Linear isoperimetric functions for surfaces in hyperbolic groups.

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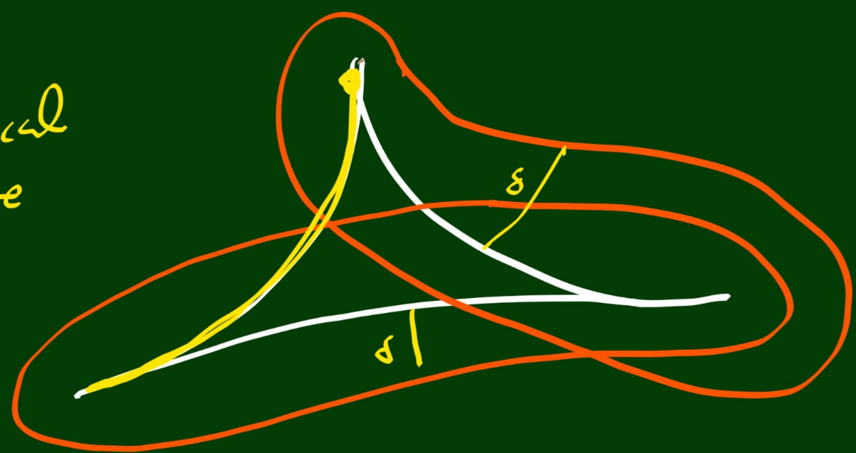
δ-thin: a triangle is δ-thin ($\delta \geq 0$) if each of its sides lies in the union of the δ-nhoods of the other 2 sides.

typical picture



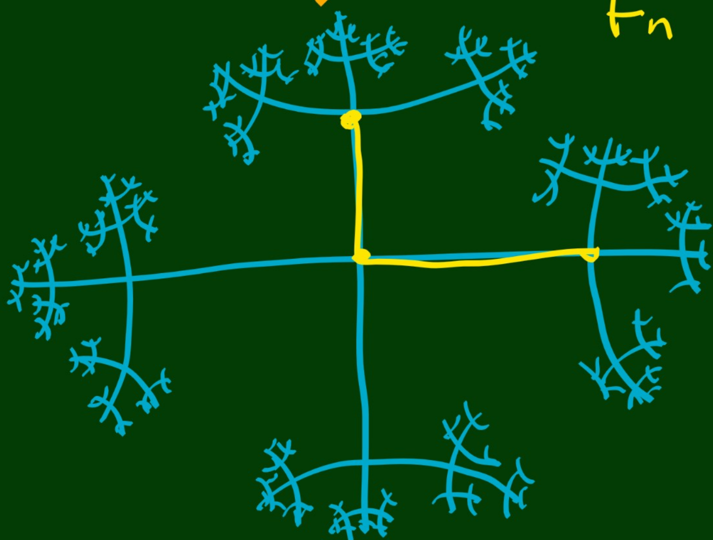
word-, Gromov-

hyperbolic groups: a group G is δ-hyperbolic if $\exists\, \delta \geq 0$ and a finite gen. set S such that all geodesic Δ's in $Cay(G,S)$ are δ-thin.

Note: hyperbolicity does not depend on the generating set! $Cay(G,S)$ is δ-hyperbolic $\Leftrightarrow$ $Cay(G,S')$ is δ'-hyperbolic

examples:

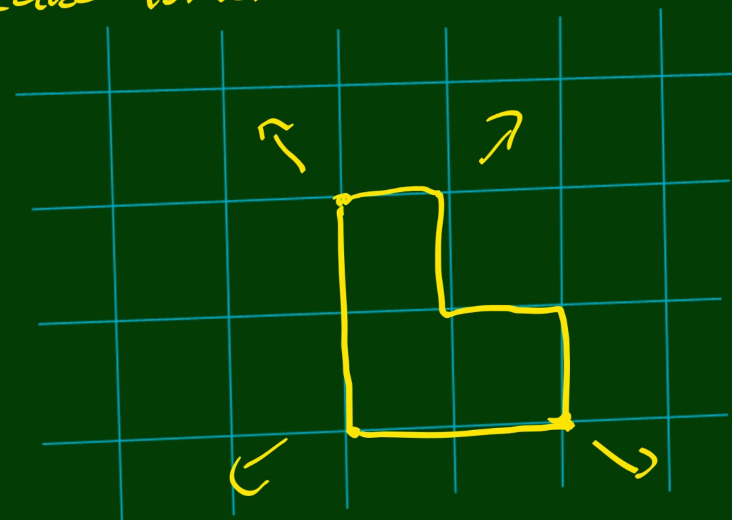
F_n



- finite gps
- surface gps
- $\pi_1 M$ 3 mnfld (closed, hyperbolic)
- some graphs of groups ✓ "some quots of hyperbolic gps"
- some small cancellation groups

