathrm{tr}\chi}, \widetilde{\hat{\chi}}\}$$

schematic equation:

$$\nabla_4 \nabla^3 \widetilde{\mathrm{tr}\chi} =_{\mathrm{RS}} \sum_{i_1+i_2+i_3 \leq 3} (1 + \nabla^{i_1}\widetilde{g} + \nabla^{\min\{i_1,2\}}\widetilde{\psi})(1 + \nabla^{i_2}\widetilde{\psi}_{H})(\nabla^{i_3}(\widetilde{\psi}_{H}, \widetilde{g}) + \Omega_{\mathcal{K}}^{-2}\nabla^{i_3}\widetilde{b}),$$

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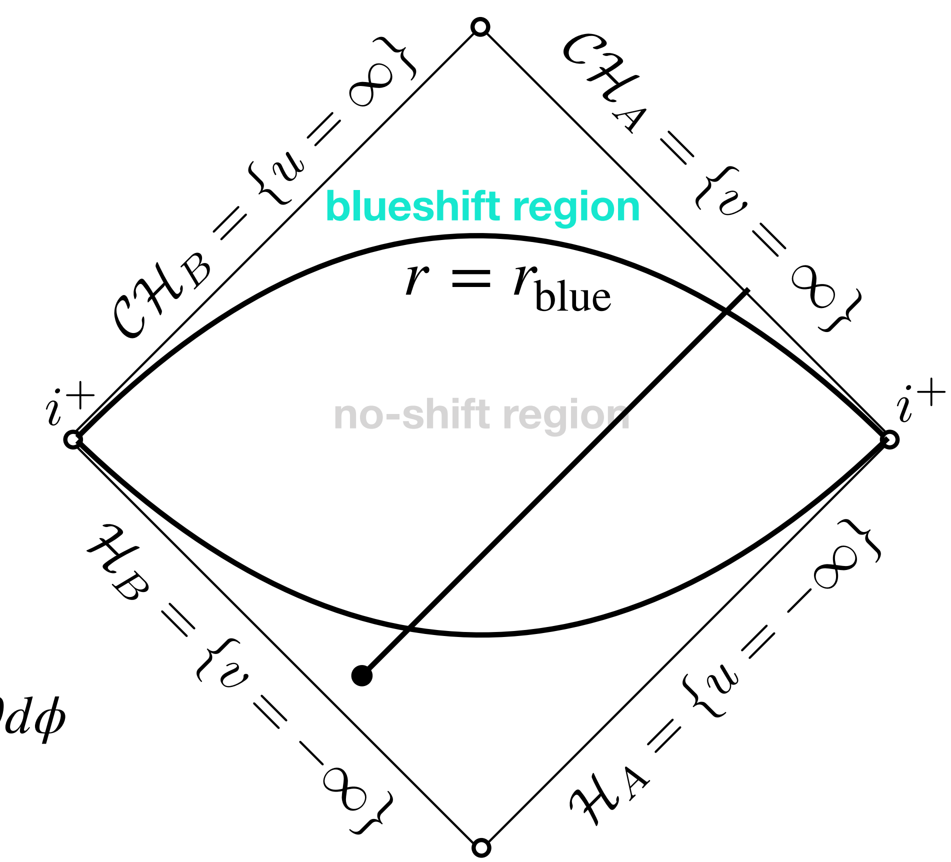
Recall from Lecture 5: the linear wave equation

- use $X = r^{2N}(u^p\partial_u + v^p\partial_v)$ with $p > 1$
- bulk non-negativity of the worst terms
- initial flux on $r = r_{\text{blue}}$ bounded from previous
- obtain finally bounds for $\int_{u=const} v^p(\partial_v\psi)^2 dv \sin\theta d\theta d\phi$



Note v is now denoted $\underline{u}$

Global weights: mimicking the X vector field

$$\|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\widetilde{\beta}\|_{L_u^\infty L_{\underline{u}}^2 L^2(S_{u,v})}$$

