CURVATURE ESTIMATES FOR STABLE MINIMAL HYPERSURFACES

Let M be an embedded smooth hypersurface of the open unit ball $B = B_1^{n+1}(0)$ of $\mathbb{R}^{n+1}$ with $0 \in M$ and $\partial M \cap B = \emptyset$. M is said to be minimal if its mean curvature vanishes everywhere. This condition is equivalent to the variational requirement that M is a critical point of the n-dimensional area functional, in the sense that for each vector field $X \in C_c^\infty(B; \mathbb{R}^{n+1})$, if we let $\varphi_t(x) = x + tX(x)$ for $x \in B$, then $\frac{d}{dt}\big|_{t=0} \mathcal{H}^n(\varphi_t(M)) = 0$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. It is of interest to understand the nature of weak (i.e. measure-theoretic) limits of sequences of such minimal hypersurface under local uniform area bounds. It is known that the "limit surface" remains a critical point of area provided we take area to mean n-dimensional Hausdorff measure weighted by a certain positive integer valued multiplicity function, but in this generality, very little is known about the regularity of the limit. All that is known is that there is a dense open subset of points of the limit surface at which it is smoothly embedded; it is not known if the limit surface must be regular $\mathcal{H}^n$ almost everywhere.

For many applications in geometry however, it has proven to be sufficient to study this question subject to the additional hypothesis that M is stable, i.e. that M satisfies $\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{H}^n(\varphi_t(M)) \geq 0$ for all vector fields $X \in C_c^\infty(B; \mathbb{R}^{n+1})$, which includes the case of locally area minimizing hypersurfaces. There is a strikingly rich regularity and compactness theory for stable minimal hypersurfaces satisfying uniform local area bounds that has been developed over the past 50 years. The goal of this project is to understand a part of this theory developed from a PDE theoretic point of view which has been extremely effective in low dimensions—specifically when $n \leq 5$. The focus of the project is to give a complete proof of the interior pointwise curvature estimate due to Schoen, Simon and Yau (Theorem 3 in [SSY]) which says that for each $\Lambda > 0$ there is a constant C that depends only on Λ such that if $n \leq 5$ and if M is a stable minimal hypersurface of B with $0 \in M$ and $\mathcal{H}^n(M) \leq \Lambda$, then

$$\sup_{M \cap B_{1/2}^{n+1}(0)} |A| \leq C$$

where A denotes the second fundamental form of M. Global geometric consequences of this estimate pointed out in [SSY] are also to be discussed. (The estimate holds also when $n = 6$ but curiously the method of proof in [SSY] does not extend to this case. Such an estimate cannot hold for stable hypersurfaces when $n \geq 7$, nor can it hold without stability even when $n = 2$).

The proof of this estimate requires (i) derivation of the PDE satisfies by $|A|$ (Simons' identity); (ii) using this PDE and the stability hypothesis to derive L^p estimates on $|A|$ for various values of p (valid in fact in all dimensions) and (iii) using a technique such as Moser iteration to arrive at the pointwise estimate. For basics of hypersurface geometry and the derivation of the stability inequality, Simons' identity and the Sobolev inequality on minimal hypersurfaces, [S] is an excellent reference. For the integral estimates on $|A|$, follow the paper [SSY]. To learn the Moser iteration technique, follow [GT].

References

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