Dynamic instability of solitons in $4 + 1$-dimensional gravity with negative cosmological constant

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Abstract

We present heuristic arguments suggesting that perturbations of Eguchi-Hanson solitons in $4 + 1$-dimensional gravity with negative cosmological constant evolve into naked singularities, while perturbations of anti-de Sitter space evolve into black holes. In support of the first, we rigorously show that perturbations of Eguchi-Hanson cannot evolve into spacetimes with horizons. Finally, we contrast these conjectures and results with an orbital stability theorem for AdS-Schwarzschild.

1 Introduction

In this note, we shall show the following

Theorem 1.1. *Consider asymptotically flat smooth initial data* $(\mathcal{S}, \bar{g}, K)$ *for the vacuum Einstein equations with cosmological constant* $\Lambda < 0$*, possessing biaxial Bianchi IX symmetry, sufficiently close to Eguchi-Hanson initial data with topology* $S^3/\mathbb{Z}_p$ *at infinity. Let* $(\mathcal{M}, g)$ *denote the maximal development of the initial-boundary value problem with standard boundary conditions at infinity preserving the mass, and let* $\pi : \mathcal{M} \to \mathcal{Q}$ *denote the projection map to the* 2*-dimensional Lorentzian quotient* $\mathcal{Q}$*. Then*

$$J^-(\mathcal{I}) = \mathcal{Q}.$$

Theorem 1.1 can be paraphrased by the statement:

Perturbations of Eguchi-Hanson cannot form horizons.

In addition to the above Theorem, we have the following

Theorem 1.2. *For small perturbations of Eguchi-Hanson, the conserved mass* M *at* $\mathcal{I}^+$ *satisfies*

$$M_{p,\Lambda} < M < 0, \tag{1}$$

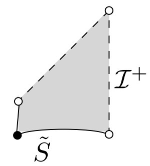
where $M_{p,\Lambda}$ *denotes the mass of the Eguchi-Hanson soliton.*

On the other hand, from previous work? we have

Theorem 1.3. *There are no static regular solutions of the Einstein equations with cosmological constant* $\Lambda < 0$ *and with topology* $S^3/\mathbb{Z}_p$ *at infinity, with mass satisfying* (1).

One thus may conjecture:

Conjecture 1.1. *The $\mathcal{Q}$ of Theorem 1.1 has Penrose diagram:*



Moreover, timelike infinity $\mathcal{I}^+$ is incomplete.

Conjecture 1.1 can be paraphrased by the statement:

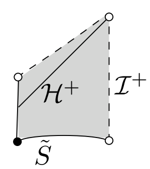
Eguchi-Hanson is dynamically unstable and perturbations of it lead to naked singularities.

We compare the situation with that of perturbations of AdS. Here, we have again by a classical theorem of Boucher, Gibbons and Horowitz:

Theorem 1.4. *There are no static regular asympototically AdS vacuum spacetimes with mass $M > 0$.*

On the other hand, horizons may now form in this case. Thus we may conjecture:

Conjecture 1.2. *Consider now asymptotically AdS smooth initial data $(\mathcal{S}, \bar{g}, K)$ possessing biaxial Bianchi IX symmetry, sufficiently close to trivial initial data, and let $\mathcal{Q}$ be as before. Then $\mathcal{Q}$ has a subset with Penrose diagram:*



Moreover, timelike infinity $\mathcal{I}^+$ is complete.

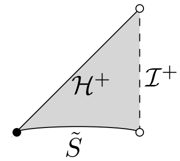
Conjecture 1.2 can be paraphrased by the statement:

Anti-de Sitter space is dynamically unstable and perturbations of it lead to black holes.

Both instabilities can be thought of as purely non-linear "memory" effects, necessitated by the conservation of mass at $\mathcal{I}^+$. In fact, somewhat ironically perhaps, their plausability is related to the validity of linear *stability*.

In contrast, in the case where trapped surfaces are present we have the following

Theorem 1.5. *Consider now asymptotically AdS data with Bianchi IX symmetry containing a trapped surface.*[1] *Then $\mathcal{Q}$ contains a subset with Penrose diagram:*



[1]In particular, this is true for data sufficiently close to Schwarzschild-AdS data.

Theorem 1.5 can be paraphrased by the statement:

Schwarzschild-AdS is orbitally stable.

One can conjecture that

Conjecture 1.3. *The solution in $J^-(\mathcal{I}^+)$ suitably approaches a Schwarzschild-AdS solution as advanced and retarded time go to infinity.*

The above conjecture can be paraphrased by the statement:

The Schwarzschild-AdS family is asymptotically stable.

Indeed, the fact that Schwarzschild-AdS is stable while Eguchi-Hanson and AdS may not be is related to the fact that the former is embedded in a one-parameter family[2]: For now, the different (conserved) mass of the perturbations does not deny them a static endstate.

2 Biaxial Bianchi IX AdS

We will say that a spacetime $(\mathcal{M}, g)$ admits biaxial Bianchi IX symmetry if topologically, $\mathcal{M} = \mathcal{Q} \times SU(2)$, for $\mathcal{Q}$ a 2-dimensional manifold possibly with boundary, and where global coordinates u and v can be chosen on $\mathcal{Q}$ such that

$$g = -\Omega^2(u,v)du\ dv + \frac{1}{4}r^2(u,v)\left(e^{2B(u,v)}\left(\sigma_1^2 + \sigma_2^2\right) + e^{-4B(u,v)}\sigma_3^2\right) \tag{2}$$

where B, Ω, and r are functions $\mathcal{Q} \to \mathbb{R}$, and the σ_i are a standard basis of left invariant one-forms on $SU(2)$, i.e. such that coordinates (θ, ϕ, ψ) can be chosen on $SU(2)$ with

$$\begin{aligned} \sigma_1 &= \sin\theta \sin\psi d\phi + \cos\psi d\theta, \\ \sigma_2 &= \sin\theta \cos\psi d\phi - \sin\psi d\theta, \\ \sigma_3 &= \cos\theta d\phi + d\psi. \end{aligned} \tag{3}$$

Geometrically, the function B measures the squashing of the three sphere. From the five-dimensional Einstein equations with a negative cosmological constant $\Lambda = -\frac{6}{l^2}$

$$R_{\mu\nu} = \frac{2}{3}\Lambda g_{\mu\nu} = -\frac{4}{l^2}g_{\mu\nu} \tag{4}$$

we can derive the two contraint equations

$$\partial_u\left(\Omega^{-2}\partial_u r\right) = -\frac{2r}{\Omega^2}(B_{,u})^2, \tag{5}$$

$$\partial_v\left(\Omega^{-2}\partial_v r\right) = -\frac{2r}{\Omega^2}(B_{,v})^2, \tag{6}$$

and a system of non-linear wave equations for the quantities r, Ω and B:

$$r_{,uv} = -\frac{1}{3}\frac{\Omega^2 R}{r} - \frac{2r_{,u}r_{,v}}{r} - \frac{\Omega^2 r}{l^2}, \tag{7}$$

$$\partial_u\partial_v \log\Omega = \frac{\Omega^2 R}{2r^2} + \frac{3}{r^2}r_{,u}r_{,v} - 3B_{,u}B_{,v} + \frac{\Omega^2}{2l^2}, \tag{8}$$

[2]Of course, one can think also of AdS as a special member of the Schwarzschild-AdS family. And Conjecture 1.3 may still be true for perturbations of AdS.

$$B_{,uv} \quad = \quad -\frac{3}{2}\frac{r_{,u}}{r}B_{,v} - \frac{3}{2}\frac{r_{,v}}{r}B_{,u} - \frac{\Omega^2}{3r^2}\left(e^{-2B} - e^{-8B}\right) \tag{9}$$

Here R is the scalar curvature of the group orbit, given by

$$R = 2e^{-2B} - \frac{1}{2}e^{-8B} \tag{10}$$

We can define a renormalized Hawking mass

$$m = \frac{r^2}{2}\left(1 + \frac{4r_{,u}r_{,v}}{\Omega^2}\right) + \frac{r^4}{2l^2} = m_{bare} + \frac{r^4}{2l^2} \tag{11}$$

Note that $m > m_{bare}$. Using the equations of motion one establishes the following monotonicity properties of the renormalized Hawking mass.

$$\partial_u m \leq 0 \tag{12}$$

$$\partial_v m \geq 0 \tag{13}$$

Note that in the asymptotically flat case, $l \to \infty$, the renormalized Hawking mass reduces to the standard Hawking mass.