$$\begin{aligned}
\mathcal{N}_{hyp} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\nabla\!\!\!/^3\widetilde{\psi}_H\|^2_{L_u^\infty L_{\underline{u}}^2 L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\nabla\!\!\!/^3\widetilde{\psi}_{\underline{H}}\|^2_{L_{\underline{u}}^\infty L_u^2 L^2(S)} + \cdots \\
\mathcal{N}_{int} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^3\nabla\!\!\!/^3\widetilde{\psi}_H\|^2_{L_u^2 L_{\underline{u}}^2 L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}\nabla\!\!\!/^3\widetilde{\psi}_{\underline{H}}\|^2_{L_u^2 L_{\underline{u}}^2 L^2(S)} \\
&\quad + \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}\nabla\!\!\!/^3\widetilde{\psi}\|^2_{L_u^2 L_{\underline{u}}^2 L^2(S)}\cdots \\
\mathcal{N}_{sph} &= \|\nabla\!\!\!/^3\widetilde{g}\|^2_{L_{\underline{u}}^\infty L_u^\infty L^2(S)} + \cdots + \|\widetilde{\psi}_{\underline{H}}\|^2_{L_u^1 L_{\underline{u}}^\infty L^2(S)} + \cdots.
\end{aligned}$$

- hyperbolic, transport and elliptic estimates under these weights
- “null condition” and the geometric properties of the solution near the Cauchy horizon ensures estimates close

Global weights: mimicking the X vector field

$$\|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\widetilde{\beta}\|_{L^\infty_u L^2_{\underline{u}} L^2(S_{u,v})}$$

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\mathcal{N}_{hyp} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\nabla^3\widetilde{\psi}_H\|^2_{L^\infty_u L^2_{\underline{u}} L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\nabla^3\widetilde{\psi}_{\underline{H}}\|^2_{L^\infty_{\underline{u}} L^2_u L^2(S)} + \cdots \\
\mathcal{N}_{int} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^3\nabla^3\widetilde{\psi}_H\|^2_{L^2_u L^2_{\underline{u}} L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}\nabla^3\widetilde{\psi}_{\underline{H}}\|^2_{L^2_u L^2_{\underline{u}} L^2(S)} \\
&\quad + \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}\nabla^3\widetilde{\psi}\|^2_{L^2_u L^2_{\underline{u}} L^2(S)} \cdots \\
\mathcal{N}_{sph} &= \|\nabla^3\widetilde{g}\|^2_{L^\infty_{\underline{u}} L^\infty_u L^2(S)} + \cdots + \|\widetilde{\psi}_{\underline{H}}\|^2_{L^1_u L^\infty_{\underline{u}} L^2(S)} + \cdots .
\end{aligned}$$

- hyperbolic, transport and elliptic estimates under these weights
- “null condition” and the geometric properties of the solution near the Cauchy horizon ensures estimates close

Global weights: mimicking the X vector field

$$\|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\widetilde{\beta}\|_{L_u^\infty L_{\underline{u}}^2 L^2(S_{u,v})}$$

$$\begin{aligned}
\mathcal{N}_{hyp} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^2\nabla\!\!\!/^3\widetilde{\psi}_H\|^2_{L_u^\infty L_{\underline{u}}^2 L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\nabla\!\!\!/^3\widetilde{\psi}_{\underline{H}}\|^2_{L_{\underline{u}}^\infty L_u^2 L^2(S)} + \cdots \\
\mathcal{N}_{int} &= \|\underline{u}^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}^3\nabla\!\!\!/^3\widetilde{\psi}_H\|^2_{L_u^2 L_{\underline{u}}^2 L^2(S)} + \||u|^{\frac{1}{2}+\delta}\varpi^N\Omega_{\mathcal{K}}\nabla\!\!\!/^3\widetilde{\psi}_{\underline{H}}\|^2_{L_u^2 L_{\underline{u}}^2 L^2(S)} \\
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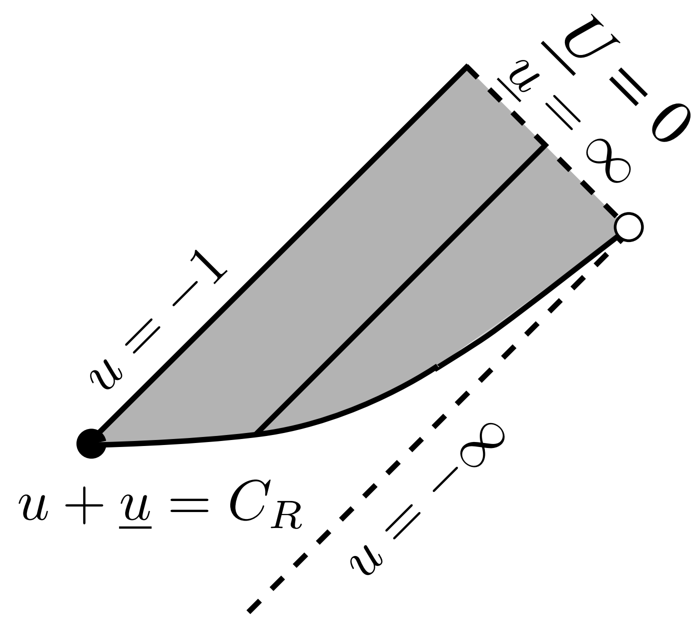
Relation of the weights with weak null singularities



