The mathematical analysis of black holes in general relativity

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Abstract. The mathematical analysis of black holes in general relativity has been the focus of considerable activity in the past decade from the perspective of the theory of partial differential equations. Much of this work is motivated by the problem of understanding the two celebrated cosmic censorship conjectures in a neighbourhood of the Schwarzschild and Kerr solutions. Recent progress on the behaviour of linear waves on black hole exteriors as well as on the full non-linear vacuum dynamics in the black hole interior puts us at the threshold of a complete understanding of the stability–and instability–properties of these solutions. This talk will survey some of these developments.

Mathematics Subject Classification (2010). Primary 83C57; Secondary 83C75.

Keywords. Einstein equations, general relativity, black holes, cosmic censorship.

1. Introduction

There is perhaps no other object in all of mathematical physics as fascinating as the black holes of Einstein's general relativity.

The notion as such is simpler than the mystique surrounding it may suggest! Loosely speaking, the black hole region $\mathcal{B}$ of a Lorentzian 4-manifold $(\mathcal{M}, g)$ is the complement of the causal past of a certain distinguished ideal boundary at infinity, denoted $\mathcal{I}^+$ and known as *future null infinity*; in symbols

$$\mathcal{B} = \mathcal{M} \setminus J^-(\mathcal{I}^+). \tag{1}$$

In the context of general relativity, where our physical spacetime continuum is modelled by such a manifold $\mathcal{M}$, this ideal boundary at infinity $\mathcal{I}^+$ corresponds to "far-away" observers in the radiation zone of an isolated self-gravitating system such as a collapsing star. Thus, the black hole region $\mathcal{B}$ is the set of those spacetime events which cannot send signals to distant observers like us.

It is remarkable that the simplest non-trivial spacetimes $(\mathcal{M}, g)$ solving the Einstein equations in vacuum

$$\mathrm{Ric}(g) = 0, \tag{2}$$

*The author is grateful to G. Holzegel, J. Luk, I. Rodnianski and Y. Shlapentokh-Rothman for comments on this manuscript and to D. Christodoulou for many inspiring discussions over the years.

the celebrated *Schwarzschild* and *Kerr* solutions, indeed contain non-empty black hole regions $\mathcal{B} \neq \emptyset$. Moreover, both these spacetimes fail to be future causally geodesically complete, i.e. in physical language, there exist freely falling observers who live for only finite proper time. The two properties are closely related in the above examples as all such finitely-living observers must necessarily enter the black hole region $\mathcal{B}$. Far-away observers in these examples, on the other hand, live forever; the asymptotic boundary future null infinity $\mathcal{I}^+$ is itself complete.

In the early years of the subject, the black hole property was widely misunderstood and the incompleteness of the above spacetimes was considered a pathology that would surely go away after perturbation. The latter expectation was shattered by Penrose's celebrated *incompleteness theorem* [68] which implies in particular that the incompleteness of Schwarzschild and Kerr is in fact a stable feature when viewed in the context of dynamics. We have now come to understand the presence of black holes not at all as a pathology but rather as a blessing, shielding the effects of incompleteness from distant observers, allowing in particular for a complete future null infinity $\mathcal{I}^+$. This motivated Penrose to formulate an ambitious conjecture known as *weak cosmic censorship* which states that for generic initial data for the Einstein vacuum equations (2), future null infinity $\mathcal{I}^+$ is indeed complete. In the language of partial differential equations, this can be thought of as a form of *global existence* still compatible with Penrose's theorem.

A positive resolution of the above conjecture would be very satisfying but would still not resolve all conceptual issues raised by the Schwarzschild and Kerr solutions. For it is reasonable to expect that our physical theory should explain the fate not just of far-away observers but of *all* observers, including those who choose to enter black hole regions $\mathcal{B}$. In the exact Schwarzschild case, such observers are destroyed by infinite tidal forces, while in the exact Kerr case, they cross a *Cauchy horizon* to live another day in a region of spacetime which is no longer determined by initial data. The former scenario is an omenous prediction indeed–but one we have come to terms with. It is the latter which is in some sense even more troubling, as it represents a failure of the notion of prediction itself. This motivates yet another ambitious conjecture, *strong cosmic censorship*, also originally due to Penrose, which says that for generic initial data for (2), the part of spacetime uniquely determined by data is inextendible. In the language of partial differential equations, this conjecture can be thought of as a statement of *global uniqueness*. For this conjecture to be true, the geometry of the interior region of Kerr black holes would in particular have to be unstable.

Despite the ubiquity of black holes in our current astrophysical world-picture, the above conjectures–even when restricted to a neighbourhood of the explicit solutions Schwarzschild and Kerr–are not mathematically understood. More specifically, we can ask the following stability and instability questions concerning the Schwarzschild and Kerr family:

> 1. *Are the <u>exteriors</u> to the black hole regions $\mathcal{B}$ in Schwarzschild and Kerr <u>stable</u> under the evolution of* (2) *to perturbation of data? In particular, is the completeness of null infinity $\mathcal{I}^+$ a stable property?*
>
> 2. *What happens to observers who enter the <u>interior</u> of the black*

hole region $\mathcal{B}$ of such perturbations of Kerr? Are the smooth Cauchy horizons of Kerr unstable?

If our optimistic expectations on these questions are in fact not realised by the theory, then this may fundamentally change our understanding of general relativity and perhaps also our belief in it!

The global analysis of solutions to the Einstein vacuum equations (2) without symmetry was largely initiated in the monumental proof [23] of the non-linear stability of Minkowski space by Christodoulou and Klainerman in 1993. As with the stability of Minkowski space, Question 1. would be a statement of global existence and stability, but now concerning a highly non-trivial geometry. Question 2., on the other hand, not only concerns a non-trivial geometry but appears to concern a regime where solutions may become unstable and in fact singular (at least, if strong cosmic censorship is indeed true!); the prospect of proving anything about such a regime seemed until recently quite remote. A number of rapid developments in the last few years, however, concerning linear wave equations on black hole backgrounds as well as the analysis of the fully non-linear Einstein equations in singular–but controlled–regimes have brought a complete resolution of Questions 1. and 2. much closer. The purpose of this talk is to survey some of these developments. In particular, we will describe the following results, which reflect the state of the art concerning our understanding of Questions 1 and 2 above, and had themselves been the subject of a number of open conjectures.

1. Linear scalar waves on Schwarzschild and Kerr backgrounds remain bounded in the black hole exterior and in fact decay polynomially. Schwarzschild is in fact linearly stable in full linearised gravity.

2. For a spherically symmetric toy model, Cauchy horizons are globally stable from the point of view of the metric in L^∞, but unstable at the level of derivatives of the metric, as the Christoffel symbols in any regular frame become singular. For the full vacuum equations (2) without symmetry, then, given the stability of the exterior, the above stability statement for the Kerr Cauchy horizon again holds.

We see in particular that the final part of 2. means that the precise understanding of Questions 1. and 2. is in fact coupled. Note that the result 2. is in fact at odds with the strongest formulations of Question 2 above and this has significant–and slightly troubling–implications as to what versions of strong cosmic censorship are indeed true. This could indicate that some of the conceptual puzzles of general relativity are here to stay!

2. Schwarzschild and Kerr

We begin by reviewing the Schwarzschild and Kerr families.

2.1. The Schwarzschild metric. The *Schwarzschild family* $(\mathcal{M}, g_M)$ represents the simplest non-trivial explicit family of solutions to the Einstein vacuum equations (2). These solutions were discovered already in December 1915 [75], the month following Einstein's final formulation of general relativity [43]. The metrics are static and spherically symmetric and can be written in local coordinates as

$$g_M = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (3)$$

Here, M is a parameter which can be identified with mass. We shall only consider the case $M > 0$. Note that the case $M = 0$ reduces to the flat Minkowski space, which is trivially a solution of (2).

