Cubulating spaces and groups; example sheet 1

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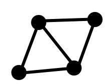
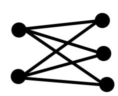
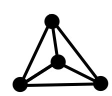
January 25, 2024

1. In how many different ways can one identify the edges of a square s so that the cube complex obtained as a result is non-positively curved? Describe each of the possibilities.

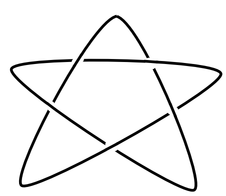
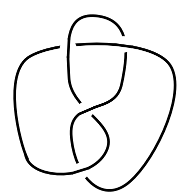
2. Show that every surface other than the sphere and the projective plane is homeomorphic to a non-positively curved square complex. What can you say about the remaining two surfaces?

3. Prove that if $X_1, \dots, X_n$ are non-positively curved cube complexes, then the product $X_1 \times \cdots \times X_n$ is a non-positively curved cube complex. (Hint: prove that the links of vertices in the product are joins[1] of the corresponding links in each of the factors.)

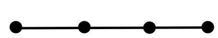
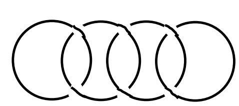
4. Identify the raags associated to the following graphs, describe (and draw if possible) their corresponding Salvetti complexes and their vertex links.

5. Construct the Dehn complexes associated to the following knot projections and find presentations for the corresponding knot groups.

6. Let Γ and L, respectively, be the graph and the link pictured below. Prove that the raag associated to Γ is isomorphic to $\pi_1(S^3 - L)$.

[1]The *join* of A and B is the space formed by taking the disjoint union $A \sqcup B$, and attaching line segments joining every point in A to every point in B.

7. The *thickness* of an edge e in a cube complex X is the number of 2-cells having e as a face. For each $n \in \mathbb{N}$, construct a (locally-finite) non-positively curved square complex X_n such that the thickness of each edge in X_n is exactly n. Now construct such an X_n having exactly one vertex.

8. For each $g \in \mathbb{N}$, let S_g denote the closed orientable surface of genus g. Determine the least number of squares in a non-positively curved square complex homeomorphic to S_g. Now do this for non-orientable surfaces. (Hint: Euler characteristic.)

9. Find a non-positively curved square complex X that has at least one square and exactly one hyperplane. What's the least number of squares needed to produce such an X?

10. Construct non-positively curved cube complexes where every hyperplane is, in each case: (a) a circle, (b) a bouquet of two circles, (c) the wedge of a torus and a circle.

11. Show that if X is the Salvetti complex of a raag, then each hyperplane of X is also the Salvetti complex of a raag; identify said raag.

12. Using only what has been covered in lectures and in the example sheet, show that if S is a surface with $\chi(S) < 0$, then $\pi_1 S$ contains a free non-abelian subgroup.

13. Prove that if $D \to X$ is a cubical disc diagram in a non-positively curved cube complex X, then D has no monogons.