3 Eguchi-Hanson AdS

The Eguchi-Hanson metric is a member of the biaxial class discussed in the last section. If we define

$$r = \rho\left(1 - \frac{a^4}{\rho^4}\right)^{\frac{1}{6}} \tag{14}$$

$$\Omega^2 = 1 + \frac{\rho^2}{l^2} \tag{15}$$

$$B = -\frac{1}{6}\ln\left(1 - \frac{a^4}{\rho^4}\right) \tag{16}$$

$$du = dt - \frac{1}{\sqrt{\left(1 + \frac{\rho^2}{l^2}\right)^2\left(1 - \frac{a^4}{\rho^4}\right)}}d\rho \tag{17}$$

$$dv = dt + \frac{1}{\sqrt{\left(1 + \frac{\rho^2}{l^2}\right)^2\left(1 - \frac{a^4}{\rho^4}\right)}}d\rho \tag{18}$$

we recover the Eguchi-Hanson AdS metric in the perhaps more familiar (t, ρ) coordinates:

$$-\left(1 + \frac{\rho^2}{l^2}\right)dt^2 + \frac{d\rho^2}{\left(1 + \frac{\rho^2}{l^2}\right)\left(1 - \frac{a^4}{\rho^4}\right)} + \frac{\rho^2\left(1 - \frac{a^4}{\rho^4}\right)}{4}\left(d\psi + \cos\theta d\phi\right)^2 + \frac{\rho^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{19}$$

Regularity at $\rho = a$, the center, enforces the coordinate ψ to have period $\frac{2\pi}{\sqrt{1+\frac{a^2}{l^2}}}$. On the other hand, ψ has to have period $\frac{4\pi}{p}$ for an integer p to remove the string singularities at the poles. The two conditions lead to the relation

$$a^2 = l^2\left(\frac{p^2}{4} - 1\right) \tag{20}$$

for $p \geq 3$. Hence the topology of the orbital space is $S^3/\mathbb{Z}_p$. (Note that there are no regular AdS Eguchi-Hanson solitons with $p = 2$.)

4 Proof of Theorem 1.1

4.1 Non-Existence of marginally trapped surfaces

It is straightforward to compute the Hawking mass of Eguchi-Hanson at infinity

$$m_\infty = \lim_{\rho \to \infty} m = -\frac{1}{6}\frac{a^4}{l^2} \tag{21}$$

We also note that the renormalized mass diverges to $-\infty$ for $\rho \to a$. These little facts immedeatly lead to the following

Lemma : Eguchi-Hanson AdS will not form (marginally) trapped surfaces under perturbations.

Proof : Under sufficiently small perturbations the renormalized mass will still be negative at infinity. By monotonicity of m, the renormalized mass is negative everywhere in the spacetime. But

$$m = \frac{r^2}{2}\left(1 + \frac{4r_{,u}r_{,v}}{\Omega^2}\right) + \frac{r^4}{2l^2} \leq 0 \tag{22}$$

can't hold together with $r_{,v} \leq 0, r_{,u} < 0$, the conditions for (marginally) trapped surfaces. Hence such surfaces must be absent.

To prove Theorem 1.1 we need to establish the stronger statement that no horizons will form. For this we will need a rigorous setup of the initial value problem in asymptotically locally AdS spaces (within the symmetry class considered) to prove an extension principle analogous to the one used in the old paper.

4.2 Initial Value Formulation for asymptotically locally AdS

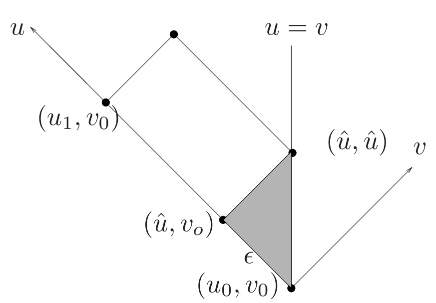
In this section we construct the initial value for asymptotically AdS spaces within the biaxial Bianchi-IX class of metrics.

4.2.1 Local existence in the interior

Can be done in the standard way (Mihalis' notes). Moreover, we have an extension principle for these "diamonds" from our old paper.

4.2.2 Local existence near (timelike) null-infinity

On the AdS-boundary, where $\widetilde{r} := \frac{1}{r} = 0$, we choose coordinates such that $u = v$. The situation is indicated in the figure. We will now specify initial data on the



$v = v_0$ ray satisfying the u-constraint and enforce the function Ω (or equivalently,

m) to be constant on the boundary. We also impose $B = 0$ on the boundary to make the spacetime asymptotically locally AdS. We shall be abler to infer local existence in at least a small region as indicated in the figure.All assumptions and the local existence theorem is made precise in the following

Proposition: Let $\Omega^{init}, \tilde{r}^{init}, B^{init}$ be C^2-functions defined on $[u_0, u_1] \times v_0$ satisfying the constraint

$$\partial_u \left(\frac{\tilde{r}^{init}_{,u}}{\left(\tilde{r}^{init}\right)^2 \left(\Omega^{init}\right)^2} \right) = \frac{2}{\left(\Omega^{init}\right)^2} \tilde{r}^{init} \left(B^{init}_{,u}\right)^2 \tag{23}$$

Assume

$$\left| B^{init}\left(\cdot , v_0\right) \right|_{C^1} \leq C \tag{24}$$

$$\left| \tilde{r}^{init}\left(\cdot , v_0\right) \right| \leq C \tag{25}$$

$$\left| \frac{\tilde{r}^{init}_{,u}}{\tilde{r}^{init}}\left(\cdot , v_0\right) \right| \leq C \tag{26}$$

$$\left| 2\log\left(\Omega^{init}\right)\left(\cdot , v_0\right) \right|_{C^1} \leq C \tag{27}$$

Then there exists an $\epsilon > 0$, dependent on the choice of C only, such that, defining $\widehat{u} = min(u_0 + \epsilon, u_1)$ there exist unique C^2-functions $(\Omega, \tilde{r}, B)$ on the region

$$\bar{\mathcal{R}} = \{(u,v) \in \mathcal{R} \ \ | \ \ u \geq v\} \quad \text{where} \quad \mathcal{R} = [u_0, \hat{u}] \times [v_0, \hat{u}] \tag{28}$$

coinciding with $(\Omega^{init}, \tilde{r}^{init}, B^{init})$ on $[u_0, \hat{u}] \times v_0$, satisfying $B = const$, $\Omega = const$ (and hence $m = const$) on the boundary and solving the evolution equations

$$\tilde{r}_{,uv} = \frac{1}{3}\Omega^2 \tilde{r}^3 \left(R + \frac{3}{l^2 \tilde{r}^2} \right) + 4\tilde{r} \frac{\tilde{r}_{,u}\tilde{r}_{,v}}{\tilde{r}^2} \tag{29}$$

$$\left(\log\Omega\right)_{,uv} = \frac{1}{2}\Omega^2 \tilde{r}^2 R + 3\frac{\tilde{r}_{,u}\tilde{r}_{,v}}{\tilde{r}^2} - 3B_{,u}B_{,v} + \frac{\Omega^2}{2l^2} \tag{30}$$

$$B_{,uv} = \frac{3}{2}\tilde{r}_{,u}\tilde{r}B_{,v} + \frac{3}{2}\tilde{r}_{,v}\tilde{r}B_{,u} - \frac{1}{3}\Omega^2\tilde{r}^2\left(e^{-2B} - e^{-8B}\right) \tag{31}$$

(Note that we have reformulated the r equation in terms of $\tilde{r} = \frac{1}{\tilde{r}}$.) as well as the constraints

$$\partial_u \left(\frac{\tilde{r}_{,u}}{\tilde{r}^2 \Omega^2} \right) = \frac{2}{\Omega^2} \tilde{r} \left(B_{,u}\right)^2 \tag{32}$$

$$\partial_v \left(\frac{\tilde{r}_{,u}}{\tilde{r}^2 \Omega^2} \right) = \frac{2}{\Omega^2} \tilde{r} \left(B_{,v}\right)^2 . \tag{33}$$

Proof: The proof proceeds by reformulating the evolution in terms of a constraction mapping between Banach spaces.
We define the Banach space

$$X := \{(\Omega, \tilde{r}, B) \quad \text{functions on } \bar{\mathcal{R}} \text{ with } \Omega \in C^0\left(\bar{\mathcal{R}}\right) \text{ and } r, B \in C^1\left(\bar{\mathcal{R}}\right)\} \tag{34}$$

with distance function

$$\begin{aligned} d\left[(\Omega_1, \tilde{r}_1, B_1), (\Omega_2, \tilde{r}_2, B_2)\right] = \\ \max\Big(\left|\log\Omega_1 - \log\Omega_2\right|, \left|\tilde{r}_1 - \tilde{r}_2\right|, \left|\partial_u \log\tilde{r}_1 - \partial_u\log\tilde{r}_2\right|, \\ \left|\partial_v \log\tilde{r}_1 - \partial_v\log\tilde{r}_2\right|, \left|B_1 - B_2\right|_{C^1} \Big) \end{aligned} \tag{35}$$

Comment: Triangle inequality needs to be checked. Motivation for this comes from the finiteness of the Hawking mass

$$m = \frac{1}{2\tilde{r}^2}\left(1 + \frac{4\tilde{r}_{,u}\tilde{r}_{,v}}{\Omega^2\tilde{r}^4}\right) + \frac{1}{2l^2\tilde{r}^4} \tag{36}$$

We see that for m to be finite at $\tilde{r} \to 0$ (where Ω is constant and $\tilde{r}_{,u} = -\tilde{r}_{,v}$) we need that $\frac{\tilde{r}_{,u}}{\tilde{r}}$ is finite for $\tilde{r} \to 0$ to cancel the cosmological term.