In discussing the Schwarzschild solution, we have not yet settled on the ambient manifold $\mathcal{M}$ on which (3) should live! Historically, this was indeed only understood later, since the correct differentiable structure of the ambient manifold is not so immediately apparent from the form (3). If we pass, however, to new coordinates (cf. Lemaitre [57]) (t^*, r, θ, ϕ) where

$$t^* = t + 2M\log(r - 2M),$$

we see that the metric expression (3) can be rewritten

$$-(1-2M/r)(dt^*)^2 + (4M/r)drdt^* + (1+2M/r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

This suggests that we may define our underlying manifold $\widetilde{\mathcal{M}}$ to be precisely

$$\widetilde{\mathcal{M}} = (-\infty, \infty) \times (0, \infty) \times \mathbb{S}^2 \qquad (5)$$

with coordinates t^*, r, θ, ϕ, on which g_M defined by (4) manifestly yields a smooth metric. Let us for now consider $(\widetilde{\mathcal{M}}, g_M)$ as our spacetime.

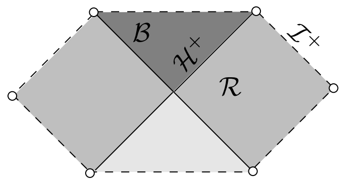
One easily sees from the form of the metric (4) that the region $\mathcal{B} \doteq \{r \le 2M\}$ has the property that future directed causal curves emanating from $\mathcal{B}$ must stay in $\mathcal{B}$ (i.e. $J^+(\mathcal{B}) = \mathcal{B}$), in particular, they cannot reach large values of r. It turns out that with a suitable definition of the asymptotic boundary future null infinity $\mathcal{I}^+$, $\mathcal{B}$ corresponds also to the black hole region defined in (1), and $\mathcal{I}^+$ is moreover complete.[1] The boundary $\mathcal{H}^+ = \{r = 2M\}$ of $\mathcal{B}$ in the spacetime $\mathcal{M}$ is known as the *event horizon*. Note that the static Killing field ∂_t of (3) extends to a Killing field ∂_{t*} on $\mathcal{M}$ which is in fact spacelike in the region $\{r < 2M\}$ and null on $\mathcal{H}^+$.

In contrast to the case of Minkowski space $M = 0$ where the above metric (4) extends from (5) to $\mathbb{R}^{3+1}$ by adding $r = 0$ to the manifold, in the case $M > 0$, the metric becomes singular as $r \to 0$ is approached. In fact, $\{r = 0\}$ can be attached as a spacelike *singular boundary* to which all *future*-incomplete causal geodesics approach. This shows that the manifold $\widetilde{\mathcal{M}}$ is future-inextendible as a suitably regular Lorentzian manifold. It is not, however, *past*-inextendible. It turns out that one can define an even larger ambient manifold $\mathcal{M}$ (by suitably pasting $\widetilde{\mathcal{M}}$ to a copy of itself) so as for (4) above to extend to a spherically symmetric solution of

[1] This means that if we define a null retarded time coordinate u such that $\partial_u r = -1$ asymptotically at $\mathcal{I}^+$, then $\mathcal{I}^+$ is covered by the u-range $(-\infty, \infty)$.

(2) which is now indeed also past-inextendible. This gives the so-called *maximally extended Schwarzschild solution* $(\mathcal{M}, g)$. See [78, 56]. In what follows, it is this $(\mathcal{M}, g)$ that we shall definitively refer to as the Schwarzschild manifold.

Note that this new manifold $(\mathcal{M}, g)$ does not admit r as a global coordinate, but can be covered by a global system of double null coordinates (U, V) whose range can be normalised to the following shaded bounded subregion $\mathcal{Q}$ of the plane $\mathbb{R}^{1+1}$:



The metric takes the form

$$-\Omega^2(U, V)dUdV + r^2(U, V)(d\theta^2 + \sin^2\theta d\phi^2)$$

where Ω and r can be described implicity. The above depiction is known as a *Carter–Penrose diagram* of $(\mathcal{M}, g)$, and gives a concrete realisation of both future null infinity $\mathcal{I}^+$ (as an open constant U-segment of the boundary of $\mathcal{Q}$ in the ambient $\mathbb{R}^{1+1}$) and the singular $\{r = 0\}$ past and future boundaries.

Note that the above manifold is *globally hyperbolic* with a Cauchy hypersurface Σ (possessing two asymptotically flat ends). That is to say, all inextendible causal curves intersect Σ exactly once. When we discuss dynamics in Section 3, this property will allow us to view Schwarzschild $(\mathcal{M}, g)$ as the maximal vacuum Cauchy development of data on Σ.

2.2. The Kerr metrics. The Schwarzschild family sits as the 1-parameter $a = 0$ subfamily of a larger, 2-parameter family $(\mathcal{M}, g_{M,a})$, discovered in 1963 by Kerr [52]. The parameter a can be identified with rotation. The latter metrics are less symmetric when $a \neq 0$–they are only *stationary* and *axisymmetric*–and are given explicitly in local coordinates by the expression

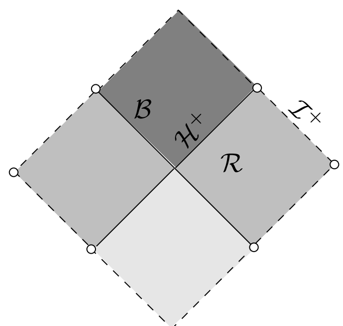
$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,dt - (r^2+a^2)d\phi\right)^2 \tag{6}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2.$$

We will only consider the case of parameter values $0 \leq |a| < M$, $M > 0$, where $\Delta = (r - r_+)(r - r_-)$ for $r_+ > r_- > 0$. The case $|a| = M$ is special and is known as the *extremal* case.

Again, by introducing $t^* = t^*(t, r)$ but now also a change $\phi^* = \phi^*(\phi, r)$, the metric can be rewritten in analogy to (4) so as to make it regular at $r = r_+$, which will again correspond to the event horizon $\mathcal{H}^+$ of a black hole $\mathcal{B}$. An additional transformation can now make the metric regular at $r = r_-$ and allows a further extension into $r < r_-$. The set $r = r_-$ will correspond to a so-called *Cauchy horizon* $\mathcal{CH}^+$ separating a globally hyperbolic region from part of the spacetime which is no longer determined by Cauchy data. Our convention will be to not include the latter extensions into our ambient manifold $\mathcal{M}$, which will, however, as in Schwarzschild, be "doubled" by appropriately pasting two $r > r_-$ regions. For us, the Kerr spacetime $(\mathcal{M}, g_{M,a})$ will thus again be globally hyperbolic with a two-ended asymptotically flat Cauchy hypersurface Σ as in the Schwarzschild case, and, in the language of Section 3, will again be the maximal vacuum Cauchy development of data on Σ. See



It is, however, precisely the existence of these further extensions to $r < r_-$ which leads to the question of strong cosmic censorship.

The Kerr solutions are truly remarkable objects with a myriad of interesting geometric properties beyond the mere fact of the presence of a black hole region $\mathcal{B}$, for instance, their having a non-trivial ergoregion $\mathcal{E}$ to be discussed in Section 4.2.1. Even the very existence in closed form of the family is remarkable, since simply imposing the symmetries manifest in the above expression (6) is by dimensional considerations clearly insufficient to ensure that the Einstein equations (2) should admit closed-form solutions. It turns out that the metrics (6) enjoy several "hidden" symmetries. For instance, they possess an additional non-trivial Killing *tensor* and they are moreover *algebraically special*. It is in fact through the latter property that they were originally discovered [52].

2.3. Uniqueness. A natural question that arises is whether there are other *stationary* solutions of (2) containing black holes $\mathcal{B}$ besides the Kerr family $g_{M,a}$.

If we impose in addition that our solutions be *axisymmetric* then indeed, the Kerr family represents the unique family of black hole solutions (with a connected horizon). See [11, 72] for the original treatments and also [24].

The expectation that the Kerr solutions are unique even without imposing axisymmetry stems from a pretty rigidity argument due to Hawking [47]. Under certain assumptions, including the real analyticity of the metric, he showed that stationary black holes are necessarily also axisymmetric, and thus, the above result applies to infer uniqueness.

The assumption of real analyticity is physically unmotivated, however, and leaves open the possibility that there may yet still be other smooth (but non-analytic) black hole solutions of (2). An important partial result has recently been proven in [1], where it is shown (generalising Hawking's rigidity argument using methods of unique continuation) that the Kerr family is indeed unique in the smooth class *provided one restricts to stationary spacetimes suitably near the Kerr family.* In particular, this means that the Kerr family is at the very least isolated in the family of all stationary solutions.

