THE COHEN–LYNDON PROPERTY IN NON-METRIC SMALL-CANCELLATION

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Abstract. We show that the Cohen–Lyndon property holds for all non-metric small-cancellation quotients. This generalises the analogous result from the metric small-cancellation setting, and answers a question asked by Lyndon in 1966 and by Wall in his 1979 problem list.

1. Introduction

The goal of this paper is to prove the *Cohen–Lyndon property* for non-metric small-cancellation quotients of free groups. That is, that given a finite presentation $P = \langle S \mid R\rangle$ satisfying the $C(6)$, $C(4)-T(4)$, or $C(3)-T(6)$ condition, the normal closure $\langle\langle R\rangle\rangle$ in the free group $F(S)$ on S has a basis consisting of certain conjugates of the elements of R. This answers a question posed by Lyndon [Lyn66, pg. 222] and Wall [Wal79, Question B5].

Theorem 1.1. *Let $P = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_k\rangle$ be a $C(6)$, $C(4)-T(4)$, or $C(3)-T(6)$ presentation, and let $N(\langle r_i\rangle)$ denote the normaliser of $\langle r_i\rangle$ in $\langle s_1, \ldots, s_n\rangle$. There exist full left transversals T_i of $N(\langle r_i\rangle)\langle\langle r_1, \ldots, r_k\rangle\rangle$ such that*

$$\langle\langle r_1, \ldots, r_k\rangle\rangle = *_{i\in I, t\in T_i} \langle r_i\rangle^t.$$

The main paper contains the $C(6)$ case of Theorem 1.1 and the underlying strategy that is used also in the $C(4)-T(4)$ and $C(3)-T(6)$ cases. The appendix, written jointly with K. Duda, contains the proofs for $C(4)-T(4)$ and $C(3)-T(6)$ small-cancellation presentations.

The analogue to Theorem 1.1 in the metric $C'(\frac{1}{6})$ case is due to Cohen and Lyndon [CL63]. Theorem 1.1 recovers and generalises Cohen and Lyndon's classical result, and what is more, pushes it beyond the negatively curved setting and into the realm of non-positive curvature. Indeed, finitely presented $C'(\frac{1}{6})$ and $C(7)$ groups are hyperbolic, but while finitely presented $C(6)$, $C(4)-T(4)$, and $C(3)-T(6)$ groups are non-positively curved by most sensible combinatorial

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criteria [GS90, Wil22, Wis03], they are not in general negatively curved in any meaningful way - indeed, these groups can contain $\mathbb{Z}^2$ subgroups.

For each $n \in \mathbb{N}$, the *metric* and *non-metric* small-cancellation conditions, denoted by $C'(\frac{1}{n})$ and $C(n)$ respectively, measure the overlaps (repeated subwords) between relators in a group presentation; we formulate these conditions precisely in Definition 2.5. The smallest numbers for which the $C'(\frac{1}{n})$ and $C(n)$ conditions produce a useful theory are $n = 6$, and $n = 7$, but there are significant differences between the properties that they are known to satisfy in each case, and between the methods that can be used to approach them.

The $C'(\frac{1}{n})$ condition implies the $C(n+1)$ condition, but in general, the converse is not true: the "purely non-metric" $C(n)$ condition does not imply the $C'(\frac{1}{n'})$ condition for any choices of $n \geq 2$ and $n' \geq 2$. For instance, for $K \geq 3$, consider the presentation

$$P = \langle a, b, t \mid w_K := ta^K t^{-1} b^{K+4} \rangle.$$

This presentation satisfies the $C(6)$ condition, but does not even satisfy the $C'(\frac{1}{2})$ condition, because b^{K+3} is a piece and $|w_K| = 2(K+3)$. Similar examples of $C(n)$–not–$C'(\frac{1}{2})$ presentations can easily be produced for any $n \geq 6$.

Another striking difference between $C(n)$ groups and their metric $C'(\frac{1}{6})$ cousins is that the $C'(\frac{1}{6})$ condition implies cocompact cubulability, while the question of whether there is any n for which all $C(n)$ groups are cubulated is still open in general [Wis04].

The $C'(\frac{1}{n})$ and $C(n)$ conditions can also be coupled with the $T(n)$ *condition*, described in Definition 4.1, to get a useful theory. Particularly, the $C(4) - T(4)$ and $C(3) - T(6)$ conditions provide similar structural results to the $C(6)$ case. Examples of groups admitting $C(4) - T(4)$ small-cancellation presentations include prime alternating link groups [Wei71] and 2-dimensional right angled Artin groups. Encompassing these two classes of examples, non-positively curved square complexes satisfy the $C(4) - T(4)$ condition. In the $C(3) - T(6)$ case, a rich family of examples was constructed by Gersten and Short in [GS90]. These examples have Property (T), and are thus very different from the $C(4) - T(4)$ examples.