non-examples:

because homotheties exist! $\mathbb{Z}^2$



- $\mathbb{Z}^n$ $n \geq 2$ } "poison"
- $BS(a,b)$
- anything containing
- any grp that isn't f.g.

algorithmic properties:

hyperbolic groups have solvable word and conjugacy problems. $w =_G 1$? w_1, w_2, $w_1 \sim_G w_2$?

topological interpretation: given X cell complex with $\pi_1 X$ hyperbolic, $\exists$ algorithms that decide if:

1. closed path $p \longrightarrow X^1$ is nullhomotopic

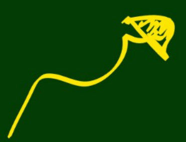
2. closed paths $p_1, p_2 \longrightarrow X^1$ are homotopic

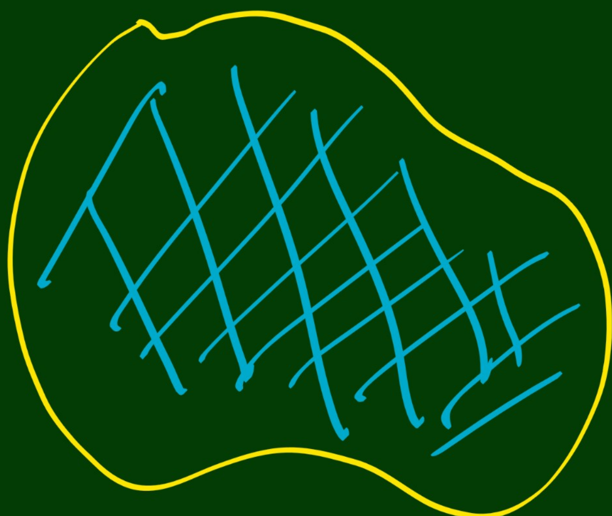
(2) $\Longrightarrow$ (1)

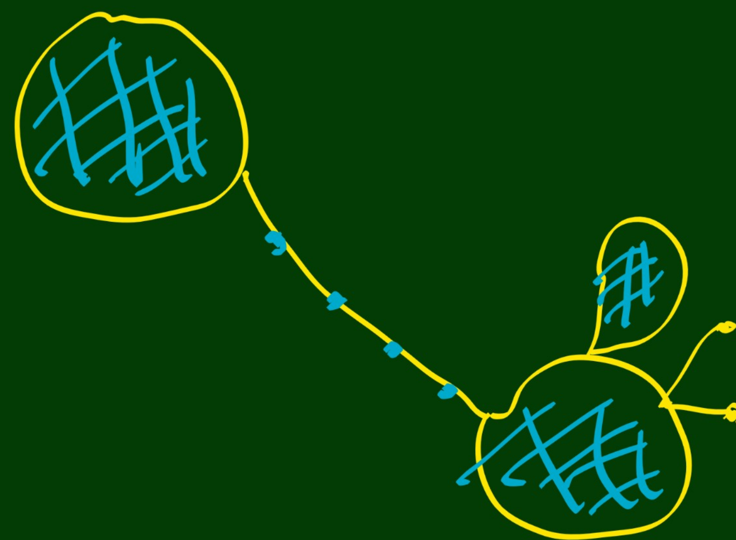
Diagrams: a <u>disc diagram</u> D is a compact, contractible 2-complex w/ embedding $D \hookrightarrow S^2$.