In view of this latter fact, it indeed makes sense to focus on the Kerr family, in particular, to entertain the question of its "asymptotic stability". Before turning to this, however, we must first make some general comments about dynamics for the Einstein equations (2).

3. Dynamics of the Cauchy problem

One of the early triumphs of the theory of partial differential equations applied to general relativity was the proof that the Einstein equations (2) indeed give rise to an unambiguous notion of *dynamics*. In the language of partial differential equations, this corresponds to the *well-posedness* of the Cauchy problem for (2), proven by Choquet-Bruhat [13] and Choquet-Bruhat–Geroch [14].

We will state the foundational well-posedness statement as Theorem 3.1 of Section 3.1 below. We will then proceed in Sections 3.2 and 3.3 to illustrate global aspects of the problem of dynamics with the statement of the stability of Minkowski space and with the formulation of the cosmic censorship conjectures, already mentioned in the introduction. This will prepare us for our study of the dynamics of black holes in Sections 4 and 5.

3.1. Well-posedness. Before formulating the well-posedness theorem, we must first understand what constitutes an initial state. In view of the fact that the Einstein equations (2) are second order, one expects to prescribe initially a triple $(\Sigma^3, \bar{g}, K)$, where $(\Sigma^3, \bar{g})$ is a Riemannian 3-manifold and K is an auxiliary symmetric 2-tensor to represent the second fundamental form. We say that a Lorentzian 4-manifold $(\mathcal{M}, g)$ is a *vacuum Cauchy development* of $(\Sigma^3, \bar{g}, K)$ if $(\mathcal{M}, g)$ solves (2) and there exists an embedding $i : \Sigma \to \mathcal{M}$ such that $i(\Sigma)$ is a Cauchy hypersurface[2] in $\mathcal{M}$ and $\bar{g}$ and K are indeed the induced metric and second fundamental form of the embedding.

The classical Gauss and Codazzi equations of submanifold geometry immediately imply the following *necessary* conditions on $(\Sigma^3, \bar{g}, K)$ for the existence of such an embedding:

$$\bar{R} + (\mathrm{tr}K)^2 - |K|^2_{\bar{g}} = 0, \qquad \overline{\mathrm{div}}K - d\,\mathrm{tr}K = 0. \tag{7}$$

We will thus call a triple $(\Sigma^3, \bar{g}, K)$ satisfying (7) a *vacuum initial data set*. In her seminal [13], Choquet-Bruhat proved that for regular $(\Sigma^3, \bar{g}, K)$, the conditions (7)

[2] In particular, developments are globally hyperbolic in the sense described at the end of Section 2.1. Global hyperbolicity is essential for the solution to be uniquely determined by data.

are also *sufficient* for the existence of a development and for a local uniqueness statement. In the langauge of partial differential equations, this is the analogue of *local well posedness*.

We are all familiar from the theory of ordinary differential equations that local existence and uniqueness immediately yields the existence of a unique *maximal* solution $x : (-T_-, T_+)$, where $-\infty \le T_- < T_+ \le +\infty$. In general relativity, maximalising Choquet-Bruhat's local statement is non-trivial as there is not a common ambient structure on which all solutions are defined so as for them to be readily compared. Such a maximalisation was obtained in

Theorem 3.1 (Choquet-Bruhat–Geroch [14]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum initial data set. Then there exists a unique smooth vacuum Cauchy development $(\mathcal{M}, g)$ with the property that if $(\widetilde{M}, \widetilde{g})$ is any other vacuum Cauchy development, then there exists an isometric embedding $i : (\widetilde{\mathcal{M}}, \widetilde{g}) \to (\mathcal{M}, g)$ commuting with the embeddings of Σ.*

The above object $(\mathcal{M}, g)$ is known as the *maximal vacuum Cauchy development.* It is indicative of the trickiness of the maximalisation procedure that the original proof [14] of the above theorem appealed in fact to Zorn's lemma to infer the existence of $(\mathcal{M}, g)$. This made the theorem appear non-constructive, a most unappealing state of affairs in view of its centrality for the theory. A constructive proof has recently been given by Sbierski [73].

For convenience, we have stated Theorem 3.1 in the smooth category, even though it follows from a more primitive result expressed in Sobolev spaces H^s of finite regularity. In the original proofs, this requisite H^s space was high and did not admit a natural geometric interpretation. In a monumental series of papers (see [54]) surveyed in another contribution to these proceedings [79], this regularity has been lowered to $\bar{g} \in H^2$, which can in turn be related to natural geometric assumptions concerning curvature and other quantities.

3.2. Global existence and stability of Minkowski space.

With the notion of dynamics well defined, we now turn to the prototype global existence and stability statement, the monumental *stability of Minkowski space* [23].

The result states that small perturbations of trivial initial data 1. lead to geodesically complete maximal vacuum Cauchy developments, with a complete future null infinity $\mathcal{I}^+$ and no black holes, 2. remain globally close to Minkowski space and in fact, 3. settle back down asymptotically to Minkowski space:

Theorem 3.2 (Stability of Minkowski space, Christodoulou and Klainerman [23]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum initial data set satisfying a global smallness assumption, i.e. suitably close to trivial initial data. Then the maximal vacuum Cauchy development $(\mathcal{M}, g)$ satisfies the following:*

1. *$(\mathcal{M}, g)$ is geodesically complete and moreover, one can attach a boundary $\mathcal{I}^+$ which is itself complete, and $\mathcal{M} = J^-(\mathcal{I}^+)$.*[3]

[3]Note that the statement $\mathcal{M} = J^-(\mathcal{I}^+)$ represents the fact that these perturbed spacetimes do not contain a non-trivial black hole region $\mathcal{B}$.

2. *$(\mathcal{M}, g)$ remains globally close to Minkowski space,*

3. *$(\mathcal{M}, g)$ asymptotically settles down to Minkowski space (at a suitably fast rate).*

In the language of partial differential equationss, the geodesic completeness of statement 1. can be thought of as a geometric formulation of "global existence". Statement 2. then corresponds to "orbital stability" while statement 3. corresponds to "asymptotic stability". Due to the supercriticality of the Einstein equations, the only known mechanism for showing long-time control of a solution is by exploiting its dispersive properties, which here arise due to the radiation of waves to null infinity $\mathcal{I}^+$. As a result, the more primitive statements 1. and 2. can only be obtained in the proof by *using* strong decay rates to flat space, i.e. the full quantitative version of 3. *Thus, the proof of all statements above is strongly coupled.*

The original proof of this theorem has been surveyed in a previous preceedings volume [19] for this conference series. Let us only briefly mention here the central role played by obtaining (in a bootstrap setting) decay of weighted energy quantities associated to the Riemann curvature tensor expressed in a null frame (which satisfies the Bianchi equations) and then coupling these with elliptic and transport estimates for the structure equations satisfied by the connection coefficients, schematically

$$\nabla\!\!\!/\,\Gamma = \Gamma \cdot \Gamma + \psi, \qquad \nabla\!\!\!/\,\psi = D\!\!\!\!/\,\psi + \Gamma \cdot \psi \tag{8}$$

where Γ denotes a generic connection coefficient and ψ denotes a generic curvature component. The problem is especially difficult precisely because the rate of decay of waves to null infinity $\mathcal{I}^+$ is borderline in $3+1$ dimensions. Thus, stability is not true for the generic equation of the degree of nonlinearity of (2), but requires identifying special, null-type[4] structure in (8). We will return to some of these aspects of the proof when we discuss black holes.