$$\underline{U} = -e^{-2\kappa_- \underline{u}}$$

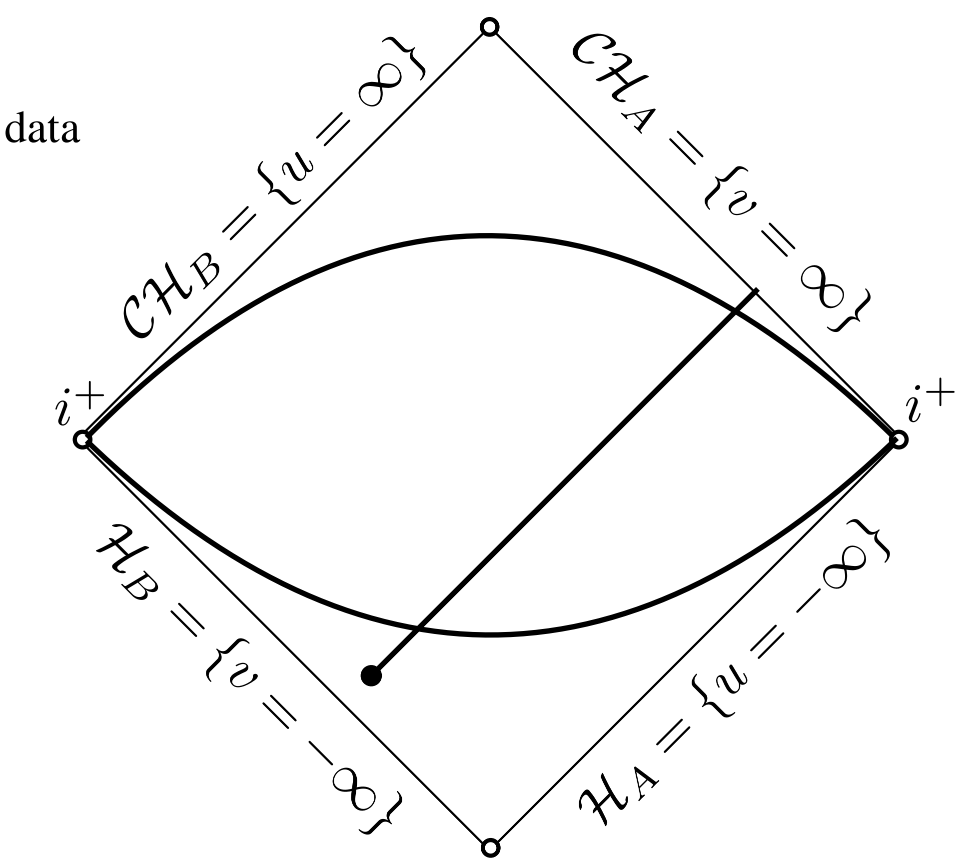
$$\| f(\underline{U}) \hat{\chi} \|_{L^2_{\underline{U}} L^2(S_{U,\underline{U}})}$$

$$f(\underline{U}) = (-\underline{U})^{\frac{1}{2}} |\log(-\underline{U})|^{\frac{1}{2}+\delta}$$

$$\| \underline{u}^{1/2+\delta} \Omega^2_{\mathscr{K}} \hat{\chi} \|_{L^2_{\underline{u}} L^2(S_{u,\underline{u}})}$$

Recall from Lecture 5: Deducing C^0 bounds

$$|\psi|^2 \leq \left(\int_1^\infty |\partial_v \psi|\right)^2 + \text{data} \leq \left(\int_1^\infty v^p |\partial_v \psi|^2 dv\right)\left(\int_1^\infty v^{-p} dv\right) + \text{data}$$