Let us define the subspace

$$\mathcal{X}_E = \{(\Omega, \tilde{r}, B) \in \mathcal{X} \mid d\,[(\Omega, \tilde{r}, B) - (1, 0, 0)]\}\,. \tag{37}$$

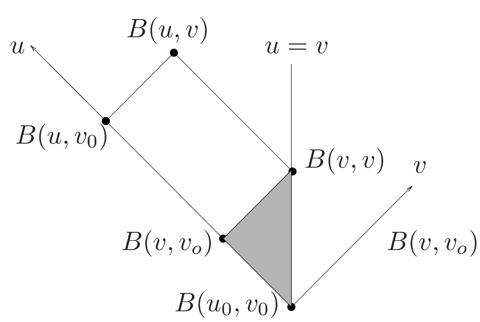
and consider the map

$$\begin{aligned} \Phi : \qquad \mathcal{X}_E &\rightarrow \mathcal{X}_E && (38)\\ (\Omega', \tilde{r}', B') &\rightarrow (\Omega, \tilde{r}, B) && (39) \end{aligned}$$

given by

$$\begin{aligned} \log\Omega'(u,v) &= \int_{v_0}^{v}\int_{v}^{u}\left[\frac{1}{2}\Omega^2\tilde{r}^2 R + 3\frac{\tilde{r}_{,u}\tilde{r}_{,v}}{\tilde{r}^2} - 3B_{,u}B_{,v} + \frac{\Omega^2}{2l^2}\right](\bar{u},\bar{v})\,d\bar{u}d\bar{v} \\ &+ \log\Omega\,(u,v_0) - \log\Omega\,(v,v_0) + \log\Omega\,(u_0,v_0) && (40)\\ \tilde{r}'(u,v) &= \int_{v_0}^{v}\int_{v}^{u}\left[\frac{1}{3}\Omega^2\tilde{r}^3\left(R + \frac{3}{l^2\tilde{r}^2}\right) + 4\tilde{r}\frac{\tilde{r}_{,u}\tilde{r}_{,v}}{\tilde{r}^2}\right](\bar{u},\bar{v})\,d\bar{u}d\bar{v} \\ &+ \tilde{r}\,(u,v_0) - \tilde{r}\,(v,v_0) + \tilde{r}\,(u_0,v_0) && (41)\\ B'(u,v) &= \int_{v_0}^{v}\int_{v}^{u}\left[\frac{3}{2}\frac{\tilde{r}_{,u}}{\tilde{r}}B_{,v} + \frac{3}{2}\frac{\tilde{r}_{,v}}{\tilde{r}}B_{,u} - \frac{1}{3}\Omega^2\tilde{r}^2\left(e^{-2B} - e^{-8B}\right)\right](\bar{u},\bar{v})\,d\bar{u}d\bar{v} \\ &+ B\,(u,v_0) - B\,(v,v_0) + B\,(u_0,v_0) && (42) \end{aligned}$$

One checks that



$$\begin{aligned} \Omega'(u,v_0) &= \Omega(u,v_0)\,, && (43)\\ \tilde{r}'(u,v_0) &= \tilde{r}(u,v_0)\,, && (44)\\ B'(u,v_0) &= B(u,v_0)\,, && (45)\\ \Omega'(v,v) &= \Omega(u_0,v_0)\,, && (46)\\ \tilde{r}'(v,v) &= \tilde{r}(u_0,v_0)\,, && (47)\\ B'(v,v) &= B(u_0,v_0)\,. && (48) \end{aligned}$$

Hence the quantities $(\Omega', \tilde{r}', B')$ satisfy the correct boundary conditions. In particular, since Ω' and $\tilde{r}'$ are constant on timelike null-infinity, the Hawking mass is also constant.
We will now show that Φ maps indeed to $\mathcal{X}_E$. We estimate

$$\log \Omega' \leq 3C + \epsilon^2 \left(\frac{1}{2} e^{2E} E^2 \frac{3}{2} + 3E^2 + 3E^2 + \frac{e^{2E}}{2l^2} \right) \tag{49}$$

$$\tilde{r}' \leq 2C + \epsilon^2 \left(\frac{1}{3} e^{2E} E^3 \frac{3}{2} + e^{2E} E \frac{1}{l^2} + 4E \cdot E^2 \right) \tag{50}$$

$$\tilde{r}' \geq 0 - \epsilon^2 \left(\frac{1}{3} e^{2E} E^3 \frac{3}{2} + e^{2E} E \frac{1}{l^2} + 4E \cdot E^2 \right) \tag{51}$$

$$|B'| \leq 3C + \epsilon^2 \left(\frac{3}{2} E \cdot E + \frac{3}{2} E \cdot E + \frac{1}{3} e^{2E} E^2 \left(e^{2E} + e^{8E} \right) \right) \tag{52}$$

Comment: Do we need a lower positive bound on $\tilde{r}$?

We can similarly estimate the u- and v-derivatives of the quantities $\Omega', \tilde{r}', B'$, e.g.

$$\left| B'_{,u} \right| \leq C + \epsilon \left(\frac{3}{2} E \cdot E + \frac{3}{2} E \cdot E + \frac{1}{3} e^{2E} E^2 \left(e^{2E} + e^{8E} \right) \right) \tag{53}$$

$$\left| B'_{,v} \right| \leq C + 2\epsilon \left(\frac{3}{2} E \cdot E + \frac{3}{2} E \cdot E + \frac{1}{3} e^{2E} E^2 \left(e^{2E} + e^{8E} \right) \right) \tag{54}$$

Thus if ϵ is small enough (depending on C and E only) Φ indeed maps to $\mathcal{X}_E$.
To show that Φ is a contraction, we have to similarly bound differences, which can be done in a completely analogous fashion.

Result: For sufficiently small ϵ the map Φ is a contraction. Banach's fixed point theorem then assures a fixed point $(\Omega, \tilde{r}, B) \in \mathcal{X}_E$.

Clearly the $(\Omega, \tilde{r}, B)$ satisfy the evolution equations and the right boundary conditions. The only thing we finally need to ensure is that the two constraint equations are also satisfied. Using the evolution equations we compute

$$\partial_v \left(\partial_u \left(\frac{\tilde{r}_{,u}}{\tilde{r}^2 \Omega^2} \right) \right) = \partial_v \left(\frac{2}{\Omega^2} \tilde{r} \left(B_{,u} \right)^2 \right) \tag{55}$$

from which we conclude that the u-constraint equation will be propagated: If it is satisfied on the initial $v = v_0$ ray, it will be satisfied everywhere in $\bar{\mathcal{R}}$. In particular, it will hold on the boundary $\tilde{r} = 0$. However, from the identity

$$\left(\partial_u + \partial_v \right) m = 0 \tag{56}$$

holding on the boundary, we easily see that the v-constraint equation is identical to the u-constraint equation on the boundary. Since

$$\partial_u \left(\partial_v \left(\frac{\tilde{r}_{,v}}{\tilde{r}^2 \Omega^2} \right) \right) = \partial_u \left(\frac{2}{\Omega^2} \tilde{r} \left(B_{,v} \right)^2 \right) , \tag{57}$$

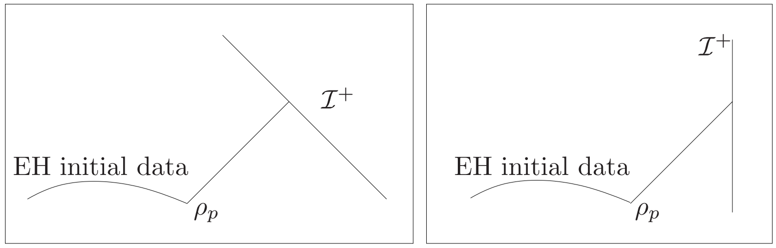
the v-constraint is propagated in the u-direction from the $\tilde{r} = 0$ boundary and will be satisfied in all of $\bar{\mathcal{R}}$.