3.3. Penrose's incompleteness theorem and the cosmic censorship conjectures.

The explicit examples of Schwarzschild and Kerr indicate that the geodesic completeness of Theorem 3.2 <u>cannot</u> hold for general asymptotically flat data if the global smallness assumption is dropped. In the early years of the subject, one could entertain the hope that this was an artifice of the high degree of symmetry of these special solutions. As mentioned already in the introduction, this was falsified by the following corollary to Penrose's 1965 incompleteness theorem:

Theorem 3.3 (Corollary of Penrose's incompleteness thoerem [68]). *Let $(\Sigma^3, \bar{g}, K)$ be a smooth vacuum data set sufficiently close to the data corresponding to Schwarzschild or Kerr. Then the maximal vacuum Cauchy development $(\mathcal{M}, g)$ is future causally geodesically <u>incomplete</u>.*

As noted already in the introduction, in the specific examples of Schwarzschild and Kerr, the above incompleteness is "hidden" in black hole regions. That is to

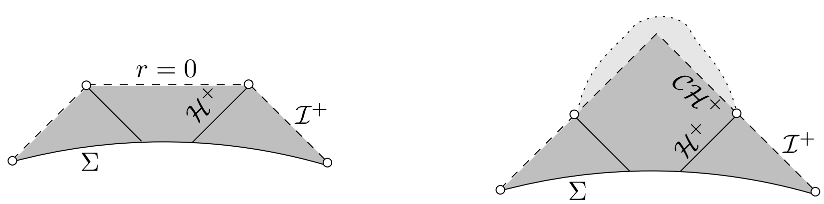
[4]In contrast, the classical null condition [53] does not hold when the Einstein equations (2) are written in harmonic gauge. See, however, the remarkable proof in [58].

say, all finitely-living observers γ must cross $\mathcal{H}^+$ into the region $\mathcal{B}$. In particular, this allows for the asymptotic boundary $\mathcal{I}^+$ to still be complete, cf. the second part of statement 1. of Theorem 3.2. This property is appealing because it means that if one is only interested in far-away observers, one need not further ponder the significance of incompleteness as the theory gives predictions for all time at $\mathcal{I}^+$. This motivates the following conjecture, originally formulated by Penrose, which, if true, would promote this feature to a generic property of solutions to (2):

Conjecture 3.4 (Weak cosmic censorship). *For* generic *asymptotically flat vacuum initial data sets, the maximal vacuum Cauchy devlopment* $(\mathcal{M}, g)$ *possesses a complete null infinity* $\mathcal{I}^+$.[5]

In the language of partial differential equations, this conjecture can be thought of as the version of *global existence* which is still compatible with Theorem 3.3.

While the above conjecture would indeed explain the possibility of far-away observation for all time, it does not do away with the puzzles opened up by the geodesic incompleteness of Theorem 3.3 from the point of view of fundamental theory. As remarked already, it is reasonable to expect that our theory gives predictions for all observers, not just "far-away" ones. The examples of Schwarzschild and Kerr tell us that the incompleteness of Theorem 3.3 may have very different origin. The Schwarzschild manifold $(\mathcal{M}, g)$ is inextendible in a very strong sense: incomplete geodesics approach what can be thought of as a spacelike singularity corresponding to $r = 0$, and not only do these observers witness infinite curvature but they are torn apart by infinite tidal forces:



Kerr, on the other hand, terminates in what can be viewed as a smooth Cauchy horizon $\mathcal{CH}^+$, across which the solution is *smoothly* extendible to a larger spacetime (the lighter shaded region) which is no longer however uniquely determined from Σ.[6] In the latter case, we see that the maximal Cauchy development is maximal *not because it is inextendible as a smooth solution of* (2) but because such extensions necessarily fail to be globally hyperbolic and thus cannot be viewed as Cauchy developments.

[5]This particular formulation is due to Christodoulou [18], who in particular, gives a precise general meaning for possessing a complete null infinity. Note also that this conjecture was originally stated without the assumption of generic. The necessity of genericity is to be expected in view of the existence of the spherically symmetric examples [16, 17].

[6]Recall that our conventions on the definition of the ambient Schwarzschild $(\mathcal{M}, g_M)$ and Kerr manifolds $(\mathcal{M}, g_{M,a})$ in Sections 2.1 and 2.2 are precisely so they be the maximal vacuum Cauchy developments of initial data $(\Sigma, \bar{g}, K)$.

As explained in the introduction, we have largely come to terms with the former possibility exhibited by Schwarzschild. It gives the theory closure as all observers are accounted for: They either live forever or are destroyed by infinite tidal forces[7]. The implications of the existence of Cauchy horizons, however, as in the Kerr case, would be quite problematic, for it restricts the ability of classical general relativity to predict the fate of macroscopic objects.

The above unattractive feature of Kerr motivated Penrose to formulate his celebrated *strong*[8] *cosmic censorship conjecture*:

Conjecture 3.5 (Strong cosmic censorship). *For generic asymptotically flat vacuum data sets, the maximal vacuum Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a suitably regular Lorentzian manifold.*

The above conjecture can be thought colloquially as saying that *"Generically, the future is determined by initial data"* since the notion of inextendibility captures the idea that there is not a bigger spacetime where the maximal Cauchy development embeds, and which would thus not be uniquely determined by Cauchy data. It can thus be considered, in the language of partial differential equations, to be a statement of *global uniqueness*.

Here the necessity of requiring genericity in the formulation of Conjecture 3.5 is clear from the start. The Kerr solutions do not satisfy the required inextendibility property. Thus, for the above conjecture to be true, this feature of Kerr must be unstable. It is not just wishful thinking that leads to Conjecture 3.5! See Section 5.1.

Finally, let us remark already that the question of how "suitably regular" should be defined in the formulation of Conjecture 3.5 is a subtle one, as will become apparent in view of Section 5.2 below.

4. The stability of the black hole exterior

To make progress on the general understanding of the theory, and in particular, the cosmic censorship conjectures of Section 3.3, we begin by looking at dynamics of (2) in a neighbourhood of the Kerr family. With the language of the Cauchy problem developed above, we may now turn to discuss what is one of the central open questions in classical general relativity–the *non-linear stability of the Kerr family* in its exterior region. This represents not only a fundamental test of weak cosmic censorship but a milestone result in itself with important implications for our current working assumption of the ubiquity of objects described by Kerr metrics in our observable universe.

4.1. The conjecture. We begin with a more precise formulation of the conjecture, taken from [29]:

[7]Speculation on what happens to their quantum ashes is beyond the scope of both classical general relativity and this article.

[8]We note that this conjecture is neither stronger nor weaker than Conjecture 3.4. See [18].

Conjecture 4.1 (Nonlinear stability of the Kerr family). *For all vacuum initial data sets* $(\Sigma, \bar{g}, K)$ *sufficiently "near" data corresponding to a subextremal* ($|a_0| < M_0$) *Kerr metric* g_{a_0,M_0}*, the maximal vacuum Cauchy development spacetime* $(\mathcal{M}, g)$ *satisfies:*

1. $(\mathcal{M}, g)$ *possesses a complete null infinity* $\mathcal{I}^+$ *whose past* $J^-(\mathcal{I}^+)$ *is bounded in the future by a smooth affine complete event horizon* $\mathcal{H}^+ \subset \mathcal{M}$*,*
2. $(\mathcal{M}, g)$ *stays globally close to* g_{a_0,M_0} *in* $J^-(\mathcal{I}^+)$*,*
3. $(\mathcal{M}, g)$ *asymptotically settles down in* $J^-(\mathcal{I}^+)$ *to a nearby subextremal member of the Kerr family* $g_{a,M}$ *with parameters* $a \approx a_0$ *and* $M \approx M_0$*.*

We have explicitly excluded the extremal case $|a| = M$ from the conjecture for reasons to be discussed in Section 4.2.5. In particular, the smallness assumption on data will depend on the distance of the initial parameters a_0, M_0 to extremality.

One can compare the above with our formulation of Theorem 3.2. Statement 1. above contains the statement of weak cosmic censorship restricted to a neighbourhood of Schwarzschild. As explained in Section 3.3, in the language of partial differential equations, this is the analogue of "global existence" still compatible with Theorem 3.3. Statement 2. can be thought to represent "orbital stability", whereas statement 3 represents "asymptotic stability". As in our discussion of the proof of the stability of Minkowski space, all these questions are coupled; it is only by identifying and exploiting the dispersive mechanism (i.e. a quantitative version of 3.) that one can show the completeness of null infinity $\mathcal{I}^+$ and orbital stability. In particular, it is essential to identify the final parameters a and M.

Like any non-linear stability result, the first step in attacking the above conjecture is to linearise the equations (2) around the Schwarzschild and Kerr solutions. The resulting system of equations is of considerable complexity; we will indeed turn to this in Section 4.3 below. But first, let us discuss what can be thought of a "poor man's" linearisation, namely the study of the *linear scalar wave equation*

$$\Box_g \psi = 0 \tag{9}$$

on a fixed Schwarzschild and Kerr background.