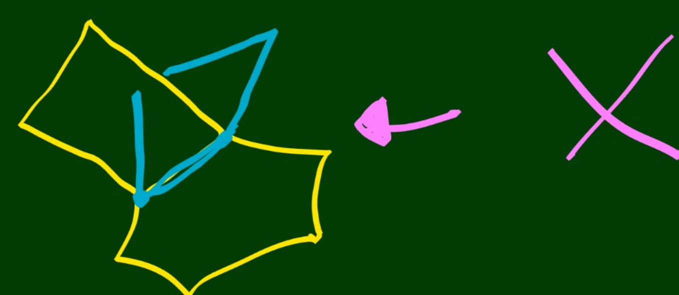
a <u>disc diagram in X</u> is a combinatorial map $D \to X$. ~~embedding~~ immersion

The <u>boundary</u> ∂D of D is ∂-path of $(S^2 - D)$









Annular diagrams: for annuli, the definitions are similar:

- An <u>annular diagram</u> A is a compact 2-complex with homotopy type of an annulus, and with embedding $\underbrace{A \hookrightarrow S^2}$

The <u>boundary</u> <u>$\partial A = C_1 \cup C_2$ of A</u> is the union of the ∂-paths of $\underbrace{S^2 - A}$.

Where's the geometry?

Problem: Solve the word / conjugacy problem "efficiently"

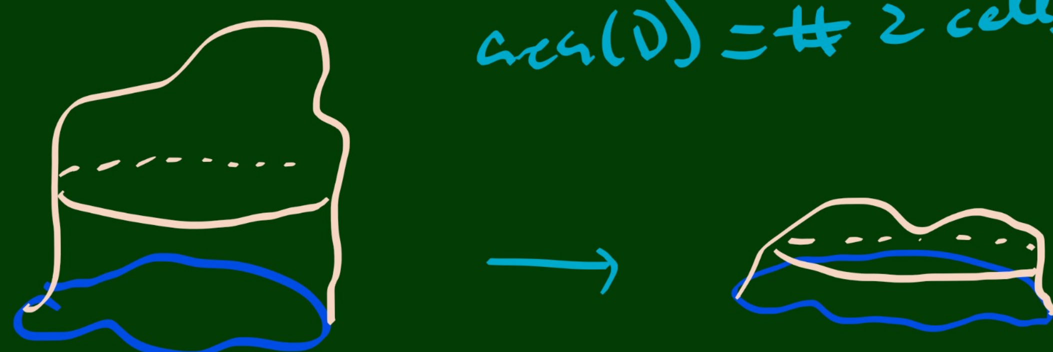
Translation: understand a group's isoperimetric properties.

$\partial D = P$

relate $|P|$ to the area(D)

$|P| = \#\,1$ cells

area$(D) = \#\,2$ cells



A map $f: \mathbb{N} \to \mathbb{R}$ is an <u>isoperimetric function</u> <u>for X</u> if $\forall$ $\rho \to X'$ nullhomotopic $\exists$ disc diagram D with $\partial D = \rho$ and

$$\underbrace{\mathrm{Area}(D) \leq \underbrace{f}(|\rho|)}$$

"G (or X) has isop. function of type <u>linear / polynomial / exponential ...</u>"

Examples: $\mathbb{Z}^2 \sim$ quadratic

$BS(1,2) \sim f(n) \sim 2^n$

finitely generated nilpotent $\sim$ polynomial.

Theorem A group is hyperbolic $\Leftrightarrow$ satisfies a linear isoperimetric function.

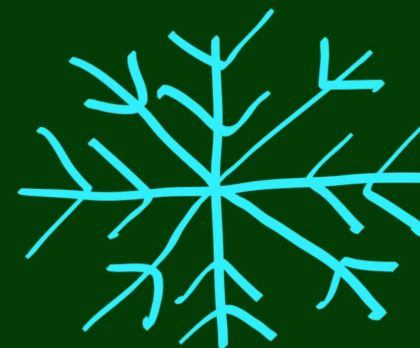
Question what growth types can be achieved?

Theorem If a group satisfies a subquadratic isoperimetric function then it satisfies a linear isoperimetric function $\Rightarrow$ hyperbolic