4.3 Extension Principle near timelike null-infinity

5 Proof of Theorem 1.2

To establish Theorem 1.4 we will prove that the Eguchi-Hanson-AdS metric minimizes the mass in the class of biaxial Bianchi IX metrics which have topology $S^3/\mathbb{Z}_p$

at infinity. This can be done in the following way. Consider the initial value prob-



lem for asymptotically locally AdS-spaces illustrated above. We assume that the initial data are prescribed both on a spacelike hypersurface extending to ρ_p and on a null-hypersurface extending from ρ_p to the timelike null-infinity of Eguchi-Hanson. Let us prescribe exactly Eguchi-Hanson initial data on the spacelike part. On the dotted null-line we can prescribe B freely. Since the mass is assumed to be constant on null-infinity for asymptotically locally AdS spaces minimizing the mass of the space time corresponds to finding a function B on the initial data slice that minimizes the mass at infinity.

5.1 Gauge conditions

We fix coordinates on the dotted null-initial data surface in the diagram by choosing

$$\rho = b \cdot v \tag{58}$$

to hold on the line. Here

$$b = \frac{1}{2}\left(1 + \frac{\rho_p}{l^2}\right)\left(1 - \frac{a^4}{\rho_p^4}\right)^{\frac{1}{2}} \tag{59}$$

is a constant. At ρ_p the Eguchi-Hanson functions take the following values

$$\Omega^2 = 1 + \frac{\rho_p^2}{l^2} \tag{60}$$

$$r_{,u} = r_{,\rho} \cdot \rho_{,u} = -\frac{1}{2} b \left(1 - \frac{a^4}{\rho_p^4}\right)^{-\frac{5}{6}} \left(1 - \frac{a^4}{3\rho_p^4}\right) \tag{61}$$

$$r_{,v} = -r_{,u}(\rho_p) \tag{62}$$

Using these initial conditions at ρ_p and the equations of motion (6), (7) we can derive the expressions for the quantities $\Omega, r_{,u}, r_{,v}$ on the whole line:

$$r = \rho \left(1 - \frac{a^4}{\rho^4}\right)^{\frac{1}{6}} \tag{63}$$

$$\Omega^2(\rho) = \left(1 + \frac{\rho_p^2}{l^2}\right) \frac{r_{,\rho}(\rho)}{r_{,\rho}(\rho_p)} \exp\left(\int_{\rho_p}^{\rho} \left(\frac{2r(\tilde\rho)}{r_{,\rho}(\tilde\rho)} (B_{,\rho}(\tilde\rho))^2\right)\right) d\tilde\rho > 0 \tag{64}$$

$$r_{,u} = \frac{1}{r(\rho)^2} \int_{\rho_p}^{\rho} \left[r(\tilde\rho)^2 \frac{1}{b} \left(-\frac{1}{3} R(B(\tilde\rho)) - \frac{r(\tilde\rho)^2}{l^2}\right) \frac{\Omega^2(\tilde\rho)}{r(\tilde\rho)} \right] d\tilde\rho + \frac{C}{r(\rho)^2} < 0 \tag{65}$$

$$r_{,v} = b \left(1 - \frac{a^4}{\rho^4}\right)^{-\frac{5}{6}} \left(1 - \frac{a^4}{3\rho^4}\right) > 0 \tag{66}$$

where we defined the constant C to be

$$C = \frac{1}{6}\frac{a^4 - 3\rho_p^4}{\rho_p^2}\left(1 + \frac{\rho_p^2}{l^2}\right) < 0 \tag{67}$$

Note that at ρ_p all functions take their Eguchi-Hanson-AdS values.

5.2 The first variation

We now want to extremize the functional

$$\begin{aligned}\mathcal{I}\left[B + t\xi\right] = b^2 \int_{\rho_p}^{\infty} \Bigg[&-\frac{4r^3\left(B_{,\rho} + t\xi_{,\rho}\right)^2}{\left(\Omega\left[B + t\xi\right]\right)^2} r_{,u}\left[B + t\xi\right] \\ &+ \frac{r r_{,\rho}}{b}\left(1 - \frac{2}{3}R\left[B + t\xi\right]\right)\Bigg] d\rho\end{aligned} \tag{68}$$

which is the mass difference between infinity and ρ_p, by varying B. Here

$$\Omega\left[B + t\xi\right] = \sqrt{\left(1 + \frac{\rho_p^2}{l^2}\right)\frac{r_{,\rho}}{r_{,\rho}(\rho_p)}} \exp\left(\int_{\rho_p}^{\rho} \frac{r}{r_{,\rho}}\left(B_{,\rho} + t\xi_{,\rho}\right)^2 d\tilde{\rho}\right) \tag{69}$$

$$r_{,u}\left[B + t\xi\right] = \frac{1}{r^2}\int_{\rho_p}^{\rho} r^2 \frac{1}{b}\left(-\frac{1}{3}R\left[B + t\xi\right] - \frac{r^2}{l^2}\right)\frac{\left(\Omega\left[B + t\xi\right]\right)^2}{r}\left(\tilde{\rho}\right) d\tilde{\rho} + C\frac{1}{r^2} \tag{70}$$

$$R\left[B + t\xi\right] = 2e^{-2(B + t\xi)} - \frac{1}{2}e^{-8(B + t\xi)} \tag{71}$$

We want

$$\begin{aligned}0 = \frac{1}{b^2}\cdot\frac{d}{dt}\bigg|_{t=0} &\mathcal{I}\left[B + t\xi\right] = \\ \boxed{\text{A}} \qquad & \int_{\rho_p}^{\infty} d\rho \frac{8r^3}{\Omega^3}\left(\frac{d}{dt}\bigg|_{t=0}\Omega\right) r_{,u}B_{,\rho}^2 \\ \boxed{\text{B}} \qquad & -\int_{\rho_p}^{\infty} d\rho \frac{4r^3}{\Omega^2}\left(\frac{d}{dt}\bigg|_{t=0} r_{,u}\right) B_{,\rho}^2 \\ \boxed{\text{C}} \qquad & -\int_{\rho_p}^{\infty} d\rho \frac{8r^3 r_{,u} B_{\rho}\xi_{,\rho}}{\Omega^2} \\ \boxed{\text{D}} \qquad & +\int_{\rho_p}^{\infty} d\rho \frac{8}{3}\frac{r\cdot r_{,\rho}}{b}\left(e^{-2B} - e^{-8B}\right)\xi\end{aligned} \tag{72}$$

We will treat the four terms seperately and refer to them by their corresponding letters given on the left. From (69) we compute

$$\frac{d}{dt}\bigg|_{t=0}\Omega\left[B + t\xi\right] = \Omega\left(\frac{2r}{r_{,\rho}}B_{,\rho}\xi\left(\rho\right) - \int_{\rho_p}^{\rho}\left(\frac{2r}{r_{,\rho}}B_{,\rho}\right)_{,\rho}\xi\left(\tilde{\rho}\right) d\tilde{\rho}\right) \tag{73}$$

and using (73) and (70) we obtain

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} r_{,u}\left[B + t\xi\right] = -2r_{,u}\int_{\rho_p}^{\rho}\left(\frac{2r}{r_{,\rho}}B_{,\rho}\right)_{,\rho}\xi\left(\tilde{\rho}\right) d\tilde{\rho} \\ +\frac{1}{r^2}\int_{\rho_p}^{\rho}\frac{r^2}{b}\left[\frac{4}{3}\frac{\Omega^2}{r}\left(e^{-2B} - e^{-8B}\right) - \frac{4\Omega^2 B_{\rho}}{r_{\rho}}\left(\frac{1}{3}R + \frac{r^2}{l^2}\right) + 2br_{,u}\left(\frac{2r}{r_{,\rho}}B_{,\rho}\right)_{,\rho}\right]\xi\left(\tilde{\rho}\right) d\tilde{\rho}\end{aligned} \tag{74}$$

