Cubulating spaces and groups; example sheet 3

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1. Let S_g be a closed orientable surface of genus $g \geq 2$. Prove that $\pi_1 S_g$ can be cocompactly cubulated (in infinitely-many ways!) by proving the following

   (a) Let L be a transverse collection of n lines in the hyperbolic plane, with $n > 1$. Then, there exists a number $R = R(L) > 0$ such any line intersecting all the lines in L, intersects the ball of radius R about the origin.

   (b) For each $k > 0$, there are finitely many $\pi_1 S_g$-orbits of transverse collections of k lines in L

   (c) There is a bound on the size of a transverse collection of lines in L.

   By choosing a suitable collection α of closed curves in S_g, and defining a wallspace structure in the hyperbolic plane using the lifts of these curves, conclude that $\pi_1 S_g$ acts cocompactly on the dual C. Show that if the collection α is *filling* in the sense that $S_g - \alpha$ is a collection of discs, then the action is also proper.

   What is the dimension of such a cubulation?

2. Let S_g be as in the previous question. Using your favourite non-positively curved square tessellation of S_g, prove that said surface is homeomorphic to a virtually special cube complex. What is the least genus needed for a surface to be homeomorphic to a special cube complex?

3. Give an example of a 2-dimensional special cube complex X that is *not* a surface or the Salvetti complex of a raag.

4. Determine the least number of cubes needed to construct a 3-dimensional special cube complex that is not a square complex.

5. Prove, amend if possible, or[1] find a counterexample for the following statements:

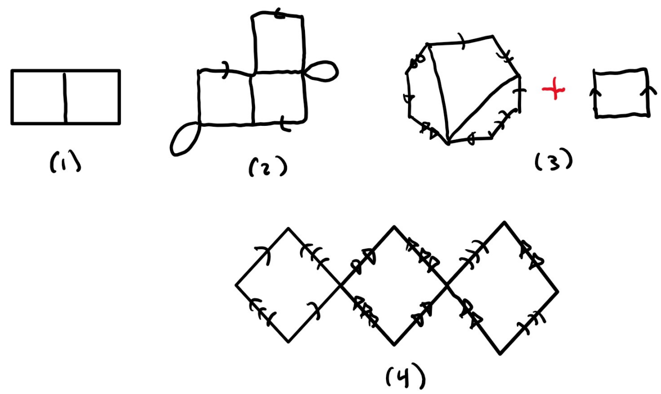
   (a) if $X_1, \ldots, X_n$ are special, then every subcomplex of $X_1 \times \cdots \times X_n$ is special,

   (b) if X is a special cube complex, then every cubical subdivision of X is special,

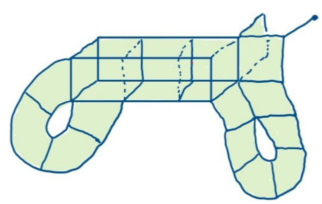
   (c) if X is a virtually special cube complex, then every hyperplane of X is special.

[1]the "or" need not be exclusive.

6. For the cube complexes X_i shown below, determine if they are special or not. For the ones that are special, construct explicitly a Salvetti complex R and a local isometry $X_i \to R$.



7. Find (i.e., draw) a special covering for the cube complex in Figure 7.



8. Let X be a non-positively curved cube complex. Can one detect from the action of $\pi_1 X$ on $\widetilde{X}$ whether X is special or not?

9. Prove that the following conditions are equivalent for a group G

   (a) For any $g \in G - \{1\}$, there is a subgroup of finite index not containing g,

   (b) for any $g \in G - \{1\}$, there is a homomorphism from G to a finite group K where the image of g is non-trivial,

   (c) the intersection of subgroups of finite index in G trivial,

   (d) the intersection of normal subgroups of finite index in G trivial.

10. Consider the free group $F_4 = \langle a, b, c, d \rangle$. Show that the subgroups $H_1 = \langle a^2bcd^{-2}bd \rangle$ and $H_2 = \langle d^3, abc, cb^4 \rangle$ are separable.