$f(n) \sim n^d$ $d \in (1,2)$ $\Rightarrow$ $f'(n) \sim n^d$ $d = 1$

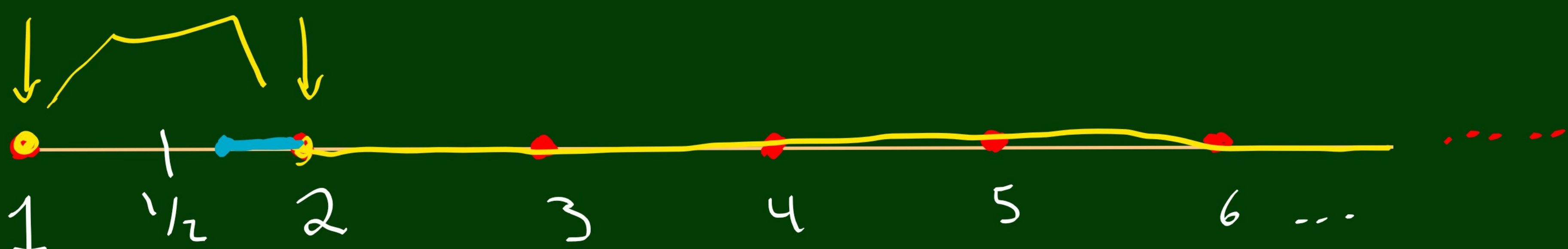
There is a gap in the "isoperimetric spectrum."

This is <u>the</u> only gap!



<u>Theorem</u> (Brady–Bridson) "snowflake groups"

The set $\{d \mid \exists \text{ group w/ minimal isop fct} \sim n^d\}$ is dense in $[2, \infty)$.



For annuli: A map $f_A : \mathbb{N} \to \mathbb{R}$ is an <u>annular isoperimetric function</u> for X if $\forall\, P_1, P_2 \to X$ homotopic, $\exists$ annular diagram $A \to X$ with $\partial A = P_1 \cup P_2$ and

$$\mathrm{Area}(A) \le f(|P_1| + |P_2|)$$

Obs classical isop function bounds from below the annular isop function

Theorem G is hyperbolic $\iff$ it satisfies a <u>linear</u> annular isoperimetric function.

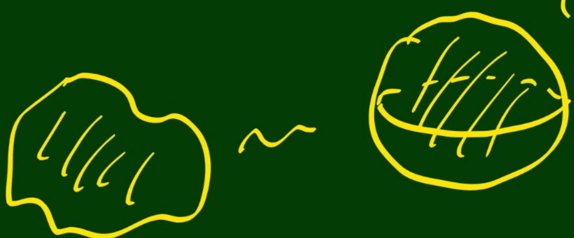
Warning: disc and annular isoperimetric functions can be very different

(Olshanskii-Sapir, 19-20)

Note There exist f.p. groups having quadratic "classical" isoperimetric function and non-recursive, annular isoperimetric function. (not quadratic)

Generalisations:

Several possibilities:



1. higher homotopy groups
2. homological isoperimetry
3. more complicated "equations" in $\pi_1 X$.

$w =_G 1$?

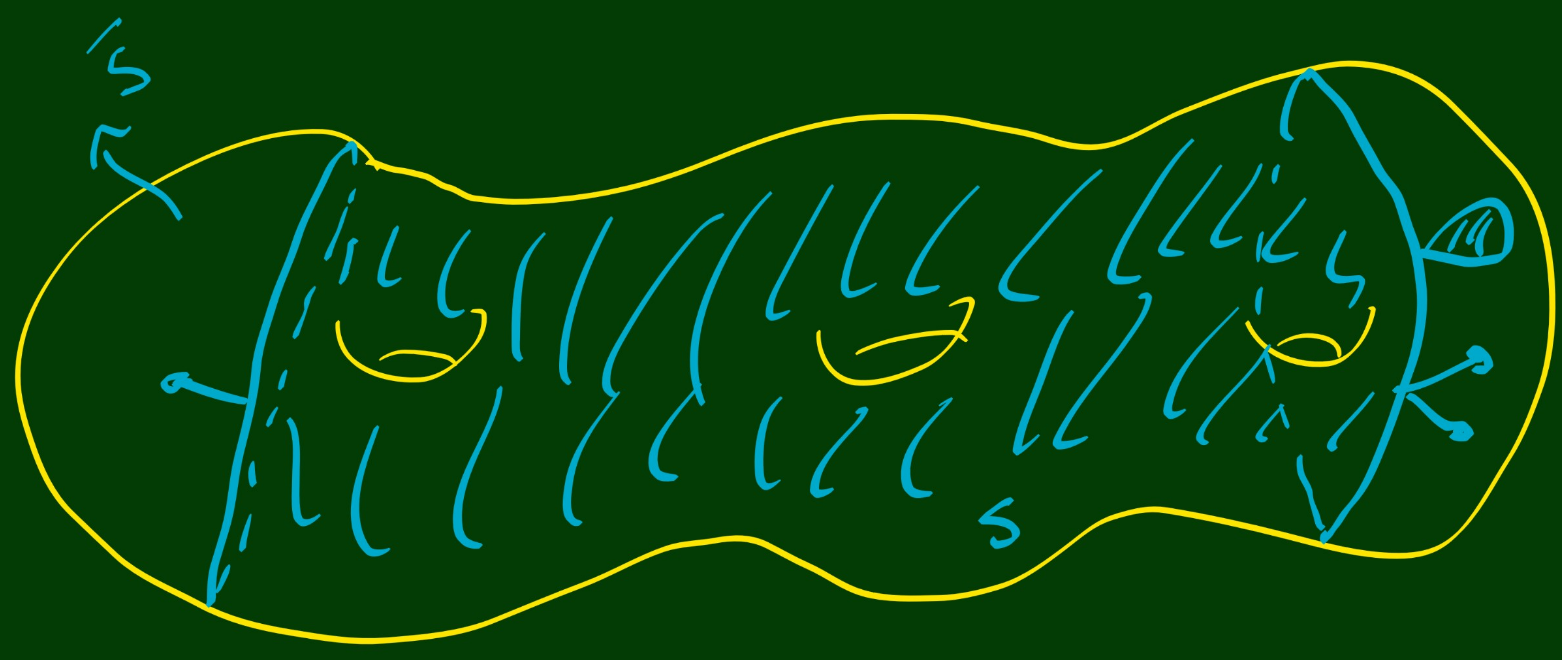
$w =_{G^{ab}} 1$?