We will refer to the expression in square brackets as $\mathcal{Q}[B(\rho)]$ in the following. (It will turn out to be a quantity that vanishes by the equation of motion.) To obtain the equation of motion we now have to isolate the ξ-terms in (72) using integration by parts. The $\boxed{\text{A}}$-term will leave us with

$$\begin{aligned} &\int_{\rho_p}^{\infty} \frac{8r^3}{\Omega^3}\left(\left.\frac{d}{dt}\right|_{t=0} \Omega\right) r_{,u} B_{,\rho}^2 = \\ \boxed{\text{A1}} \qquad &\int_{\rho_p}^{\infty} \left[\frac{16 r^4 r_{,u} \left(B_{,\rho}\right)^3}{\Omega^2 r_{,\rho}}\right] \xi(\rho)\, d\rho \\ \boxed{\text{A2}} \qquad & -\int_{\rho_p}^{\infty} \left(\frac{8r^3 r_{,u}\left(B_{,\rho}\right)^2}{\Omega^2}\right)\left(\int_{\rho_p}^{\rho} \left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho} \xi(\tilde{\rho}) d\tilde{\rho}\right) d\rho \end{aligned} \tag{75}$$

Let us consider the $\boxed{\text{B}}$ term.

$$\begin{aligned} & -\int_{\rho_p}^{\infty} \frac{4r^3}{\Omega^2}\left(\left.\frac{d}{dt}\right|_{t=0} r_{,u}\right) B_{,\rho}^2\, d\rho = \\ \boxed{\text{B1}} \qquad & \int_{\rho_p}^{\infty} \frac{8r^3 r_{,u}\left(B_{,\rho}\right)^2}{\Omega^2}\left(\int_{\rho_p}^{\rho}\left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho} \xi(\tilde{\rho})\, d\tilde{\rho}\right) d\rho \\ \boxed{\text{B2}} \qquad & -\int_{\rho_p}^{\infty} \frac{4r\left(B_{,\rho}\right)^2}{\Omega^2}\left(\int_{\rho_p}^{\rho} \frac{r^2}{b} \mathcal{Q}[B(\tilde{\rho})]\, \xi(\tilde{\rho})\, d\tilde{\rho}\right) d\rho \end{aligned}$$

The term $\boxed{\text{B1}}$ cancels $\boxed{\text{A2}}$. The term $\boxed{\text{B2}}$ can be simplified by an equation by parts, using the constraint (6):

$$\boxed{\text{B2}} = -\int_{\rho_p}^{\infty} \frac{2r^2 r_{,\rho}}{b}\left(\left.\frac{1}{\Omega^2}\right|_{\infty} - \frac{1}{\Omega^2}\right) \mathcal{Q}[B(\rho)]\, \xi(\rho)\, d\tilde{\rho} \tag{76}$$

The terms $\boxed{\text{C}}$ and $\boxed{\text{D}}$ of (72) simply read

$$\boxed{\text{C}} + \boxed{\text{D}} = \int_{\rho_p}^{\infty} \left[\left(\frac{8r^3 r_{,u} B_{\rho}}{\Omega^2}\right)_{,\rho} + \frac{8}{3}\frac{r \cdot r_{,\rho}}{b}\left(e^{-2B} - e^{-8B}\right)\right] \xi(\rho)\, d\rho \tag{77}$$

The term $\boxed{\text{C}}$ can be combined with the term $\boxed{\text{A1}}$ again using the constraint (6) to yield

$$\boxed{\text{A1}} + \boxed{\text{C}} = \int_{\rho_p}^{\infty} \left(\frac{8r^3 r_{,u} B_{,\rho}}{r_{,\rho}}\right)_{,\rho} \frac{r_{,\rho}}{\Omega^2} \xi(\rho)\, d\rho \tag{78}$$

We now add up the terms $\boxed{\text{A1}}$, $\boxed{\text{B2}}$, $\boxed{\text{C}}$ and $\boxed{\text{D}}$ to arrive at the equations of motion.

$$\begin{aligned} 0 = & \left(\frac{8r^3 r_{,u} B_{,\rho}}{r_{,\rho}}\right)_{,\rho} \frac{r_{,\rho}}{\Omega^2} + \frac{8}{3}\frac{r \cdot r_{,\rho}}{b}\left(e^{-2B} - e^{-8B}\right) + \\ & \frac{r^2 r_{,\rho}}{b}\left(-\left.\frac{1}{\Omega^2}\right|_{\infty} + \frac{1}{\Omega^2}\right)\left[\frac{8}{3}\frac{\Omega^2}{r}\left(e^{-2B} - e^{-8B}\right) - \frac{8\Omega^2 B_{,\rho}}{r_{\rho}}\left(\frac{1}{3}R + \frac{r^2}{l^2}\right) + 4b r_{,u}\left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho}\right] \end{aligned} \tag{79}$$

This expression can be simplified considerably, if we use equation (7), which in our gauge can be written as

$$\partial_{\rho}\left(r^2 r_{,u}\right) = \frac{\Omega^2 r}{b}\left(-\frac{1}{3}R - \frac{r^2}{l^2}\right) \tag{80}$$

The result is

$$\frac{1}{\Omega^2}\bigg|_\infty \left[\frac{4}{3}\frac{\Omega^2}{r}\left(e^{-2B}-e^{-8B}\right)-\frac{4\Omega^2 B_{,\rho}}{r_\rho}\left(\frac{1}{3}R+\frac{r^2}{l^2}\right)+2br_{,u}\left(\frac{2r}{r_{,\rho}}B_{,\rho}\right)_{,\rho}\right]=0 \quad (81)$$

Since

$$\frac{1}{\Omega^2}\bigg|_\infty \neq 0\,, \tag{82}$$

which is a consequence of equation (64), the equation of motion is equal to the term in square brackets and can be written as

$$B_{,\rho\rho}=-B_{,\rho}\left(f(\rho)-\frac{\Omega^2}{rbr_{,u}}\left(\frac{R}{3}+\frac{r^2}{l^2}\right)\right)-\frac{r_{,\rho}\Omega^2}{3br^2r_{,u}}\left(e^{-2B}-e^{-8B}\right) \tag{83}$$

where

$$f(\rho)=\frac{(\rho^4+a^4)(3\rho^4+a^4)}{\rho(\rho^4-a^4)(3\rho^4-a^4)} \tag{84}$$

There is also a neat way of writing this equation:

$$\partial_\rho\left(\frac{r^3r_{,u}B_{,\rho}}{r_{,\rho}}\right)=-\frac{\Omega^2 r}{3b}\left(e^{-2B}-e^{-8B}\right) \tag{85}$$

which upon integration leads to the result

$$\int_{\rho_p}^{\infty}\frac{\Omega^2 r}{3b}\left(e^{-2B}-e^{-8B}\right)d\rho=\frac{a^4}{3\rho_p^2} \tag{86}$$

The mass is *extremized* if B satisfies (83). Of course the equation is supplemented by boundary conditions. At ρ_p the function B has to assume its Eguchi-Hanson value whereas at ∞ it has to vanish to satisfy the criterion of asymptotically locally AdS. It is straightforward to check that

$$B_{EH}=-\frac{1}{6}\log\left(1-\frac{a^4}{\rho^4}\right) \tag{87}$$

indeed satisfies (83) and the boundary condtions.
To prove Theorem 1.4 it suffices to show that we are in a local minimum. This can be done by calculating the second variation, which we are going to do in the next subsection.

Comment: However, one could be more ambitious and prove that Eguchi-Hanson is the *global* minimizer. One way to achieve this would be to show uniquess of the solution to (83) via the maximum principle. This work is currently in progress.

An important property of extremizers follows directly from the maximum principle applied to the differential equation (83):

Lemma 1: Any extremizer satisfies

$$B\geq 0 \qquad B_{,\rho}\leq 0 \qquad B_{,\rho\rho}\geq 0 \tag{88}$$

everywhere in $[\rho_p,\infty)$.

Proof: At an extremum of a solution B the differential equation (83) reads

$$B_{,\rho\rho}=-\frac{r_{,\rho}\Omega^2}{3br^2r_{,u}}\left(e^{-2B}-e^{-8B}\right)\ . \tag{89}$$

The term $\left(e^{-2B} - e^{-8B}\right)$ has the sign of B, the term $-\frac{r_{,\rho}\Omega^2}{3br^2 r_{,u}}$ is always positive because $r_{,u}$ is always negative. By the maximum principle, there can only be maxima ($B_{,\rho\rho} < 0$) if B itself is negative and minima if B is positive. Since B starts at ρ_p with a positive value (namely its Eguchi-Hanson value) and is zero at infinity, the function B must go to zero monotonically. Thus we conclude $B_{,\rho} < 0$ and $B > 0$. The statement about the second derivative then follows from applying these facts to (83). □

Corollary: Different extremizers can only intersect at their boundary points.

Furthermore, using an asymptotic expansion, one can show

Lemma 2: Nontrivial solutions near infinity behave like

$$B \propto \frac{1}{\rho^4} \tag{90}$$

Proof: From equation (63) we can derive that

$$r = \rho + \mathcal{O}\left(\frac{1}{\rho^3}\right) \tag{91}$$

In our gauge, the Hawking mass can be written as

$$m = \frac{r^2}{2}\left(1 - \frac{br_{,\rho}}{\kappa}\right) + \frac{r^4}{2l^2} \tag{92}$$

with $\kappa = \frac{-\Omega^2}{4r_{,u}}$.

