ANALYSIS OF PDE'S (CCA) 2016
MID-TERM ASSIGNMENT

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There are five assignments, to be studied in groups of 2–3. As a result each group will present a 45' talk to the whole class on dates TBD. The preparation should include a presentation of the theorem(s) in reasonable generality, with the necessary definitions, and if possible with an example on how it is used in PDEs. Note that the order of the topics listed below will correspond to the order of the talks.

1. Holmgren's uniqueness theorem and Lewy's counter-example

For ODE's, the Cauchy–Kovalevskaya theorem can be interpreted merely as an *a posteriori regularity theorem*: all solutions constructed by Picard–Lindelöf are indeed analytic as soon as the coefficients are. A natural question is whether the same occurs for PDE's. The answer is yes for *linear systems*, and this is Holmgren's uniqueness Theorem (which can be again interpreted as an a posteriori regularity result). The reference is F. John *Partial differential equations*, pp. 80–87.

When the coefficients are no longer analytic but only smooth (C^∞) the next natural question is whether there always exist local solutions, just like what happens with Cauchy-Kovalevskaya Theorem for analytic coefficients. The answer is in general no, even for linear PDEs. This is Lewy's counter-example, the reference is his 1957 paper *An Example of a Smooth Linear Partial Differential Equation Without Solution*.

The presentation should present both results with their proofs.

2. Hille–Yosida Theorem

This theorem is a powerful tool from the theory of semigroups for constructing solutions to linear PDE's, and identifies sharp conditions on the associated *generator*, i.e. the operator L defining the PDE $\partial_t u = Lu$, for the semigroup to exist.

The notions of unbounded operators with dense domain and that of spectrum and resolvent for such operators on Banach space should be introduced. Then, the Hille–Yosida theorem should be stated and proved first for contraction semigroups, giving also in this case the formulation of Lumer–Philips using the dual space.

Next, the theorem should be stated and proved under the more general conditions $\|(\lambda - L)^{-n}\| \leq M(\lambda - \nu)^{-n}$ for any $n \geq 1$, some fixed $\nu > 0$ and $M \geq 1$, and any $\lambda > \nu$. Conversely, the presentation should also explain the conditions on a semigroup for the existence of a generator, together with its formula in terms of the semigroup.

The main reference is Pazy's book *Semigroups of Linear Operators and Applications to Partial Differential Equations*, although some material can also be found in Evans' book.

3. Sobolev inequalities

The Sobolev inequalities are one of the most fundamental tools in all parts of PDE's.

This assignement consists in introducing the general notions of Hölder and Sobolev spaces, and presenting the proof of Sobolev inequalities, including Gagliardo-Nirenberg-Sobolev inequality, Morrey's inequality, and the general Sobolev inequalities. If time permits, trace inequalities could also be discussed.

The main references are the book of Evans, the book of Lieb–Loss and the book of Brézis.

4. Riesz–Thorin Theorem and interpolation inequalities

This is one of the oldest and most important *interpolation theorems*, which plays a key role in analysis in general and PDE's in particular. The theorem shows how to interpolate between bounds obtained in different Lebesgue spaces for a linear operator.

There are several proofs: the original proof of Riesz based on a reduction to ℓ^p, discrete spaces and intricate calculations, the proof of Thorin which started the so-called "complex interpolation method" (see Thorin's paper or Rudin's book *Complex Analysis*, chap. 12) later extended by Calderón and others, and the abstract real-interpolation method (J- and K-methods), which can be found in Bergh & Löfström's book *Interpolation Spaces: An Introduction*.

The presentation should present the theorem, include a detailed presentation of one of its proofs, convey some ideas of the others if possible, and introduce the general notion of interpolation spaces.

5. The Littlewood-Paley decomposition

The 1-variable theory was introduced by Littlewood and Paley in the 1930s and developed further by Polish mathematicians Zygmund and

Marcinkiewicz using complex function theory; more recently Stein later extended the theory to higher dimensions. The idea is to decompose a function into blocks localised in Fourier variable, in an approximately orthogonal way.

The assignment should first present the basic definitions, together with how derivatives interact with the decomposition, a proof of the characterization of Lebesgue and Sobolev spaces in terms of Lebesgue norms of the dyadic blocs, Bernstein inequalities, and the introduction of Besov spaces with their main embeddings into Sobolev spaces.

Second it should present the key application of this decomposition: the estimates on products and composition of functions in Sobolev spaces allowing to avoid "distributing" derivatives. If time permits the Marcinkiewicz multiplier theorem could be presented, together with the notion of pseudo-differential operators. The main references are the book *Pseudo-differential operators and Nash-Moser theorem* of Serge Alinhac and Patrick Gérard (Chapter II A plus beginning of B) and the lecture notes of Isabelle Gallagher.