Measuring complexity of curves on surfaces

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based on joint work with Max Neumann-Coto

Zoom

Topology Students Workshop
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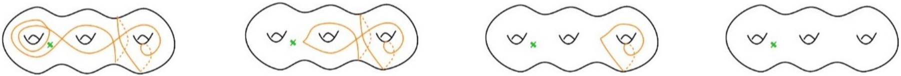
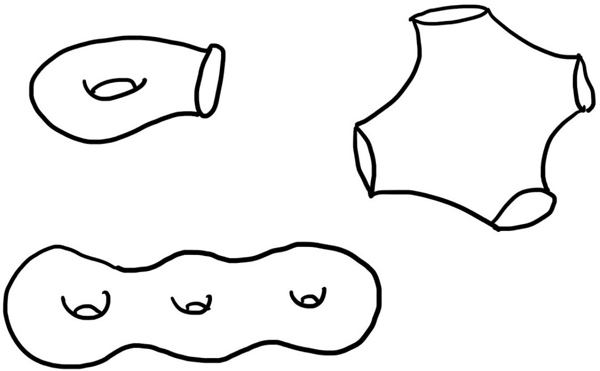


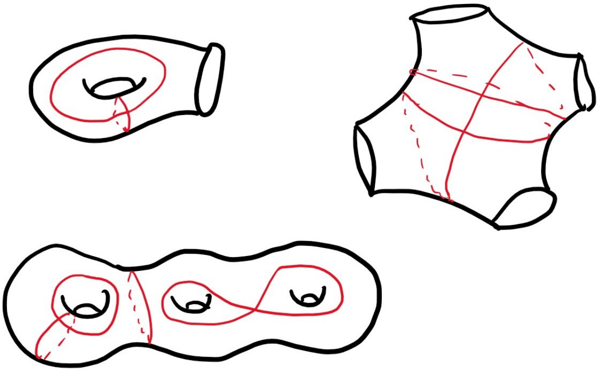
Introduction: Objects

-Σ: an orientable, compact, hyperbolic surface.

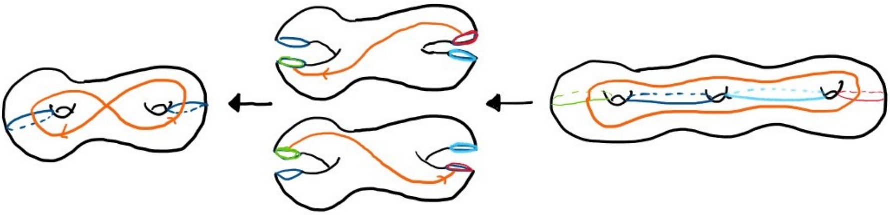


Introduction: Objects

- α: a closed, essential curve.



Introduction



In this talk, we show that the minimum degree of a covering to which a curve lifts to a closed embedding is bounded above linearly by the self-intersection number of the curve.

Introduction: measures of complexity

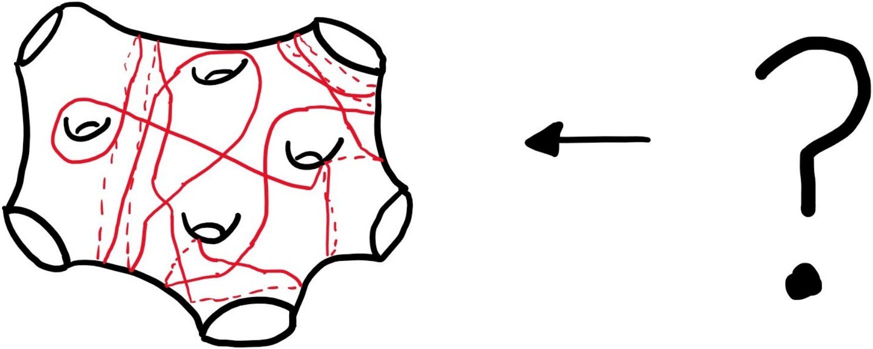
Let α be a curve in Σ and let $[\alpha]$ be its free homotopy class, consider:

- $i(\alpha)$: the minimum self-intersection number of a curve in $[\alpha]$.
- $d(\alpha)$: the minimum degree of a covering of Σ to which α lifts as a closed embedding.

A (corollary of a) theorem of Scott

Scott, 1978

Let Σ be a surface and α a curve with minimal self-intersections in Σ, then there is a covering of finite degree where α lifts as a closed embedding.



Question

Is the degree of such a covering of Σ bounded in some reasonable way?

Related results

Theorem (Patel, 2014)

There is a hyperbolic metric on Σ and a constant C for which every closed geodesic α of length k lifts as an embedding in a covering of degree $\leq Ck$.

Theorem (Aougab-Gaster-Patel-Sapir, 2017)

$d(\alpha)$ is bounded above by a function that depends only on the topology of Σ and on $i(\alpha)$.

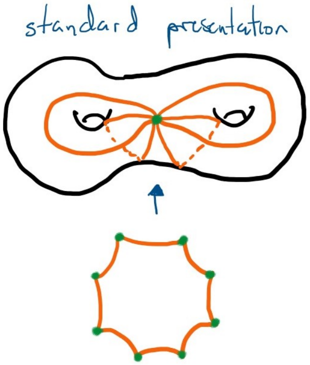
Question:

Is it possible to bound the minimum degree $d(\alpha)$ by a function of the self-intersection number *only*?

Yes - in our Main Theorem, we show this by explicitly constructing coverings with this property.

Presenting $\pi_1(\Sigma)$ via *cutting graphs*

- A way of cutting Σ into a disc corresponds to a presentation for $\pi_1(\Sigma)$.