Consider first the flat case, $l \to \infty$. To ensure finite mass at infinity, $\frac{1}{\kappa}$ should asymptotically behave like

$$\frac{1}{\kappa} = \frac{1}{b} + \mathcal{O}\left(\frac{1}{\rho^2}\right) \tag{93}$$

Hence

$$\kappa = b + \mathcal{O}\left(\frac{1}{\rho^2}\right) \tag{94}$$

Using that $R \to \frac{3}{2}$ near infinity, the differential equation (83) written out to lowest order in $\frac{1}{\rho}$ reads

$$B_{,\rho\rho} = -\frac{3}{\rho}B_{,\rho} + \frac{8}{\rho^2}B \tag{95}$$

This Euler-type equation has a unique solution that decays at infinity, $B = \frac{1}{\rho^4}$.

For the AdS-case we need a slightly different argument. In this case the behaviour of $\frac{1}{\kappa}$ to ensure finite Hawking mass at infinity is changed to

$$\frac{1}{\kappa} = \frac{1}{b}\left(\frac{\rho^2}{l^2} + 1 + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right) \tag{96}$$

$$\kappa = \frac{bl^2}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \tag{97}$$

Hence, to lowest order, the diffential equation (83) in the AdS-case reads

$$B_{,\rho\rho} = -\frac{5}{\rho}B_{,\rho} \tag{98}$$

Again this equation has a unique solution that decays, namely $B = \frac{1}{\rho^4}$. □

5.3 The second variation

The second variation is

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{I}\left[B+t\xi\right] =$$

$$\begin{aligned}
&\boxed{1} \quad \int_{\rho_p}^{\infty} \left(\frac{-24r^3}{\Omega^4} r_{,u} (B_{,\rho})^2 \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right)^2 \right) d\rho \\
&\boxed{2} \quad + \int_{\rho_p}^{\infty} \left(\frac{8r^3}{\Omega^3} r_{,u} (B_{,\rho})^2 \left(\left.\frac{d^2}{dt^2}\right|_{t=0} \Omega\left[B+t\xi\right] \right) \right) d\rho \\
&\boxed{3} \quad + \int_{\rho_p}^{\infty} \left(\frac{16r^3}{\Omega^3} (B_{,\rho})^2 \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right) \left(\left.\frac{d}{dt}\right|_{t=0} r_{,u}\left[B+t\xi\right] \right) \right) d\rho \\
&\boxed{4} \quad + \int_{\rho_p}^{\infty} \frac{32r^3}{\Omega^3} r_{,u} \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right) B_{,\rho}\xi_{,\rho} d\rho \\
&\boxed{5} \quad - \int_{\rho_p}^{\infty} \frac{4r^3}{\Omega^2} \left(\left.\frac{d^2}{dt^2}\right|_{t=0} r_{,u}\left[B+t\xi\right] \right) (B_{,\rho})^2 \, d\rho \\
&\boxed{6} \quad - \int_{\rho_p}^{\infty} \frac{16r^3}{\Omega^2} \left(\left.\frac{d}{dt}\right|_{t=0} r_{,u}\left[B+t\xi\right] \right) B_{,\rho}\xi_{,\rho} d\rho \\
&\boxed{7} \quad - \int_{\rho_p}^{\infty} \frac{8r^3}{\Omega^2} r_{,u} \left(\xi_{,\rho}\right)^2 d\rho \\
&\boxed{8} \quad - \int_{\rho_p}^{\infty} \frac{16}{3} \xi^2 \frac{r r_{,\rho}}{b} \left(e^{-2B} - 4e^{-8B}\right) d\rho
\end{aligned} \tag{99}$$

We will refer to these terms on the right by their corresponding boxed numbers given on the left. To evaluate this expression we recall the formulae (73) and (74), where we can simplify the latter using the vanishing of the square bracket by the equation of motion. Note that we then have the relation

$$\left.\frac{d}{dt}\right|_{t=0} r_{,u}\left[B+t\xi\right] = 2\frac{r_{,u}}{\Omega} \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right) - 4\frac{r r_{,u}}{r_{,\rho}} B_{,\rho}\xi(\rho) \quad . \tag{100}$$

Furthermore, we will need the expressions

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \Omega\left[B+t\xi\right] = \frac{1}{\Omega} \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right)^2 + \Omega \int_{\rho_p}^{\rho} \frac{2r}{r_{,\rho}} \xi_{,\rho}^2 \left(\tilde{\rho}\right) d\tilde{\rho} \tag{101}$$

and

$$\begin{aligned}
&\left.\frac{d^2}{dt^2}\right|_{t=0} r_{,u}\left[B+t\xi\right] = \\
&\quad \frac{1}{r^2} \int_{\rho_p}^{\rho} r^2 \Bigg[-\frac{8}{3}\xi^2 \frac{1}{b} \left(e^{-2B} - 4e^{-8B}\right) \frac{\Omega^2}{r} \\
&\quad + \frac{16}{3}\xi \frac{1}{b} \left(e^{-2B} - e^{-8B}\right) \frac{\Omega}{r} \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right) \\
&\quad + \frac{2}{rb} \left(-\frac{1}{3}R - \frac{r^2}{l^2} \right) \left(\Omega \left(\left.\frac{d^2}{dt^2}\right|_{t=0} \Omega\left[B+t\xi\right] \right) + \left(\left.\frac{d}{dt}\right|_{t=0} \Omega\left[B+t\xi\right] \right)^2 \right) \Bigg] d\tilde{\rho}
\end{aligned} \tag{102}$$

We insert the relation (100) into term $\boxed{3}$ and $\boxed{6}$ of (99) and equation (101) into term $\boxed{2}$ leading to some cancellations. Finally, we insert (73) and use partial integration to arrive at the following simplified expression for the first three terms of (99).

$$\begin{aligned}\boxed{1{+}2{+}3} = &-\int_{\rho_p}^{\infty} \frac{64r^5 r_{,u}}{\Omega^2 \left(r_{,\rho}\right)^2} \left(B_{,\rho}\right)^4 \xi^2 d\rho \\ &+\int_{\rho_p}^{\infty} \frac{16r^3}{\Omega^2} r_{,u} \left(B_{,\rho}\right)^2 \left(\int_{\rho_p}^{\rho} \left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho} \xi\left(\tilde{\rho}\right) d\tilde{\rho}\right)^2 d\rho \\ &+\int_{\rho_p}^{\infty} \frac{8r^3}{\Omega^2} r_{,u} \left(B_{,\rho}\right)^2 \left(\int_{\rho_p}^{\rho} \frac{2r}{r_{,\rho}} \left(\xi_{,\rho}\right)^2 d\tilde{\rho}\right)^2 d\rho\end{aligned} \tag{103}$$

The terms $\boxed{4}$ and $\boxed{6}$ together simplify to

$$\boxed{4{+}6} = \int_{\rho_p}^{\infty} \frac{64r^4 r_{,u} \xi}{\Omega^2 r_{,\rho}} \left(B_{,\rho}\right)^2 \xi_{,\rho} d\rho \tag{104}$$

Term $\boxed{5}$ is the most painful. We have to insert (102) and use several partial integrations, therby applying (6) continuously. Let us do this separately for each term in the square bracket of (102) and refer to the results as $\boxed{5\text{a}}$, $\boxed{5\text{b}}$, $\boxed{5\text{c}}$.

$$\boxed{5\text{a}} = -\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} \int_{\rho_p}^{\infty} \frac{16\Omega^2 r \xi^2}{3b} \left(e^{-2B} - 4e^{-8B}\right) d\rho + \int_{\rho_p}^{\infty} \frac{16 r r_{,\rho} \xi^2}{3b} \left(e^{-2B} - 4e^{-8B}\right) d\rho \tag{105}$$

The second term obviously cancels the term $\boxed{8}$ in (99).

$$\boxed{5\text{a} + 8} = -\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} \int_{\rho_p}^{\infty} \frac{16\Omega^2 r \xi^2}{3b} \left(e^{-2B} - 4e^{-8B}\right) d\rho \tag{106}$$

The $\boxed{5\text{b}}$-term will consist of two terms because we have to insert (73). We refer to these two terms as $\boxed{5\text{bI}}$ and $\boxed{5\text{bII}}$.