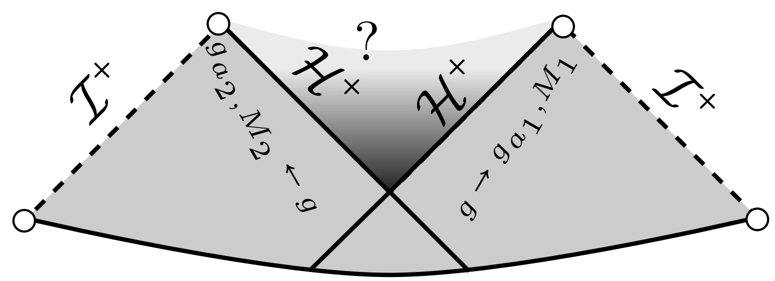


- now integrate the above over the spheres
- Since $p > 1$, the second factor is finite
- commute with $\mathbf{\Omega}_i$
- and apply Sobolev on the spheres

Showing continuity of the metric

- Similar in view of first variation relations $\mathcal{L}_{e_4}\gamma = 2\chi$, etc…

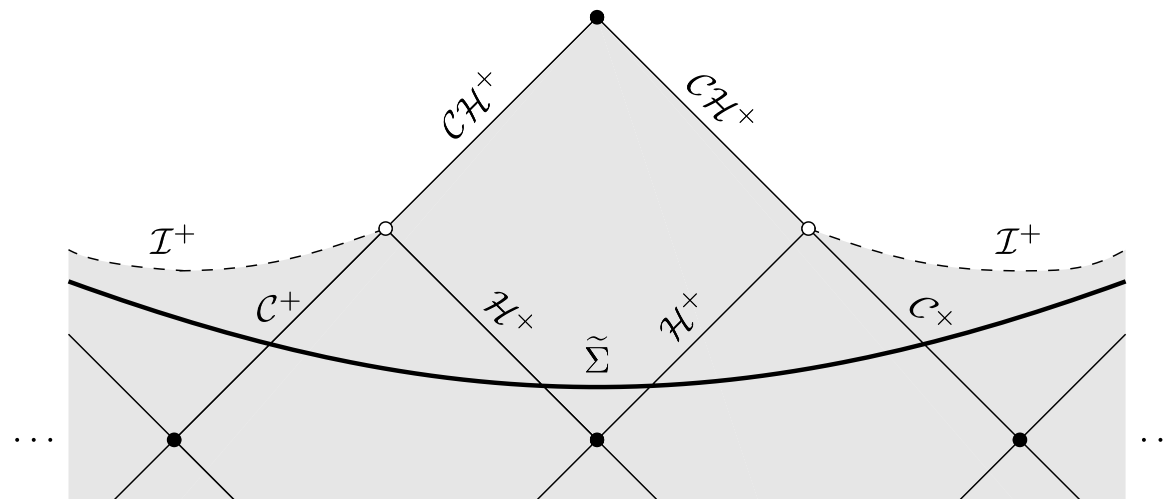
Open problem I



$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$

Finish the proof of the nonlinear exterior stability of Kerr conjecture!

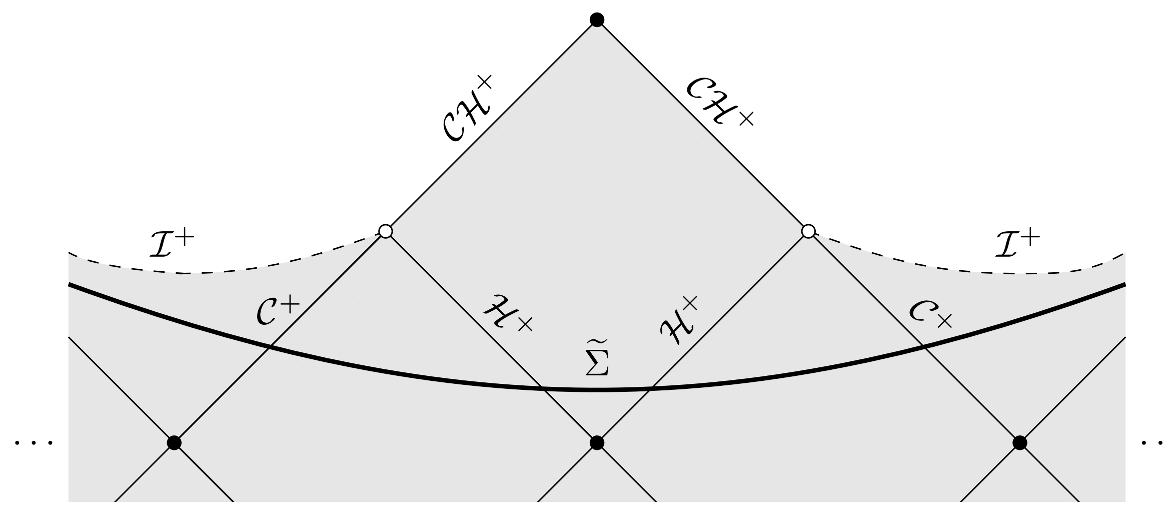
Remark:
the $\Lambda > 0$ analogue



Theorem. (Hintz–Vasy) *The region between the event and cosmological horizons is stable for very slowly rotating Kerr–de Sitter, with exponential decay along* $\mathscr{CH}^+$.