The Cohen–Lyndon property was introduced in [CL63], where it was shown to hold for one-relator presentations and $C'(\frac{1}{6})$ small-cancellation presentations. It was later established for certain presentations of Fuchsian groups by Zieschang in [Č66]. Informally, the Cohen-Lyndon property encodes when the relators in the presentation are "as independent as possible". When none of the relators are proper powers, it implies that the relation module $\langle\langle \mathcal{R} \rangle\rangle / \langle\langle \mathcal{R} \rangle\rangle'$ is a sum of $|R|$ cyclic modules [LS77, 10.3] and therefore immediately implies that such presentations have no relation gap, and that the corresponding presentation complexes are aspherical. The Cohen-Lyndon property was also used by Baumslag [Bau67] to obtain a partial characterisation of Hopfian one-relator groups, and by Zieschang to study automorphisms of Fuchsian groups.

We derive the Cohen–Lyndon property from a homotopical statement, in that we essentially produce am explicit homotopy equivalence between the Cayley graph associated to the small-cancellation presentation and a wedge of cycles representing the relators and their translates, ranging over left transversals as in the statement of Theorem 1.1. The main technical statement is thus Lemma 3.1. This text therefore serves two purposes: it extends the results in [CL63] to the setting of non-metric small-cancellation and it introduces a topological viewpoint to proving that the Cohen–Lyndon property holds – a viewpoint which, we hope, can be used in other settings.

Remark 1.2. While in the present paper we only concern ourselves with group presentations, we note that a version of the Cohen–Lyndon property can be defined for any pair $(G, \mathcal{H})$, or equivalently, for any quotient $G/\langle\langle \mathcal{H} \rangle\rangle$, where G is an arbitrary group and $\mathcal{H}$ is a collection of subgroups of G. It has been shown to hold in various settings (see [KS72, MCS86, EH87, DH94, Sun20]), but none of these variations can be used to deduce Theorem 1.1.

The results in the present work extend almost verbatim to the more general setting of graphical $C(6)$ quotients (see for instance [Gru15]) of free groups, by replacing the bouquet B_n associated to a generating set S in a "classical" group presentation with the defining graph in the graphical presentation. To avoid excessive technicalities, we have chosen not to structure our exposition around that version of the theory. In [Are24], we prove versions of these results for sufficiently good cubical small-cancellation quotients (in the sense of [Wis21]); we note that the graphical and classical versions of the theory are special cases of the cubical version.

1.1. **Structure and strategy:** In Section 2 we present the necessary background regarding the Cohen–Lyndon property and small-cancellation theory. In Section 3 we describe an ordering on the cycles of the Cayley graph that takes into account the structure of a certain simplicial complex associated to the presentation. A key property of this ordering is that it respects (graph-theoretical) distance with respects to a fixed "origin"; this is proven in Lemma 3.11. The main technical result is Lemma 3.12 which allows us to inductively "rebuild" the Cayley graph in a manner consistent with the ordering. These results, together with a couple more lemmas, are then assembled to deduce Theorem 1.1. In the Appendix (Section 4), we explain how to adapt the results in the previous sections to the $C(4) - T(4)$ and $C(3) - T(6)$ cases.

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2. Background

2.1. **Small-cancellation notions.** We adopt a topological, rather than combinatorial, viewpoint for defining classical small-cancellation theory. This is mostly a matter of convenience – the topological viewpoint is more suitable for the method or our proof, and generalises more naturally to other versions of the theory. We emphasise that in the classical setting, both viewpoints are equivalent.

Let $P = \langle S \mid R \rangle$ be a presentation for a group G, and let $\mathcal{X}(P)$ denote its presentation complex. This is a 2-complex that has a single vertex, an edge for each $s \in S$, and a 2-cell for each $r \in R$, so that $\pi_1\mathcal{X}(P) = G$. The Cayley graph $Cay(G, S)$ is the 1-skeleton of the universal cover $\widetilde{\mathcal{X}(P)} := \widetilde{\mathcal{X}}(P)$. The definitions below are stated in terms of arbitrary 2-complexes, but the reader may take $X = \widetilde{\mathcal{X}}(P)$ for the remainder of this section.

A map $f : X \longrightarrow Y$ between 2-complexes is *combinatorial* if it maps open cells homeomorphically to open cells. A complex is *combinatorial* if all attaching maps are combinatorial (possibly after subdividing).