(g,n)-surface diagrams:

- a <u>surface diagram</u> is a compact 2-complex S having homotopy type of a surface with boundary and w/ embedding $S \hookrightarrow \bar{S}$ where $\bar{S}$ closed surface.

- Surface diagram S in X is a map $S \to X$.

- Boundary $\partial S = \bigcup_i C_i$ is union of ∂-paths of $\bar{S} - S$.



Homology:

<u>Question</u>: Given a nullhomologous k-cycle, how "efficiently" can it be filled in with a (k+1)-chain?

C cycle, $\partial D = C$

relate k-Vol(C) with (k+1)-Volume(D)

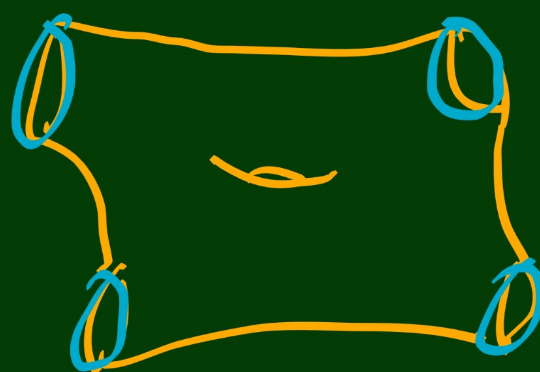
k-Volume = # k-dim'l cells

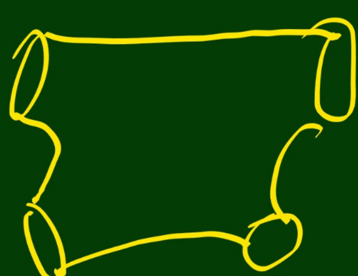
(k+1)-Volume = # (k+1)-dim'l cells

Definition A map $f:\mathbb{N}\to[0,\infty)$ is a <u>weak</u> homological isoperimetric function for X if $\forall\ C\to X$ nullhomologous, $\exists\ D\to X$ chain with $\partial D = C$ and:

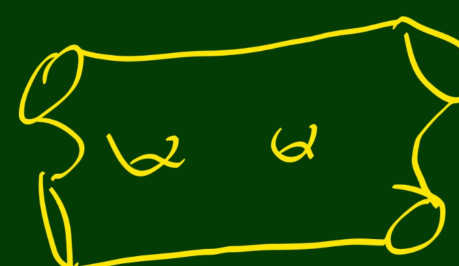
$$(k+1)\text{-Volume}(D)\leq f(k\text{-vol}(C))$$

When $k=1$:





control the area but lose control over the topology of surfaces



Gersten $k=1$, Lang, Mineyev in all other cases

Theorem If G is torsion-free and hyperbolic $\Rightarrow$ it satisfies linear homological isoperimetric functions $\forall\, k \geq 1$ $(\mathbb{Q}, \mathbb{R}, \mathbb{Z})$

Note: there exist groups whose "classical" and weak isoperimetric functions are inequivalent!