$$\boxed{5\text{bI}} = \int_{\rho_p}^{\infty} \left(\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} - \frac{r_{,\rho}}{\Omega^2}\right) \frac{64r^2\Omega^2 B_{,\rho}}{3b r_{,\rho}} \left(e^{-2B} - e^{-8B}\right) \xi^2 d\rho \tag{107}$$

$$\boxed{5\text{bII}} = \int_{\rho_p}^{\infty} \left(-\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} + \frac{r_{,\rho}}{\Omega^2}\right) \frac{32r\Omega^2}{3b} \left(e^{-2B} - e^{-8B}\right) \xi \left(\int_{\rho_p}^{\rho} \left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho} \xi\left(\tilde{\rho}\right) d\tilde{\rho}\right) d\rho \tag{108}$$

The $\boxed{5\text{c}}$ term will me made of four terms because we have to insert the relation (101) and then the square of (73). We will refer to these four terms as $\boxed{5\text{cI}}$ ... $\boxed{5\text{cIV}}$.

$$\boxed{5\text{cI}} = \int_{\rho_p}^{\infty} \left(\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} - \frac{r_{,\rho}}{\Omega^2}\right) \frac{32\Omega^2 r^3 \left(B_{,\rho}\right)^2}{b\left(r_{,\rho}\right)^2} \left(-\frac{1}{3}R - \frac{r^2}{l^2}\right) \xi^2 d\rho \tag{109}$$

$$\boxed{5\text{cII}} = \int_{\rho_p}^{\infty} \left(-\frac{r_{,\rho}}{\Omega^2}\bigg|_{\infty} + \frac{r_{,\rho}}{\Omega^2}\right) \frac{32\Omega^2 r^2 B_{,\rho}}{b r_{,\rho}} \xi \left(-\frac{1}{3}R - \frac{r^2}{l^2}\right) \left(\int_{\rho_p}^{\rho} \left(\frac{2r}{r_{,\rho}} B_{,\rho}\right)_{,\rho} \xi\left(\tilde{\rho}\right) d\tilde{\rho}\right) d\rho \tag{110}$$

$$\boxed{\text{5cIII}} = \int_{\rho_p}^{\infty} \left(\left.\frac{r_{,\rho}}{\Omega^2}\right|_{\infty} - \frac{r_{,\rho}}{\Omega^2} \right) \frac{8\Omega^2 r}{b} \left(-\frac{1}{3}R - \frac{r^2}{l^2} \right) \left(\int_{\rho_p}^{\rho} \left(\frac{2r}{r_{,\rho}} B_{,\rho} \right)_{,\rho} \xi(\tilde{\rho})\, d\tilde{\rho} \right)^2 d\rho \tag{111}$$

$$\boxed{\text{5cIV}} = \int_{\rho_p}^{\infty} \left(\left.\frac{r_{,\rho}}{\Omega^2}\right|_{\infty} - \frac{r_{,\rho}}{\Omega^2} \right) \frac{4\Omega^2 r}{b} \left(-\frac{1}{3}R - \frac{r^2}{l^2} \right) \left(\int_{\rho_p}^{\rho} \frac{2r}{r_{,\rho}} (\xi_{,\rho})^2 (\tilde{\rho})\, d\tilde{\rho} \right) d\rho \tag{112}$$

We now find that the second term of (103) cancels $\boxed{\text{5bII}} + \boxed{\text{5cII}} + \boxed{\text{5cIII}}$ using the equation of motion (83):

$$\boxed{1+2+3}_{second} + \boxed{\text{5bII}} + \boxed{\text{5cII}} + \boxed{\text{5cIII}} = 0 \tag{113}$$

Using the equation (80) we can subsume the term $\boxed{\text{5cIV}}$ together with the last term of (103) and $\boxed{7}$ to give

$$\boxed{1+2+3}_{last} + \boxed{\text{5cIV}} + \boxed{7} = -\left.\frac{r_{,\rho}}{\Omega^2}\right|_{\infty} \int_{\rho_p}^{\infty} \frac{8r^3 r_{,u}}{r_{,\rho}} (\xi_{,\rho})^2\, d\rho \tag{114}$$

Finally, the sum of $\boxed{\text{5bI}}$, $\boxed{\text{5cI}}$, the first term of (103) and the term (104) simplifies to give

$$\boxed{\text{5bI}} + \boxed{\text{5cI}} + \boxed{1+2+3}_{first} + \boxed{4+6} = \left.\frac{r_{,\rho}}{\Omega^2}\right|_{\infty} \int_{\rho_p}^{\infty} \frac{64 r^4 r_{,u} (B_{,\rho})^2}{(r_{,\rho})^2} \xi \xi_{,\rho} d\rho \tag{115}$$

We are ready to state the final result:

$$\begin{aligned} &\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{I}[B + t\xi] = \\ &\quad \left.\frac{8r_{,\rho}}{\Omega^2}\right|_{\infty} \int_{\rho_p}^{\infty} d\rho \left[\frac{-r^3 r_{,u}}{r_{,\rho}} (\xi_{,\rho})^2 + \frac{8r^4 r_{,u} (B_{,\rho})^2}{(r_{,\rho})^2} \xi\xi_{,\rho} - \frac{2\Omega^2 r}{3b} \left(e^{-2B} - 4e^{-8B}\right) \xi^2 \right] \\ &= \left.\frac{8r_{,\rho}}{\Omega^2}\right|_{\infty} \int_{\rho_p}^{\infty} d\rho \left[\frac{-r^3 r_{,u}}{r_{,\rho}} (\xi_{,\rho})^2 + \left(-4\partial_\rho \left(r^2 r_{,u} \left(\frac{B_{,\rho} r}{r_{,\rho}} \right)^2 \right) - \frac{2\Omega^2 r}{3b} \left(e^{-2B} - 4e^{-8B}\right) \right) \xi^2 \right] \end{aligned} \tag{116}$$

5.4 Positivity of the second variation

We want to show that the integral on the right hand side is positive. Unfortunately, the integrand is not manifestly positive. However, for large enough ρ_p we can show the following

Proposition 1: For any extremizer, we can choose ρ_p large enough such that the integrand of (116) is manifestly positive.
Proof (sketched): The $(\xi_{,\rho})^2$ in (116) is already manifestly positive. Using the asymptotic behaviour of an extremizer (Lemma 2 above) and of $r_{,u}, \Omega$, it is straightforward to show that for large enough ρ the ξ^2 term in (116) will also become positive. More precisely, let us compute the weight w of the ξ^2 term in (116). Using the equation of motion (83) we obtain

$$w = \frac{8}{3} \frac{r^2 \Omega^2 B_{,\rho}}{b r_{,\rho}} \left(e^{-2B} - e^{-8B}\right) - 4 \frac{r^3 \Omega^2 (B_{,\rho})^2}{(r_{,\rho})^2 b} \left(\frac{R}{3} + \frac{r^2}{l^2} \right) - \frac{2\Omega^2 r}{3b} \left(e^{-2B} - 4e^{-8B}\right) \tag{117}$$

For the general case, we will have to work harder. Let us plug in the Eguchi-Hanson values into (116) and use Hardy inequalities to prove positivity of (116) at least for the Eguchi-Hanson case.
The Eguchi-Hanson values are

$$r = \rho \left(1 - \frac{a^4}{\rho^4}\right)^{\frac{1}{6}} \tag{118}$$

$$\Omega(\rho) = \sqrt{2b}\frac{\rho}{\left(\rho^4 - a^4\right)^{\frac{1}{4}}} \tag{119}$$

$$r_{,u} = \frac{1}{6}\left(1 + \frac{\rho^2}{l^2}\right)\frac{a^4 - 3\rho^4}{\rho^4}\left(1 - \frac{a^4}{\rho^4}\right)^{-\frac{1}{3}} \tag{120}$$

and the positivity of (116) at Eguchi-Hanson is equivalent to the positivity of

$$\begin{aligned} \int_{\rho_p}^{\infty} d\rho \Bigg(& \left[\frac{1}{2}\left(1 + \frac{\rho^2}{l^2}\right)\frac{\rho^4 - a^4}{\rho}\right](\xi_{,\rho})^2 \\ & + \left[\frac{12\rho\left(3\rho^8 - 6a^4\rho^4 - a^8\right)}{\left(a^4 - 3\rho^4\right)^2} - \frac{32a^8\rho^3}{l^2\left(a^4 - 3\rho^4\right)^2}\right]\xi^2\Bigg) > 0 \end{aligned} \tag{121}$$

This is an inequality of Hardy-type. We are going to prove it seperately for the flat case and the additional $\frac{1}{l^2}$-terms in case of a cosmological constant. Since $\rho_p > a$ we will set the left boundary equal to a and prove positivity for all functions ξ vanishing at $x = a$ and infinity.
The flat case: For the "flat" terms, we need to prove that

$$\int_a^{\infty} d\rho \left[\frac{12\rho\left(-3\rho^8 + 6a^4\rho^4 + a^8\right)}{\left(a^4 - 3\rho^4\right)^2}\right]\xi^2 < \int_a^{\infty} d\rho \left[\frac{1}{2}\frac{\rho^4 - a^4}{\rho}\right](\xi_{,\rho})^2 \tag{122}$$

holds for any ξ vanishing at a and at infinity. We can scale out the parameter a by setting $\rho = x \cdot a$:

$$\int_1^{\infty} dx \left[\frac{12x\left(-3x^8 + 6x^4 + 1\right)}{\left(1 - 3x^4\right)^2}\right]\xi^2 < \int_1^{\infty} dx \left[\frac{1}{2}\frac{x^4 - 1}{x}\right](\xi_{,x})^2 \tag{123}$$

Note that for $x \geq c = \left(\frac{3+2\sqrt{3}}{3}\right)^{\frac{1}{4}}$ the inequality is trivially satisfied.