4.2. A poor man's stability result: $\Box_g \psi = 0$ on Kerr.

The study of (9) in the Schwarzschild case goes back to the classic paper of Regge and Wheeler [71] which considered the formal analysis of fixed modes. The first definitive result about actual solutions of (9) is due to Kay and Wald [51] and gives that solutions of $\Box_g \psi = 0$ on Schwarzschild arising from regular localised initial data remain uniformly *bounded* in the exterior, up to and including $\mathcal{H}^+$.

The last decade has seen a resurgence in interest in this problem so as to prove not just boundedness but *decay* and to handle not just Schwarzschild but the *general subextremal Kerr* case. Many researchers have contributed to this understanding [32, 5, 33, 7, 44, 36, 81, 2] which progressed from the Schwarzschild case $a = 0$ to the very slowly rotating case $|a| \ll M$ and finally to the general subextremal case $|a| < M$. This programme has culminated in the following result:

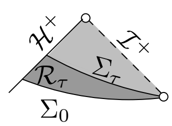
Theorem 4.2 ("Poor man's" linear stability of Kerr [39, 41]). *For Kerr exterior backgrounds in the full subextremal range $|a| < M$, general solutions ψ of* (9) *arising from regular localised data remain bounded and decay at a sufficiently fast polynomial rate through a hyperboloidal foliation of spacetime.*

See also [8, 34, 42, 50] for analysis of the wave equation on (Schwarzschild) Kerr-(anti) de Sitter backgrounds.

A complete survey of the proof of Theorem 4.2 is beyond the scope of this article, but it is worth discussing briefly the salient geometric properties of the Schwarzschild and Kerr families which enter into the analysis.

4.2.1. The conserved energy and superradiance. The existence of conserved energy identities is often crucial for boundedness results. Recall that to every Killing field X^μ, by Noether's theorem, there is a corresponding conserved 1-form associated to solutions ψ of (9) formed by contracting X^μ with the energy-momentum tensor $T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\psi\partial_\alpha\psi$. If the Killing field is causal, then the flux terms on suitably oriented spacelike or null hypersurfaces are non-negative definite. Let us examine this in the context of our problem.

We first consider the Schwarzschild case $a = 0$. As explained in Section 2.1, the static Killing field ∂_t is then timelike in the black hole exterior, becoming null at the horizon $\mathcal{H}^+$. The associated energy identity applied in a region $\mathcal{R}_\tau$

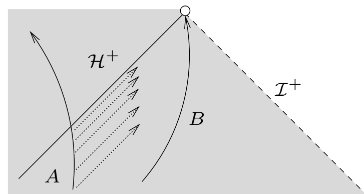


indeed gives nonnegative definite flux terms, and thus yields a useful conservation law for solutions ψ of (9)–*but barely!* After obtaining higher order estimates via further commutations of (9) by Killing fields and applying the usual Sobolev estimates, this is sufficient to estimate ψ and its derivatives pointwise *away from the horizon*. Since this energy is degenerate where ∂_t becomes null, it is, however, insufficient to obtain uniform pointwise control of the solution and its derivatives *up to and including* $\mathcal{H}^+$. The original boundedness proof of Kay and Wald [51] overcame this problem in a clever manner, but using very fragile structure associated to the exact Schwarzschild metric.

In the Kerr case, for all non-zero values $a \neq 0$, things become much worse. For there is now a region $\mathcal{E}$ in the black hole exterior where the stationary Killing field ∂_t is spacelike! This is known as the *ergoregion*. As a result, the energy flux corresponding to ∂_t is no-longer non-negative definite and thus does not yield even a degenerate global boundedness in the exterior. This is the phenomenon of *superradiance*; there is in particular no *a priori* bound on the flux of radiation to null infinity $\mathcal{I}^+$.

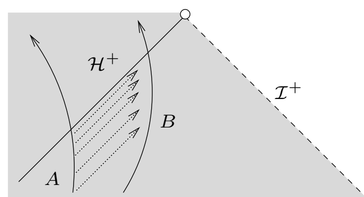
Before understanding how this problem is overcome, we must first discuss two other phenomena, the celebrated *red-shift effect* and the difficulty caused by the presence of *trapped null geodesics*.

4.2.2. The redshift. The *red-shift effect* was first discussed in a paper of Oppenheimer–Snyder [64]. One considers two observers A and B as depicted:



The more adventurous observer A falls in the black hole whereas observer B for all time stays outside. Considering a signal emited by A at a constant frequency according to her watch, in the geometric optics approximation, the frequency of the signal as measured by observer B goes to zero as B's proper time goes to infinity–i.e. it is shifted infinitely *to the red* in the electromagnetic spectrum.

For general sub-extremal black holes, there is a localised version of this effect at the horizon $\mathcal{H}^+$:



If both observers A and B fall into the black hole and are connected by time translation $A = \phi_\tau B$ where ϕ_τ is the Lie flow of the Killing field ∂_t, then the frequency measured by B is shifted to the red by a factor exponential in τ.

It turns out that the above geometric optics argument can be captured by the *coercivity properties* of a physical space energy identity near $\mathcal{H}^+$, corresponding to a well-chosen transversal vector field N to $\mathcal{H}^+$. Such a vector field was introduced in [33] and the construction was generalised in the Epilogue of [38] to arbitrary Killing horizons with positive surface gravity $\kappa > 0$.[9] The good coercivity properties do not hold globally however, and thus to obtain a useful estimate one must combine the energy identity of N with additional information.

In the Schwarzschild case $|a| = 0$, it is precisely the conserved energy estimate discussed in Section 4.2.1 with which one can combine the above red-shift estimate to obtain finally the uniform boundedness of the non-degenerate N-energy. One can moreover further *commute* (9) with N preserving the red-shift property at the horizon [37, 38] to again obtain a higher order N-energy estimate, from which then pointwise boundedness follows using standard Sobolev inequalities. This gives a simpler and more robust understanding of Kay and Wald's original [51]. See [38].

In the Kerr case $a \neq 0$, however, in view of the absense of any global a priori energy estimate, it turns out that in order to apply the N identity, one needs some understanding of dispersion. *Thus, the problems of boundedness and decay are*

[9]Note that the above positivity property breaks down in the extremal case $|a| = M$ as this is characterized precisely by $\kappa = 0$. See Section 4.2.5 below.

coupled. For the latter, however, it would seem that we have to understand a certain high-frequency obstruction to decay caused by so-called *trapped null geodesics*.

4.2.3. Trapped null geodesics. Again, we begin with the Schwarzschild case. It is well known (cf. [47]) that the hypersurface $r = 3M$ is generated by null geodesics which neither cross the horizon $\mathcal{H}^+$ nor escape to null infinity $\mathcal{I}^+$. They are the precise analogue of *trapped rays* in the classical obstacle problem. In the context of the latter, the presence of a single such ray is sufficient to falsify certain quantitative decay bounds [70]. A similar result holds in the general Lorentzian setting [74]. Weaker decay bounds can still hold, however, if the dynamics of geodesic flow around trapping is "good", that is to say, the trapped null geodesics are themselves dynamically unstable in the context of geodesic flow.

It turns out that Schwarzschild geometry indeed exhibits "good" trapping. The programme of capturing this by *local integrated energy decay estimates* with degeneration was initiated by [5]. See [33, 7, 35]. From these and the red-shift identity of Section 4.2.2, the full decay statement of Theorem 4.2 in the $a = 0$ case can now be inferred directly by a black box method [36]. See also [80].

The Schwarzschild results [33, 7, 35] exploited the fact that not only is the structure of trapping "good" from the point of view of geodesic flow in phase space, but it is localised at the codimensional-1 hypersurface $r = 3M$ of *physical space.* The latter feature is broken in Kerr for all $a \neq 0$. Nonetheless, in the case $|a| \ll M$, analogues of local integrated energy decay could still be shown using either Carter's separability [38, 40], complete integrability of geodesic flow [81], or, commuting the wave equation with the non-trivial Killing tensor [2]. Each of these methods effectively frequency localises the degeneration of trapping and uses the hidden symmetries of Kerr discussed in Section 2.2; implicitly, these proofs all show that when viewed in phase space, the structure of trapping remains "good".[10] The above [38, 40, 81, 2] all use the assumption $|a| \ll M$ in a <u>second</u> essential way, so as to treat superradiance as a small parameter; in particular, this allows one to couple integrated local energy decay with the red-shift identity of Section 4.2.2 and obtain, simultaneously, both boundedness and decay.