Concretely: Abrams – Brady – Dani – Young

construct group with the following properties:

1) CAT(0)

2) whose weak isop functions has strictly smaller growth than classical one.

Note: $\exists$ groups for which weak isop functions are defined but the classical ones are not!

Generalising to other "equations" in $\pi_1 X$:

Def Let $p_1 \longrightarrow X_1, \cdots p_n \longrightarrow X$, be a collection of paths, its g-Area is given by:

$$\mathrm{Area}_g\left(\bigsqcup_i p_i\right) = \inf\left(\mathrm{Area}(S) : \begin{array}{l} S \to X \text{ has genus } g \\ \text{and } \partial S = \bigsqcup_i p_i \end{array}\right)$$

Generalised isoperimetric functions:

The (g,n)-isoperimetric function for X is:

$$f_{g,n}(m) = \sup\left(\mathrm{Area}_g\left(\bigsqcup_{i=1}^{n} P_i\right) : \left|\bigsqcup_{i=1}^{n} P_i\right| = m\right)$$

$$\bigsqcup_i P_i \;\; \text{-}\!\!\rangle\text{-} \;\; \mathrm{Area}_g\left(\bigsqcup P_i\right) < \infty$$

Theorem (A-Wise, 2020)

Let G be a torsion-free hyperbolic group,
(g,n)-isoperimetric functions are linear
$\forall\, g \geq 0,\ n \geq 1,$

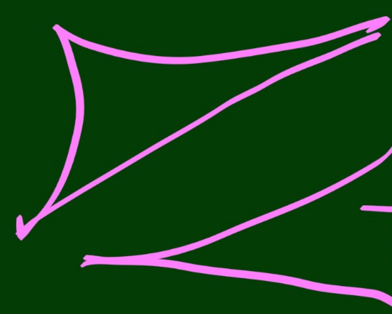
Motivating example (Gromov's trick):

- M - neg curved riemannian mfld w/ bdry
- $S \to M$ surface with bdry and $\underline{\chi(S) < 0}$
- $\partial S \to \infty$-order elements in $\pi_1 M$

$\exists$ $S' \to M$ with $\partial S = \partial S'$ and

$$\mathrm{Area}(S') \leq \underbrace{k(\chi(S) + |\partial S|)}_{\text{area is bounded by an annular isop function}}$$

Δ's depends on $\chi(S)$

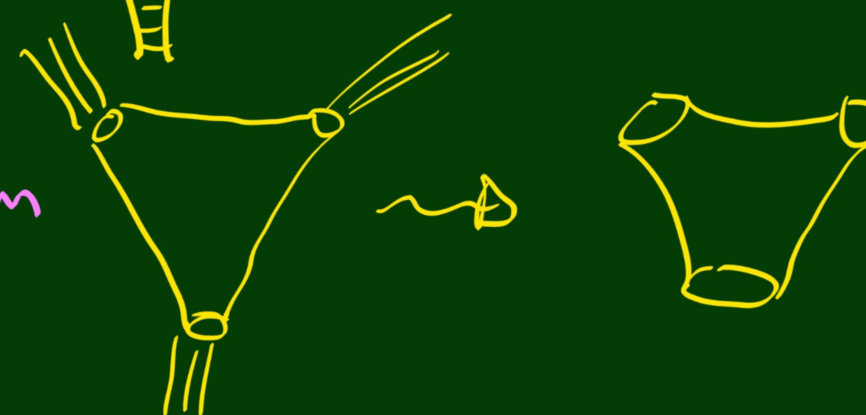


has constant area

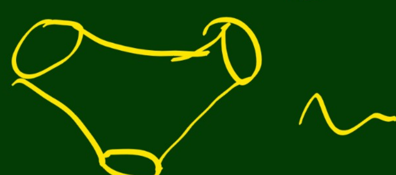
Some ingredients:

(a) Gromov's trick

(b) Trimming diagram



(c) Chopping into pieces



disc diagrams
annular diagrams

(d) Graph of spaces decomposition

that governs the combinatorics of the decomposition

Further directions:

- Compute examples – CAT(0)?

- Handle torsion – homology?

- Uniformise $\exists\ F(m) \geq fg_n(m)$

  "strict weight test" $\sim f_n(m)$