- $l(\alpha)$ = minimum word-length of α in a presentation with a single vertex.

A possible way forward

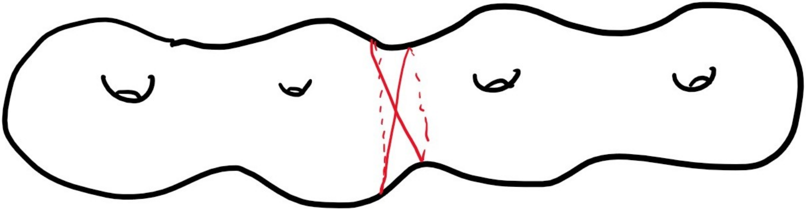
Lemma (Scott, 1978)

Let α be an immersed curve in a surface with non-empty boundary, then $d(\alpha) \leq l(\alpha)$. Let α be an immersed arc, then $d(\alpha) \leq l(\alpha) + 1$.

Can we bound $l(\alpha)$ effectively?

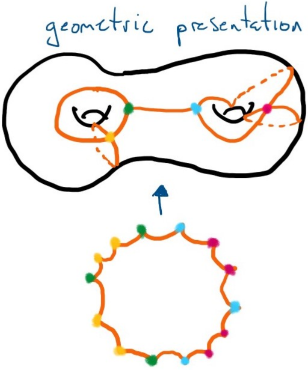
No!

There are curves with a single self-intersection and whose word-length in a standard presentation grows with the genus of the surface:



Presenting $\pi_1(\Sigma)$ via *cutting graphs* pt 2

- A graph with several vertices yields a *geometric* presentation with fundamental polygon P.



- $n(\alpha)$ = minimum $|\alpha \cap \partial P|$ among all such P.

Results

Theorem (A.-Neumann-Coto, 2020)

Let α be a curve or an arc in a compact surface Σ, then $n(\alpha) \leq i(\alpha) + 1$.

Proof sketch for closed surfaces and filling curves:

Let $i(\alpha) = s$. We need to find a graph G with

- $|G \cap \alpha| \leq s + 1$
- $\Sigma - G$ is homeomorphic to a disc.

Results

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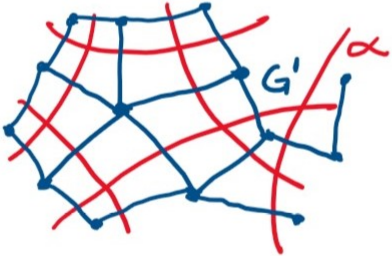
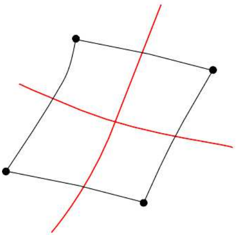
Results

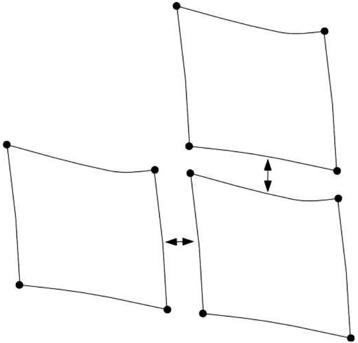
Let G' be the dual graph to α:



Results

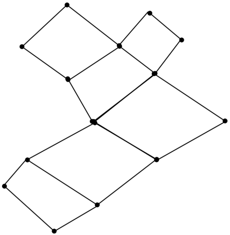
- G' intersects α once at each edge.
- G' cuts Σ into quadrilaterals $C_1, C_2, ..., C_s$.
- Glue $C_1, C_2, ..., C_s$ along $s-1$ sides to get a polygonal disc P that is a fundamental domain for Σ.





Results

- ∂P has $2(s+1)$ sides:



-Define G to be the image of ∂P in Σ. Then G is the desired graph.

Results

Main theorem (A.-Neumann-Coto, 2020)

Let α be a curve with minimal self-intersections in Σ, then

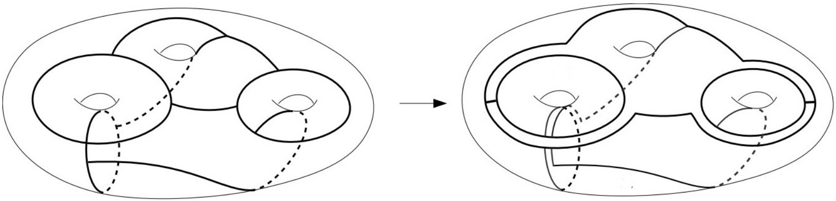
1. $d(\alpha) \leq i(\alpha) + 1$, if Σ is planar.
2. $d(\alpha) < 5(i(\alpha) + 1)$.

Comments

To prove the main theorem we combine several ingredients:

-Use a version of Scott's Lemma relating $d(\alpha)$ and $l(\alpha)$, but modified to work instead with $n(\alpha)$.

-For closed surfaces more work is needed: cut Σ to obtain a surface with boundary Σ' as in the figure.



Comments

-The curve becomes a set of arcs, each of which has to be lifted separately to a "partial covering".

-Glue these partial coverings to a covering of Σ where the curve lifts.

-One has to be careful when doing this, since in each partial covering, boundary circles might cover with distinct degrees.

Related question

A theorem due to Hempel states that all surface groups are residually finite, i.e., that for each $g \in \pi_1(\Sigma)$ there is a a finite index subgroup that does not contain g.

Is there a good bound for the minimum index of such a subgroup?

Equivalently:

Is there a good bound for the minimum degree of a covering of Σ where a lift of α does *not* close?

Our results imply that this degree is $\leq 5(i(\alpha)+1)$, but this does not seem sharp.