Although the problems of boundedness and decay are indeed coupled, a more careful examination shows that one need not understand *trapping* in order to obtain boundedness. Our earlier result [37] had in fact showed that, exploiting the property that superradiance is governed by a small parameter and the ergorergion lies well within the region of coercivitiy properties of the red-shift identity, one could prove boundedness using dispersion *only for the "superradiant part"* of the solution, which is itself *not trapped.* This in fact allowed one to infer boundedness for (9) on suitable metrics only assumed C^1 close to Schwarzschild, for which one cannot appeal to structural stability of geodesic flow.

It turns out that it is the above insight which holds the key to the general $|a| < M$ case. Remarkably, one can show that, for the entire subextremal range, not only is trapping always good, but *the superradiant part is never trapped.* The latter

[10] Note that the latter fact can also be inferred from structural stability properties of geodesic flow. See [84].

is particularly suprising since when viewed in physical space, there do exist trapped null geodesics in the ergorergion for a close to M. The above remarks are sufficient to construct frequency localised vector field multipliers yielding integrated local energy decay in the high frequency regime. See the original treatment in [39].

4.2.4. Finite frequency obstructions. There is one final new difficulty that appears in the general $|a| < M$ case: excluding the possibility of finite frequency exponentially growing superradiant modes or resonances.

The absense of the former was proven in a remarkable paper of Whiting [83]. Whiting's methods were very recently extended to exclude resonances on the axis by Shlapentokh-Rothman in [76]. These proofs depend heavily on the algebraic symmetry properties of the resulting radial o.d.e. associated to Carter's separation of (9)–*yet another miracle of the Kerr geometry!* Using a continuity argument in a, it is sufficient in fact to appeal to the result [76] on the real axis. This is the final element of the proof of Theorem 4.2. See [41] for the full details.

4.2.5. The extremal case and the Aretakis instability. Let us finally note that the precise form (see [41]) of Theorem 4.2 does *not* in fact hold without qualification for the extremal case $|a| = M$. This is related precisely to the degeneration of the red-shift of Section 4.2.2.

Theorem 4.3 (Aretakis [3, 4]). *For extremal Kerr $|a| = M$, for generic solutions of ψ, translation invariant transversal derivatives on the horizon fail to decay, and higher-order such derivatives grow polynomially.*

Decay results for axisymmetric solutions of (9) in the case of $|a| = M$ have been obtained in [4], but the non-axisymmetric case is still open and may be subject to additional instabilities. It is on account of Theorem 4.3 that we have excluded $|a| = M$ from Conjecture 4.1. The nonlinear dynamics around extremality promise many interesting features! See [63].

4.3. The full linear stability of Schwarzschild.

We have motivated our study of (9) as a "poor man's" linearisation of (2). Let us turn now to the actual linearisation of (2) around black hole backgrounds, that is to say, the true problem of linear stability.

Very recently, with G. Holzegel and I. Rodnianski, we have obtained the full analog of Theorem 4.2 for the linearised Einstein equations around Schwarzschild.

Theorem 4.4 (Full linear stability of Schwarzschild [30]). *Solutions for the linearisation of the Einstein equations around Schwarzschild arising from regular admissible data remain bounded in the exterior and decay (with respect to a hyperboloidal foliation) to a linearised Kerr solution.*

The additional difficulties of the above thorem with respect to the scalar wave equation (9) lie in the highly non-trivial structure of the resulting coupled system equations. As in the non-linear stability of Minkowski space, a fruitful way of capturing this structure is with respect to the structure equations and Bianchi equations captured by a null frame. Linearising (8), we schematically obtain

$$\nabla\!\!\!/\,\Gamma^{(1)} = \Gamma^{(1)} \cdot \Gamma^{(0)} + \psi^{(1)}, \qquad \nabla\!\!\!/\,\psi^{(1)} = \mathcal{D}\!\!\!/\,\psi^{(1)} + \Gamma^{(1)} \cdot \psi^{(0)} + \Gamma^{(0)} \cdot \psi^{(1)}, \tag{10}$$

where $\Gamma^{(1)}$, $\psi^{(1)}$ now denote linearised spin coefficients and curvature components, respectively, and $\Gamma^{(0)}$, $\psi^{(0)}$ now denote background terms. Note that in the case of Minkowski space, $\psi^{(0)} = 0$ and thus the equations for $\psi^{(1)}$ decouple from those for $\Gamma^{(1)}$ and admit a coercive energy estimate via contracting the Bel-Robinson tensor with ∂_t [22]. Already in the Schwarzschild case, however, $\psi^{(0)} \neq 0$ and the two sets of equations in (10) are coupled. A fundamental difficulty is the absense of an obvious coercive energy identity for the full system (10), or even just the Bianchi part. Thus, even obtaining a degenerate boundedness statement, cf. Section 4.2.1, is now non-trivial.

Our approach expresses (10) with respect to a suitably normalised null frame associated to a double null foliation. We then introduce a novel quantity, defined explicitly as

$$P = \mathcal{D}\!\!\!/_2^\star \mathcal{D}\!\!\!/_1^\star \left(-\rho^{(1)}, \sigma^{(1)}\right) + \frac{3}{4}\rho_0 (tr\chi)_0 \left(\hat{\chi}^{(1)} - \underline{\hat{\chi}}^{(1)}\right)$$

together with a dual quantity $\underline{P}$. Here $\rho^{(1)}$, $\sigma^{(1)}$ denote particular linearised components of the Riemann tensor, $\hat{\chi^{(1)}}$ and $\underline{\hat{\chi}}^{(1)}$ denote the linearised shears of the foliation, ρ_0 and $\mathrm{tr}\chi_0$ are Schwarzschild background terms and $\mathcal{D}\!\!\!/_2^\star$ and $\mathcal{D}\!\!\!/_1^\star$ denote the first order angular differential operators of [23].

The quantity P decouples from (10) and satisfies the Regge–Wheeler equation

$$\Omega\nabla\!\!\!/_3(\Omega\nabla\!\!\!/_4(r^5 P)) - (1-2Mr^{-1})\Delta\!\!\!/(r^5 P) + (4r^{-2} - 6Mr^{-3})(1-2Mr^{-1})(r^5 P) = 0 \tag{11}$$

Like (9), the above equation does indeed admit a conserved coercive energy estimate. The first part of our proof obtains a complete understanding of P, which is a relatively easy generalisation of Theorem 4.2 restricted to $a = 0$;

Proposition 4.5. *Solutions P of* (11) *arising from regular localised data satisfy boundedness and integrated local energy decay (non-degenerate at the horizon and with "good weights" at infinity, cf. [36]) and decay polynomially with respect to a hyperboloidal foliation.*

See also [6]. Given Proposition 4.5, one can then exploit a hierarchial structure in (10) to estimate, one by one, all other quantitites, schematically denoted $\Gamma^{(1)}$,$\psi^{(1)}$, by integration as transport equations in L^2. From integrated local energy decay and boundedness for P, one obtains integrated local energy decay and boundedness for each quantity, after a suitable linearised Kerr solution is subtracted. It is essential here that one uses the full strength of Proposition 4.5 with respect to the non-degeneration at the horizon and the "good" weights at infinity.

It is interesting to compare our approach to the formal mode analysis of the physics literature (see [12]). There one attempts to recover everything from the linearised curvature components $\alpha^{(1)}$ and $\underline{\alpha}^{(1)}$, which also decouple and satisfy the so-called Bardeen–Press equation[11]. In contrast to (11), however, this equation does not admit an obvious coercive conserved energy, but it can nonetheless be shown that it does not admit growing modes. From this one can in principle formally recover control of other quantities for fixed modes [12]. This approach, however, fails to yield an estimate beyond fixed modes, precisely because of the absense of a mode-independent energy estimate for Bardeen–Press. Note that when viewed in frequency space, our P can be related to $\alpha^{(1)}$ by the transformation theory of Chandrasekhar [12].

