A cubical Rips construction

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GaTO Seminar

0. Intro

Let $\heartsuit$ = "nice" class of groups

👻 = pathological property

Is there a $G \in \heartsuit$ and $H \leq G$ having 👻 ?

$\heartsuit = \Bigg\{$ eg: hyperbolicity
cubulability

👻 $= \Bigg\{$ eg: unsolvable membership problem
incoherence
undecidable rank problem

Rips construction

Thm (Rips) Given f.p. group Q, $\exists$ SES:

$$1 \to N \to G \to Q \to 1$$

where:
- N is fg
- G is hyperbolic (& cubulated)

Wise

Why is this useful???

•) In many cases, can lift  from Q to subgroup of G.

•) Very flexible (many variations exist)

I. Some background stuff

A. Small-cancellation theory

Idea: small-cancellation measures overlaps between relators in a group presentation

Def. $P = \langle S \mid R \rangle$ finite symmetrised presentation.

A piece is a subword p that appears in at least two different ways amongst relators.

ex $\langle a,b,c \mid a^2 b^2 a^2 c^2 \rangle$ $\quad$ a^2 is a piece

$\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$ length-1 subwords are only pieces

(standard presentation of any closed surface w/ $g \geq 2$)

$C'(\frac{1}{n})$ small cancellation condition if p is a piece in some $r \in R$, then $|p| < \frac{1}{n}|r|$

<u>Theorem</u> If a presentation for G satisfies $C'(\frac{1}{6})$, then:

(1) hyperbolic

(2) aspherical (if the relators are not proper powers)

(3) cubulable

(4) "Random" groups satisfy the $C'(\frac{1}{6})$ condition

Gromov, Ollivier

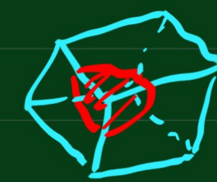
<u>Corollary</u> "Random" groups are 2-dimensional

B. Cube complexes and cubulability

A cube complex is non-positively curved ("NPC") if all links of vertices are flag complexes

→ simplicial

→ if (n−1)-skeleton of a simplex is in link(v), then so is the simplex.



CAT(0) cube complex are simply connected NPC cube complexes

⇝ CAT(0) cube complexes are CAT(0) spaces, so

⟹ Contractible

⟹ NPC cube complex X is a class. space for $\pi_1 X$

G is [cocompactly] cubulated if it acts properly and cocompactly on a CAT(0) cube complex

II. Back to Rips' construction

Thm Given f.p. group Q, and $\lambda > 0$, $\exists$ $C'(\frac{1}{\lambda})$ group G and SES $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ where N is f.g.

Sketch Write $Q = \langle a_1 \dots a_n \mid r_1 \dots r_m \rangle$

$$G = \langle a_1, \dots a_n, x, y \mid r_1 *, \dots, r_m *, \begin{matrix} \{a_i^{\pm 1} x a_i^{\mp 1} *\} \\ \{a_i^{\pm 1} y a_i^{\mp 1} *\} \end{matrix} \rangle$$

each "$*$" is a different "nice" word on x and y, for instance:

$$*_1 = xyxy^2 \dots xy^{200\lambda}$$

$$*_2 = xy^{200\lambda+1} \dots xy^{400\lambda}$$

etc...

$G \xrightarrow{\varphi} Q$ is the map that kills x, y; conjugation rels imply $\operatorname{Ker}\varphi = N = \langle x, y \rangle$.

Rmk G obtained here has $cd = 2$

Applications:

A. There exist hyperbolic gps with unsolvable membership problem:

Choose Q f.p. having unsolvable word problem (examples exist)

input in Rips construction, get $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$.

G is hyperbolic, and $w \in N \trianglelefteq G$ iff $w =_Q 1$, which is undecidable.

B. " " hyperbolic incoherent groups

Use $F_2 \times F_2$

C. " " hyperbolic gps with undecidable rank problem

Similar

D, ..., Z... many more things !!!

III. Beyond the 2nd dimension

Q: how do we produce large-dimensional hyperbolic groups?

(1) Manifolds

Gromov Bestvina

Question/conjecture: Are all constructions arithmetic/number theo?

False; work of lots of people:

(2) Coxeter groups (Januszkiewicz-Świątkowski, Osajda)

(3) Pseudomanifolds (Fujiwara-Manning, Mosher-Sageev)

etc...

More flexibility? Rips-type technique?

IV. Cubical presentations ...

$$\langle a_1 \cdots a_n \mid r_1 \cdots r_m \rangle \longrightarrow \langle X \mid \{ Y_i \to X \} \rangle := X^*$$

"coned-off space"

SVK $\Rightarrow \pi_1 X^* \cong \pi_1 X / \langle\langle \{\pi_1 Y_i\} \rangle\rangle$

→ There is a cubical small-cancellation theory where:

Pieces are overlaps between higher dimensional subcomplexes in X^*

Small-cancellation conditions are defined similarly to classical case

and:

$C'(\frac{1}{14})$ + $\pi_1 X$ hyperbolic $\Rightarrow$ hyperbolicity for X^*

$\Uparrow$

$\underbrace{B(6)}$ + extra stuff $\Rightarrow$ cocompact cubulability of X^*

wallspace structure

V. ... and a variation of the Rips construction

Thm (A, 22) Given Q f.p. and X special cube complex with $\pi_1 X := G$ hyperbolic and non-elementary, $\exists$ SES:

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

where:

- $N \cong G/K$
- Γ is hyperbolic and cocompactly cubulated
- $\max\{cd\, G, 2\} \geq cd\, \Gamma \geq cd\, G - 1$

Thanks!