Cubulating spaces and groups; example sheet 2

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1. Prove that a path $\sigma \to \widetilde{X}$ is a geodesic if and only if each edge of σ is dual to a distinct hyperplane of $\widetilde{X}$. (Hint: consider an innermost "bigon" formed by a segment of σ and a hyperplane that crosses it twice, and a minimal area disc diagram determined by these.)

2. Show that the intersection of two convex subcomplexes of a CAT(0) cube complex is a convex subcomplex.

3. Let $\widetilde{X}$ be a CAT(0) cube complex, and H be a hyperplane. The *open carrier* $N^o(H)$ of H is the union of all open cubes intersecting H. A *frontier* is a connected component of $N(H) - N^o(H)$, a *halfspace* is a connected component of $\widetilde{X} - H$, and the *major and minor halfspaces* of H are, respectively, the smallest subcomplex containing a halfspace and the largest subcomplex contained in a halfspace of H. Prove that

   (a) each frontier of H is convex.

   (b) each major and minor halfspace is convex

   (c) the carrier $N(H)$ is convex.

4. Show that CAT(0) cube complexes have the *Helly property*. Namely, prove that if X is a CAT(0) cube complex and $Y_1, \dots, Y_n$ are finitely many convex subcomplexes satisfying $Y_i \cap Y_j \neq \emptyset$ for each i, j, then $\bigcap_i Y_i \neq \emptyset$.

5. Show that if G acts on a CAT(0) cube complex $\widetilde{X}$ cocompactly, then the action is proper if and only if it is metrically proper; show that if G acts on $\widetilde{X}$ properly, then the action is cocompact if and only if the quotient $G \setminus X$ is compact.

6. Prove that if G acts cocompactly on a CAT(0) cube complex $\widetilde{X}$, then every hyperplane is acted on cocompactly by its stabiliser.

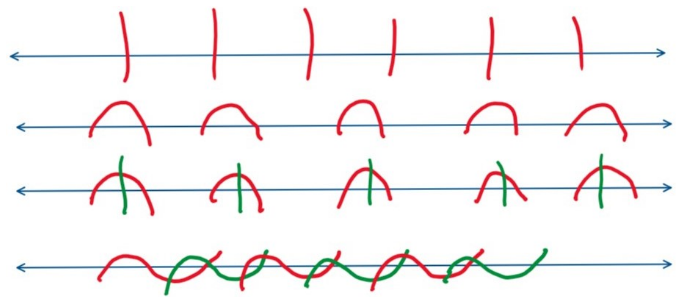
7. Let C be the dual cube complex to a wallspace $(S, \mathcal{W})$.

   (a) Show that the maximal dimension of a cube in C is equal to the maximal size of a collection of pairwise crossing walls in $(S, \mathcal{W})$,

   (b) can the dual C to a wallspace be infinite-dimensional? If yes, give an example, if no, prove or otherwise justify.

8. For each $n \in \mathbb{N}$, give an example of a wallspace structure on the hyperbolic plane whose dual has dimension n.

9. Determine the cube complexes dual to each of the following wallspaces, and their quotients under the action by translations of $\mathbb{Z}$:



10. Determine the cube complexes dual to the wallspaces obtained by taking a collection of n pairwise crossing lines and their integer translates in $\mathbb{E}^2$ (see Figure 1). Describe the non-canonical 0-cubes.

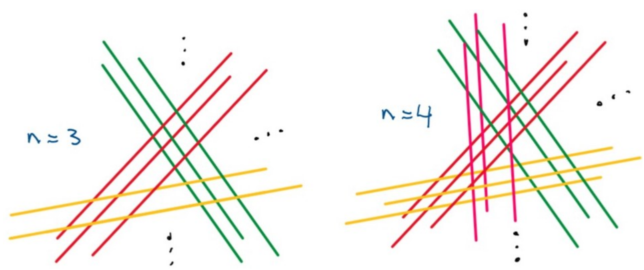


Figure 1: Collections of 3 and 4 pairwise crossing lines, and some of their translates.