We reiterate finally that in the above argument, obtaining even boundedness for the full system (10) required the dispersive part of Proposition 4.5. Thus we see that, even at the linear level, there does not appear to be a pure "orbital stability" result; just as in the non-linear theory, boundedness is coupled to showing quantitative decay.

4.4. The road to Conjecture 4.1. Before turning in Section 5 to the black hole interior, let us revisit our fully nonlinear problem of Conjecture 4.1.

The issue of using decay rates as in Theorem 4.2 in a nonlinear setting satisfying a null condition has been addressed in a scalar problem by Luk [59]. See also [49].

As we described in Section 4.1, to prove Conjecture 4.1, one must identify (and linearise around) the asymptotic parameters to which the solution will asymptote–and for every open set of initial data, these parameters will generically have $a \neq 0$. It follows that until the analogue of Theorem 4.4 has been obtained for Kerr, at the very least for the very slowly rotating regime $|a| \ll M$, then one expects that there is no open set in the moduli space of initial data which can be handled.

It is worth mentioning, however, that there is a restricted version of Conjecture 4.1 which can in principle be studied using only the Schwarzschild linear stability result. If axisymmetry is imposed on the initial data and one moreover imposes that the initial angular momentum vanishes, then, since angular momentum does not radiate to null infinity under the assumption of axisymmetry, one expects that the solution should approach a Schwarzschild black hole and thus should be amenable to study using only Theorem 4.4. This is the content of ongoing work.

We mention finally that under *spherical symmetry*, one can formulate an analogous problem to that of Conjecture 4.1 concerning the Einstein–scalar field system (see [15]) or the Einstein–Maxwell–scalar field system (to be discussed in the next section).[12] The analogue of Conjecture 4.1 is then proven in [15, 26, 32]. The above problem retains few of the difficulties described in Section 4.2–in particular, it does not exhibit superradiance or trapping. Moreover, on the nonlinear side, it is interesting to note that spherical symmetry breaks the supercriticality of the Einstein equations, so in particular, allows 1., 2. and 3. to be proven separately. Nonetheless, the above models have been especially important as a source for intuition on the stability and instability properties of black hole *interiors*. We turn to this now.

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[11] In the Kerr case, this generalises to the Teukolsky equation. See [12].

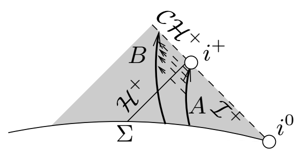
[12] Recall that in view of Birkhoff's theorem [47], the only spherically symmetric vacuum solutions are Schwarzschild.

5. The black hole interior and singularities

We now turn to the interior of Kerr black holes and strong cosmic censorship.

5.1. The blue-shift instability. In Section 3.3, we motivated Penrose's strong cosmic censorship by little other than wishful thinking–the possibility of Cauchy horizons is so problematic that we hope that generically they cannot form. There is indeed, however, a heuristic argument that suggests that at least the Kerr Cauchy horizon may be unstable.

The argument, due to Penrose [67], goes as follows. Let A and B be again two observers, where B now enters the black hole whereas A remains for all time outside. If A sends a signal to B, then the frequency measured by B becomes infinitely high as B's proper time approaches his Cauchy horizon-crossing time.



That is to say, the signal is infinitely shifted *to the blue.*

As with the red-shift effect discussed in Section 4.2.2, this effect should be reflected in the behaviour of waves, but now as an instability. This was in fact studied numerically in [77] for the related case of the scalar wave equation (9) on Reissner–Nordström background.[13] In view of the role of (9) as a "poor-man's linearisation" of (2), the above heuristic arguments were the first indication that the smooth Cauchy-horizon behaviour of Kerr could be unstable.[14]

A general result due to Sbierski [74] shows that the geometric optics argument is sufficient to falsify a quantitative energy boundedness result analogous to the precise statement of Theorem 4.2 in the exterior. Suprisingly, however, it turns out that the blue-shift instability is <u>not</u> strong enough for ψ to blow up in L^∞.

Theorem 5.1 (Franzen [45]). *Solutions ψ of the wave equation (9) as in Theorem 4.2 remain pointwise bounded $|\psi| \leq C$ on sub-extremal Kerr for $a \neq 0$ (or Reissner–Nordström $Q \neq 0$) in the black hole interior, up to and including $\mathcal{CH}^+$.*

This result, whose proof uses as an input the result of Theorem 4.2 restricted to $\mathcal{H}^+$, can be thought of as the first indication that rough stability results hold all the way to $\mathcal{CH}^+$. To explore this, however, let us first turn to certain spherically symmetric toy models.

5.2. Spherically symmetric toy-models. With the Schwarzschild case as the only example to go by, Penrose had originally speculated [67] that the

[13] Reissner–Nordström $(\mathcal{M}, g_{M,Q})$ is a spherically symmetric family of solutions to the Einstein–Maxwell equations and for $Q \neq 0$ has a Cauchy horizon similar to Kerr.

[14] For an another manifestation of the blue-shift instability when solving the Einstein equations *backwards* in the exterior, see [29].

blue-shift instability in the fully non-linear setting would give rise to a spacelike singularity[15].

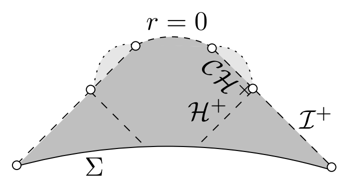
The simplest toy model with a true wave-like degree of freedom where this can be studied is the Einstein–Maxwell[16]–real scalar field system

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \doteq 8\pi(\frac{1}{4\pi}(F_\mu{}^\lambda F_{\lambda\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) + \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi) \tag{12}$$

$$\nabla^\mu F_{\mu\nu} = 0, \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0, \qquad \Box_g\psi = 0, \tag{13}$$

under spherical symmetry. It turns out that for this toy model, Penrose's expectation does not hold as stated: At least a part of the boundary of the maximal development is a null Cauchy horizon through which the metric is at least continuously extendible:

Theorem 5.2 (C^0-stability of a piece of the Cauchy horizon, [25, 27]). *For all two-ended asymptotically flat spherically symmetric initial data for* (12)–(13) *with non-vanishing charge, the maximal development can be extended through a non-empty Cauchy horizon* $\mathcal{CH}^+$



as a spacetime with C^0 *metric.*

The above theorem depends in fact also on joint work with Rodnianski [32] on the exterior region (cf. the end of Section 4.4) which obtains upper polynomial bounds for the decay of ψ on $\mathcal{H}^+$. Heuristic and numerical [46, 10] work suggests a precise asymptotic tail, in particular, polynomial *lower bounds* on $\mathcal{H}^+$. With this as an *assumption*, one can obtain the following

Theorem 5.3 (Weak null singularities, [27]). *For spherically symmetric initial data as above where a pointwise lower bound on* $\partial_v\psi$ *is assumed to hold asymptotically along the event horizon* $\mathcal{H}^+$ *that forms, then the above Cauchy horizon* $\mathcal{CH}^+$ *is singular: The Hawking mass (thus the curvature) diverges and, moreover, the extension of Theorem 5.2 fails to have locally square integrable Christoffel symbols.*

The above two theorems confirmed a scenario which had been suggested on the basis of previous arguments of Hiscock [48], Israel–Poisson [69] and Ori [65] as well as numerical studies of the above system [9, 10]. In view of the blow up of the

[15] In fact, one still often sees an alternative formulation of Conjecture 3.5 as the statement that "Generically, singularities are spacelike".

[16] The pure scalar field model, whose study was pioneeered by Christodoulou [15], does not admit Cauchy horizons emanating from i^+. The system (12)–(13) is the simplest generalisation that does, in view of the fact that it admits Reissner–Nordström as an explicit solution.

Hawking mass, the phenomenon was dubbed *mass inflation*. The type of singular boundary exhibited by the above theorem, where the Christoffel symbols fail to be square integrable but the metric continuously extends, is known as a *weak null singularity*.