Step 1: We show that the inequality (123) holds for any ξ, which vanishes at $x = 1$ and takes arbitrary values at the point $x = 1.04$.

$$\begin{aligned} & \int_1^{1.04} dx \left[\frac{12x\left(-3x^8 + 6x^4 + 1\right)}{\left(1 - 3x^4\right)^2}\right]\xi^2 = -5\xi^2(1.04) - \int_1^{1.04} dx \left(\frac{12\left(x^2 + x^6\right)}{2 - 6x^4} + 6\right)2\xi\xi_{,x} \\ & \leq \sqrt{\int_1^{1.04} \frac{12x\left(-3x^8 + 6x^4 + 1\right)}{\left(1 - 3x^4\right)^2}\xi^2 dx}\sqrt{\int_1^{1.04} \frac{12\left(1 + x^2 - 3x^4 + x^6\right)^2}{x\left(1 + 6x^4 - 3x^8\right)}(\xi_{,x})^2 dx} \end{aligned} \tag{124}$$

using Hoelder's inequality and therefore

$$\int_1^{1.04} dx \left[\frac{12x\left(-3x^8 + 6x^4 + 1\right)}{\left(1 - 3x^4\right)^2}\right]\xi^2 \leq \int_1^{1.04} \frac{12\left(1 + x^2 - 3x^4 + x^6\right)^2}{x\left(1 + 6x^4 - 3x^8\right)}(\xi_{,x})^2 dx \tag{125}$$

holds for any ξ vanishing at $x=1$. It is easily shown that

$$\frac{12\left(1+x^2-3x^4+x^6\right)^2}{x\left(1+6x^4-3x^8\right)} < \frac{1}{2}\frac{x^4-1}{x} \tag{126}$$

holds in the interval $[1, 1.04]$.

Step 2: We show that (123) holds for any function ξ supported in $[1.04, c+0.1]$ with $\xi(c+0.1)=0$ and $\xi(1.04)$ arbitrary. Since any ξ vanishing at $x=1$ and $x=\infty$ can be decomposed into a ξ_1 vanishing at $x=1$ and $x=c+0.1$ and a ξ_2 having support only in $[c,\infty)$ (and therefore trivially satisfying (123)), this step will prove that (123) holds for all ξ vanishing at $x=1$ and $x=\infty$.
Define the weight functions

$$w(x) = \begin{cases} \frac{12x\left(-3x^8+6x^4+1\right)}{(1-3x^4)^2} & \text{if } x\in[1.04, c] \\ 0 & \text{if } x\in[c, c+0.1] \end{cases} \tag{127}$$

and

$$v(x) = \frac{1}{2}\frac{x^4-1}{x} \tag{128}$$

We have to show

$$\int_{1.04}^{c+0.1} w(x)\xi^2 dx < K\cdot\int_{1.04}^{c+0.1} v(x)\left(\xi_{,x}\right)^2 dx \tag{129}$$

holds for functions ξ vanishing at the right boundary and a constant $K\leq 1$. For such weighted Hardy-inequalities there is the following

Theorem: (cf Theorem 6.2 in [**?**]) Let $w(x)$ and $v(x)$ be positive, measurable functions in the interval $[a,b]$ and ξ be an absolutely continuous function also defined in the interval $[a,b]$ and vanishing at the point b. Then the inequality

$$\int_a^b w(x)\xi^2(x)dx \leq C^2\cdot\int_a^b v(x)\left(\xi_{,x}\right)^2(x)dx \tag{130}$$

holds for some constant C if and only if

$$B := \sup_{a<x<b} F(x) := \sup_{a<x<b}\sqrt{\int_a^x w(x)dx\cdot\int_x^b \frac{1}{v(x)}dx} < \infty \tag{131}$$

Furthermore, the "optimal constant" C in (130) satisfies

$$B\leq C\leq 2B \tag{132}$$

In our case, we can compute the function $F(x)$ explictly from the given weights (128) and (127). The plot of $F(x)$ in $[1.04, c+0.1]$ is given in Figure1. Clearly, $F(x)<\frac{1}{2}$ everywhere. By the theorem, the ideal constant C in (130) is smaller than one for the weights considered. Hence (129) holds – even for some K smaller than 1.

The AdS terms
For the additional AdS terms in (121) we proceed along the same lines. We have to show

$$\int_a^\infty d\rho\frac{32a^8\rho^3}{\left(a^4-3\rho^4\right)^2}\xi^2 < \int_a^\infty \frac{1}{2}\rho\left(\rho^4-a^4\right)\left(\xi_{,\rho}\right)^2 d\rho \tag{133}$$

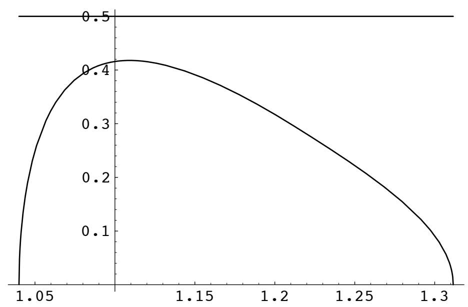


Figure 1: The function F(x)

or equivalently, using rescaled variables

$$\int_1^{\infty} \frac{32x^3}{\left(1-3x^4\right)^2}\xi^2 dx < \int_1^{\infty} \frac{1}{2}x\left(x^4-1\right)\left(\xi_{,x}\right)^2 dx \tag{134}$$

The latter inequality should hold for any function ξ vanishing at 1 and infinity.
Step 1: We show that (134) holds in $[1, 1.07]$ for functions ξ vanishing at $x = 1$ and admitting arbitrary values at $x = 1.07$:

$$\int_1^{1.07} \frac{32x^3}{\left(1-3x^4\right)^2}\xi^2 dx \leq -0.9\xi^2(1.07) - \int_1^{1.07}\left(\frac{8}{3-9x^4}+\frac{4}{3}\right)2\xi\xi_{,x}dx \tag{135}$$

$$\leq \sqrt{\int_1^{1.07} \frac{32x^3}{\left(1-3x^4\right)^2}\xi^2\ dx} \cdot \sqrt{\int_1^{1.07} \frac{2\left(x^4-1\right)^2}{x^3}\left(\xi_{,x}\right)^2 dx} \tag{136}$$

and therefore

$$\int_1^{1.07} \frac{32x^3}{\left(1-3x^4\right)^2}\xi^2 dx \leq \int_1^{1.07} \frac{2\left(x^4-1\right)^2}{x^3}\left(\xi_{,x}\right)^2 dx \tag{137}$$

Now since

$$\frac{2\left(x^4-1\right)^2}{x^3} < \frac{1}{2}x\left(x^4-1\right) \tag{138}$$

holds for $x \in [1, 1.07]$ the first step is complete.

Step 2: We show that (134) holds in $[1.07, M]$ for functions ξ, arbitrary at $x = 1.07$ and vanishing at $x = M$, where M is an arbitrary large number. To do this we again construct the critical function in the theorem above and show that it is everywhere smaller than $\frac{1}{2}$. Figure 2 shows a plot of the critical function $\tilde{F}$ for the case $M = 10$.

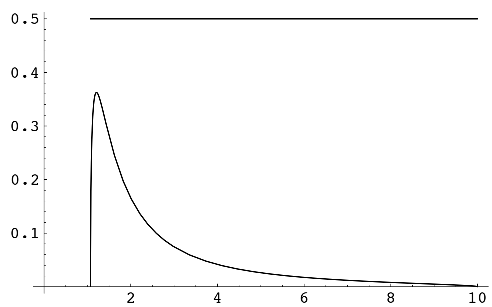


Figure 2: The function $\tilde{F}(x)$