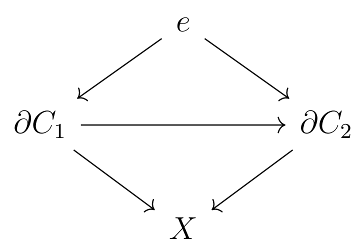
Definition 2.1 (Pieces)**.** Let X be a combinatorial 2-complex. A non-trivial combinatorial path $p \to X$ is a *piece* if there are 2-cells C_1, C_2 such that $p \to X$ factors as $p \to \partial C_1 \to X$ and $p \to \partial C_2 \to X$ but there does not exist a homeomorphism $\partial C_1 \to \partial C_2$ such that the following diagram commutes

$$\begin{array}{ccc} p & \longrightarrow & \partial C_1 \\ \downarrow & \swarrow & \downarrow \\ \partial C_2 & \longrightarrow & X \end{array}$$

Definition 2.2 (Disc diagram)**.** A *disc diagram* D is a compact contractible combinatorial 2-complex, together with an embedding $D \hookrightarrow S^2$ that induces a cell structure on S^2. Viewing the sphere as the 1-point compactification of $\mathbb{R}^2$, this cellular structure consists of the 2-cells of D together with an additional 2-cell containing the point at infinity. The *boundary path* ∂D is the attaching map of the 2-cell at infinity. A *disc diagram in a complex* X is a combinatorial map $D \to X$. The *area* of a disc diagram D is the number of 2-cells in D.

A disc diagram D might map to X in an 'ineffective' way: it might, for instance, be quite far from an immersion. Sometimes it is possible to replace a given diagram with a simpler diagram D' having the same boundary path as D, as we now explain.

Definition 2.3. A *cancellable pair* in D is a pair of 2-cells C_1, C_2 meeting along a path e such that the following diagram commutes:



A cancellable pair leads to a new disc diagram by removing $e \cup Int(C_1) \cup Int(C_2)$ and then glueing together the paths $\partial C_1 - e$ and $\partial C_2 - e$. This procedure results in a diagram D' with $\mathsf{Area}(D') = \mathsf{Area}(D) - 2$ and $\partial D' = \partial D$. A diagram is *reduced* if it has no cancellable pairs.

Definition 2.4 (Annular diagrams and collared diagrams)**.** An *annular diagram* A is a compact combinatorial 2-complex homotopy equivalent to S^1, together with an embedding $A \hookrightarrow S^2$, which induces a cellular structure on S^2. The *boundary paths* $\partial_{in} A$ and $\partial_{out} A$ of A are the attaching maps of the two 2-cells in this cellulation of S^2 that do not correspond to cells of A. An *annular diagram in a complex* X is a combinatorial map $A \to X$. A disc diagram $D \to X$ is *collared* by an annular diagram $A \to X$ if $\partial D = \partial_{in} A$.

An *arc* in a diagram is a path whose internal vertices have valence 2 and whose initial and terminal vertices have valence ≥ 2. A *boundary arc* is an arc that lies entirely in ∂D.

Definition 2.5 (Small-cancellation conditions)**.** A complex X satisfies the $C(n)$ condition if for every reduced disc diagram $D \to X$, the boundary path of each 2-cell in D either contains a non-trivial boundary arc, or is not the concatenation of less than n pieces. The complex X satisfies the $C'(\frac{1}{n})$ condition if for each 2-cell $R \to X$, and each piece $p \to X$ which factors as $p \to R \to X$, then $|p| < \frac{1}{n}|\partial R|$.

When a complex X satisfies sufficiently good small-cancellation conditions, then it is possible to classify the reduced disc diagrams $D \to X$ in terms of a few simple behaviours exhibited by their cells.

Definition 2.6 (Ladders, shells, and spurs)**.** A disc diagram L is a *ladder* if it is the union of a sequence of closed 1-cells and 2-cells $C_1, \ldots, C_n$, such that for $1 < j < n$, there are exactly two components in $L - C_j$, and exactly one component in $L - C_1$ and $L - C_n$. Moreover, if C_i is a 1-cell then it is not contained in any other C_j.

A *shell* of D is a 2-cell $C \to D$ whose boundary path $\partial C \to D$ is a concatenation $qp_1 \cdots p_k$ for some $k \leq 3$ where q is a boundary arc in D and $p_1, \ldots, p_k$ are non-trivial pieces in the interior of D. The arc q is the *outerpath* of C and the concatenation $p_1 \cdots p_k$ is the *innerpath* of C. A *spur* is a vertex of degree 1 on ∂D.

We now state the two fundamental results of small-cancellation theory. Proofs can be found in [Lyn66, MW02], for instance.

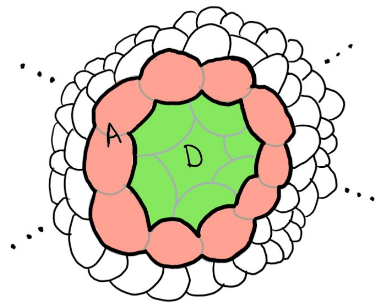


FIGURE 1. An annular diagram collaring a disc diagram in a $C(6)$ complex.