Thus our theorem can be applied to yield stability of the above Penrose diagram and the falsification of the C^0 formulation of SCC for $\Lambda > 0$.

Remark: the $\Lambda > 0$ analogue



Theorem. (Hintz–Vasy) *The region between the event and cosmological horizons is stable for very slowly rotating Kerr–de Sitter, with exponential decay along* $\mathscr{CH}^+$.

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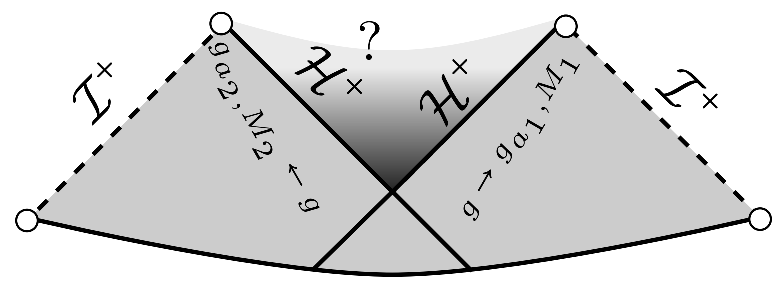
Open problem II

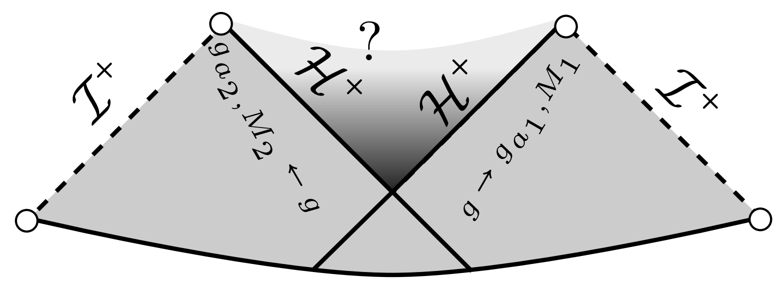


$$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$$

In the context of the nonlinear exterior stability of Kerr conjecture, show in addition that there is a generic lower bound on the rate of approach to Kerr.

See Luk–Oh, Angelopoulos–Aretakis–Gajic

Open problem II



$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$

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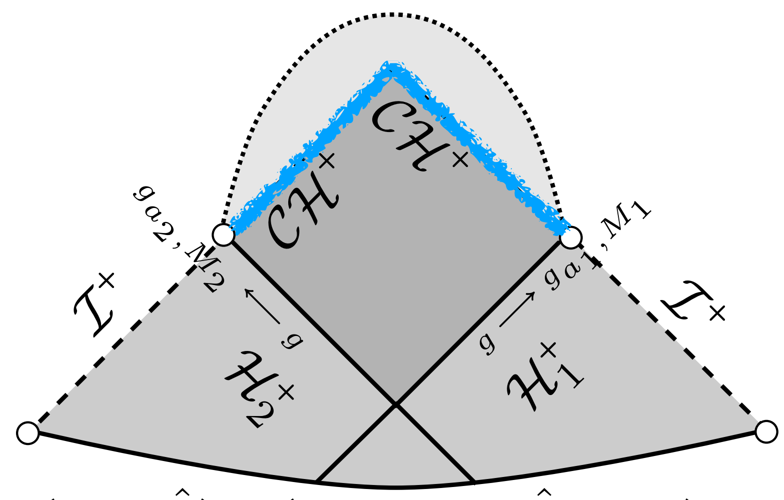
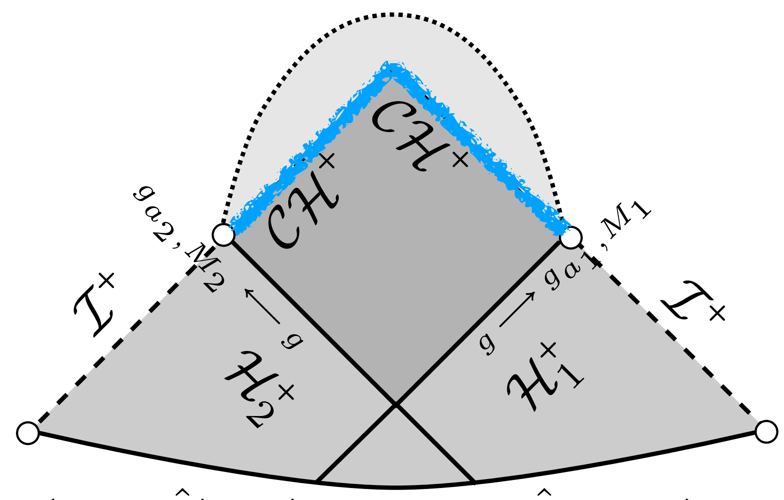
Open problem III