The above results apply to general solutions, not just small perturbations of Reissner–Nordström. In the stability context, it turns out that the $r = 0$ piece is absent, and the entire bifurcate Cauchy horizon is globally stable:

Theorem 5.4 (Global stability of the Reissner–Nordström Cauchy horizon [28]). *For small, spherically symmetric perturbations of Reissner–Nordström, the maximal development is extendible beyond a bifurcate null horizon as a manifold with continuous metric. The Carter–Penrose diagramme is as in the Reissner–Nordström case. In particular, there is no spacelike part of the singularity.*

Note that the above is precisely the result that one obtains by naively extrapolating Theorem 5.1 to the fully non-linear theory, identifying ψ with the metric.

Corollary 5.5 (Bifurcate weak null singularities, [28]). *Under the assumptions of Theorem 5.4 and the additional asssumption of Theorem 5.3 on both event horizons, the Cauchy horizons $\mathcal{CH}^+$ represent bifurcate weak null singularities and the extensions fail to have locally square integrable Christoffel symbols.*

The ultimate spherically symmetric toy model is that of the Einstein-Maxwell–*charged* scalar field system, that is when the scalar field is complex-valued and carries charge and is directly coupled with the Maxwell field through this charge, besides the gravitational coupling through the Einstein equations (as in (12)). In his Cambridge Ph.D. thesis [55], J. Kommemi has shown an analogue of Theorem 5.2 for this model, *given an a priori decay assumption on the horizon.*

5.3. Beyond toy models: Einstein vacuum equations without symmetry.

Whereas the above work [32, 27, 55] more or less definitively resolves the issue of the appearance of weak null singularities in spherically symmetric toy models, one could still hold out hope that the vacuum Einstein equations (2) do not allow for the formation of such singularities but favour spacelike singularities as in the Schwarzschild case. In contrast to the spherically symmetric "toy" world, for the Einstein vacuum equations without symmetry there is really no numerical work available on this problem and very little heuristics (see however [66]).

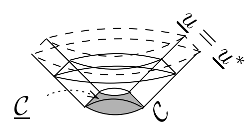
5.3.1. Luk's vacuum weak null singularities. The first order of business is thus to construct examples of local patches of vacuum spacetime with a weak null singular boundary. This has recently been accomplished in a breakthrough paper of J. Luk [60], based in part on his previous work with Rodnianski [61, 62] on impulsive gravitational waves.

Luk's spacetimes have no symmetries and are constructed by solving a characteristic initial value problem with characteristic data of a prescribed singular behaviour. The problem reduces to showing existence in a rectangular domain as well as propagation of the singular behaviour. This is given in:

Theorem 5.6 (Luk [60]). *Consider characteristic initial data for the Einstein vacuum equations on a bifurcate null hypersurface* $\mathcal{C} \cup \underline{\mathcal{C}}$ *whose spherical sections are parameterised by affine* $\underline{u} \in [0, \underline{u}^*))$ *and* $u \in [0, u^*))$*, resepectively, and where the outgoing shear* $\hat{\chi}$ *(and sufficient angular derivatives) satisfies*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}. \tag{14}$$

Then the maximal development can be covered by a double null foliation terminating in a null boundary $\underline{u} = \underline{u}^*$



through which the metric is continuously extendible. The singular behaviour (14) *propagates, making this boundary a* *<u>weak null singularity</u>.*

Moreover, in analogy with the Luk–Rodnianski theory of two interacting impulsive gravitational waves [62], Luk obtained

Theorem 5.7 (Luk [60]). *Consider again characteristic data as above but such that both outgoing shears* $\hat{\chi}$ *and* $\hat{\underline{\chi}}$ *(and sufficient angular derivatives) satisfy*

$$|\hat{\chi}| \sim |\log(\underline{u}^* - \underline{u})|^{-p} |\underline{u}^* - \underline{u}|^{-1}, \qquad |\hat{\underline{\chi}}| \sim |\log(u^* - u)|^{-p} |u^* - u|^{-1}, \tag{15}$$

and moreover, the data satisfies an appropriate smallness condition. Then the maximal development can be covered by a double null foliation which terminates in a bifurcate null hypersurface $\{u^*\} \times [0, \underline{u}^*] \cup [0, u^*] \times \{\underline{u}^*\}$ *through which the metric is continuously extendible. Relations* (15) *propagate, making the boundary of spacetime a* *<u>bifurcate weak null singularity</u>.*

Note that in Luk–Rodnianski theory [61, 62], (14) is replaced by the assumption that $\hat{\chi}$ is discontinuous but *bounded*. Thus, it was possible in [61, 62] to interpret the Einstein equations beyond these null hypersurfaces, which interact simply passing through each other, leaving in their wake a regular spacetime. Here, however, the boundaries are much more singular ($\hat{\chi}$ is not in any L^p for $p > 1$), and thus, the solution cannot be interpreted beyond them, even as a weak solution of (2).[17]

In the short space of this article, it is impossible to give an overview of the proofs of the above theorems. As in several of the results we have discussed, the proof expresses (8) with respect to a null frame attached to a double null foliation, and moreover, relies on a renormalisation of this system which removes the most singular components (extending ideas from [61, 62]). This does not completely regularise the system, however, and a fundamental role is played by a hierarchy of largeness/smallness which is preserved in evolution by special null structure of (8). These ideas are in turn related to the seminal work of Christodoulou [20] on the dynamic formation of trapped surfaces, surveyed in another article in these proceedings [21], and his *short pulse method*.

[17] In particular, the name "weak null singularity" is in some sense unfortunate!

5.3.2. The global stability of the Kerr Cauchy horizon. Putting together essentially all the ideas form Sections 5.2–5.3.1, we have very recently obtained the following result in upcoming joint work with J. Luk.

Theorem 5.8 (Global stability of the Kerr Cauchy horizon [31]). *Consider characteristic initial data for* (2) *on a bifurcate null hypersurface* $\mathcal{H}^+ \cup \mathcal{H}^+$*, where* $\mathcal{H}^\pm$ *have future-affine complete null generators and their induced geometry is globally close to and dynamically approaches that of the event horizon of Kerr with* $0 < |a| < M$ *at a sufficiently fast polynomial rate. Then the maximal development can be extended beyond a bifurcate Cauchy horizon* $\mathcal{CH}^+$ *as a Lorentzian manifold with* C^0 *metric. All finitely-living observers pass into the extension.*

Let us note explicitly that a corollary of the above theorem *together with a successful resolution of Conjecture 4.1* would be the following definitive statement

Corollary 5.9. *If Conjecture is 4.1 is true then the Cauchy horizon of the Kerr solution is globally stable and the* C^0*-inextendibility formulation and the "generically, spacetime singularities are spacelike" formulation of strong cosmic censorship are both false.*

5.3.3. The future for strong cosmic censorship. In view of the toy-model results of Theorem 5.3 and Corollary 5.5, all is not lost for strong cosmic censorship. A version of the inextendibility requirement in the formulation of strong cosmic censorship which is compatible with the result of Theorem 5.3 for the toy problem and may still be true for the vacuum without symmetry is the statement that *"*$(\mathcal{M}, g)$ *be inextendible as a Lorentzian manifold with locally square integrable Christoffel symbols"*. This formulation is due to Christodoulou [20] and would guarantee that there be no extension which can be interpreted as a weak solution of (2). It is an interesting open problem to obtain this in a neighbourhood of the Kerr family. This naturally separates into the following two statements:

Conjecture 5.10. *1. Under a suitable assumption on the data on* $\mathcal{H}^+$ *in Theorem 5.8, then* $\mathcal{CH}^+$ *is a weak null singularity, across which the metric is inextendible as a Lorentizian manifold with locally square integrable Christoffel symbols. 2. The above assumption on* $\mathcal{H}^+$ *holds for the data of Conjecture 4.1, provided the latter are generic.*

One can in fact localise the result of Theorem 5.8 to apply to spacetimes with one asympotically flat end, provided they satisfy the assumption on $\mathcal{H}^+$, and one can infer again a non-empty piece of null singular boundary $\mathcal{CH}^+$. Thus, all black holes which asymptotically settle down in their exterior region to Kerr with $0 < |a| < M$ will have a non-empty C^0-Cauchy horizon, which, assuming a positive resolution to Conjecture 5.10, will correspond to a weak null singularity.

Do the above Cauchy horizons/weak null singularities "close up" the whole maximal development as in the above two-ended case? Or will they give way to a spacelike (or even more complicated) singularity? These questions may hold the key to understanding strong cosmic censorship beyond a neighbourhood of the Kerr family.

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