Use the above to show that generically, the Cauchy horizon is indeed a weak null singularity and Christodoulou's formulation of strong cosmic censorship is true

See M.D., Luk–Oh, Luk–Sbierski

Open problem III



Use the above to show that generically, the Cauchy horizon is indeed a weak null singularity and Christodoulou's formulation of strong cosmic censorship is true

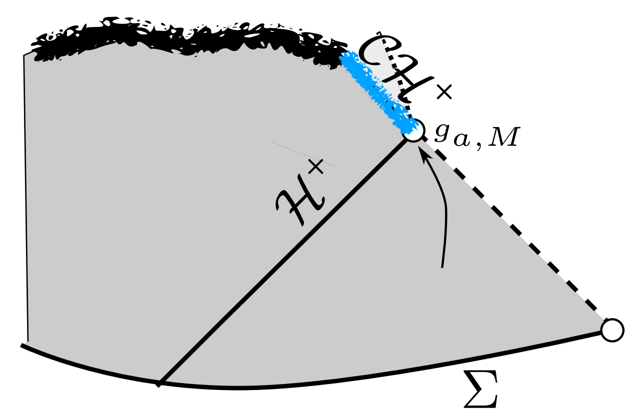
See M.D., Luk–Oh, Luk–Sbierski

Strong cosmic censorship (Christodoulou formulation)

Conjecture. (R. Penrose, 1973) *The Kerr Cauchy horizon is a fluke! For generic asymptotically flat initial data* $(\Sigma, \bar{g}, K)$ *for the vacuum equations, the maximal future Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a* C^0 *Lorentzian manifold with locally square integrable Christoffel symbols.*

This formulation is sufficiently strong to assure that there is no extension even as a weak solution.

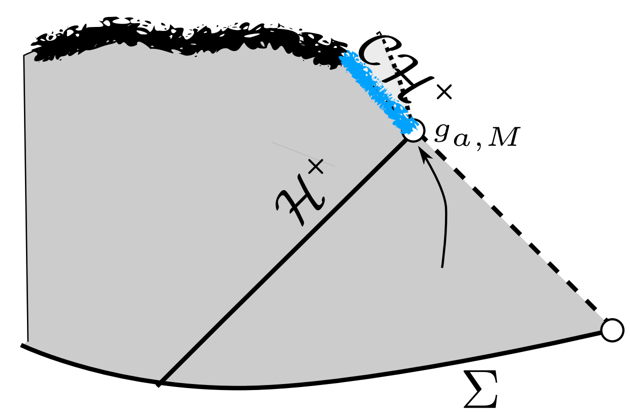
Open problem IV



Is there an open set in the moduli space of vacuum initial data leading to a spacelike singularity?

See Rodnianski–Speck for Einstein-scalar field

Open problem IV



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References for Lecture 6

- L. Burko and A. Ori, ed. “Internal Structure of Black Holes and Spacetime Singularities” Annals of the Israel Physics Society, volume 13
- M.D. and J. Luk “The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon”, arXiv:1710.01772
- J. Luk. Weak null singularities in general relativity. JAMS, 31:1-63, 2018.
- J. Luk and I. Rodnianski. Local propagation of impulsive gravitational waves. Comm. Pure Appl. Math., 68(4):511–624, 2015.
- I. Rodnianski and J. Speck “A regime of linear stability for the Einstein-scalar field system with applications to nonlinear Big Bang formation”, Ann. Math. 187 (2018) 65–156