Theorem 2.7 (Greendlinger's Lemma)**.** *Let X be a $C(6)$ complex and $D \to X$ be a reduced disc diagram, then either*

(1) D is a single cell,
(2) D is a ladder,
(3) D has at least three shells and/or spurs.

Theorem 2.8 (The Ladder Theorem)**.** *Let X be a $C(6)$ complex and $D \to X$ be a reduced disc diagram. If D has exactly 2 shells or spurs, then D is a ladder.*

We say a diagram D in X has *minimal area* if it has minimal area amongst all diagrams $D' \to X$ with $\partial D = \partial D'$. Note that a minimal area disc diagram must be reduced, but that the converse is not necessarily true.

3. Main Theorem

We start with the following remark, which we state and prove in a way that is tailored exactly to our applications, and which is known to experts in other similar settings (see for instance [Wis21, 5.6+5.7]):

Lemma 3.1. *Let X be a simply-connected $C(6)$ small-cancellation complex. Let C_1, C_2 be 2-cells of X. Then either $\partial C_1 = \partial C_2$, or $C_1 \cap C_2 = \emptyset$, or $C_1 \cap C_2$ is contractible.*

Proof. Assume that $C_1 \cap C_2 \neq \emptyset$. To show that $C_1 \cap C_2$ is connected, we proceed by contradiction.

For vertices $v, v' \in X^{(1)}$, let $\mathsf{d}(v, v')$ denote the usual graph metric, i.e., $\mathsf{d}(v, v')$ is the least number of edges in a path connecting v and v' in X (note that such a path exists because X is simply connected). Let a, b be vertices of $C_1 \cap C_2$ in distinct connected components satisfying that $\mathsf{d}(a, b)$ is minimised amongst any such pairs of points. Let $\alpha \to C_1, \beta \to C_2$ be paths with endpoints a, b. Furthermore, choose α, β to be locally geodesic (so that they have no spurs) and so that the disc diagram D bounded by $\alpha\beta^{-1}$ has minimal area amongst all possible such choices. Note that if D contains either C_1 or C_2, then either $\alpha = \beta^{-1}$, in which case a, b lie in the same connected component of $C_1 \cap C_2$, contradicting the initial hypothesis, or we can find a smaller area disc diagram satisfying the conditions above and not containing the corresponding C_i by "pushing across" to expel C_i from D. Consider the disc diagram D_+ obtained by attaching C_1 and C_2 to D along α and β. Then D_+ is a ladder by Theorem 2.8. Indeed, if D_+ were not reduced, then since C_1, C_2, and D are reduced, a cancellable pair would contain one of C_1, C_2 and a 2-cell in D, but then, after performing the cancellation, we would obtain a new diagram D' with the same properties as D and smaller area, contradicting our previous choice. Now, since D was assumed to be minimal area, then Greendlinger's Lemma implies that D is either a single 0-cell, a ladder, or has at least three shells or spurs.

If D has a shell C this contradicts the $C(6)$ condition, as its outerpath is then the concatenation of at most 2 pieces (since neither C_1 and C_2 are contained in D, then the pieces are components of the intersection of C with C_1 and C_2). Thus, D is either a single vertex, in which case $a = b$, or D has exactly two spurs (in the ladder case) or at least three spurs (in the general case). If D has ≥ 3 spurs, then at least one of these lies either on α or β, contradicting that these paths are locally geodesic. Thus D has exactly 2 spurs. In this case, by removing the spurs from the diagram we find vertices a', b' in the same connected components of $C_1 \cap C_2$ as a, b but such that $\mathsf{d}(a', b') < \mathsf{d}(a, b)$, contradicting the hypothesis that a, b had been chosen to minimise the distance between points in the respective connected components of $C_1 \cap C_2$.

Since in all cases we arrive at a contradiction, then we conclude that $C_1 \cap C_2$ has a single connected component, which is either a single piece (and hence simply connected and contractible) or, up to removing backtracks, $\partial C_1 = \partial C_2$. ○

We return to the setting of group presentations. A presentation $P = \langle S \mid R \rangle$ is *symmetrised* if whenever $r \in R$, then $\bar{r} \in R$ whenever $\bar{r}$ is a cyclic permutation of r or $\bar{r} = r^{-1}$; the presentation P is *cyclically reduced* if no $r \in R$ contains a subword $w = ss^{-1}$ where $s \in S$.

A finite presentation P is a $C(n)$ *small-cancellation presentation* if it is symmetrised and cyclically reduced, and $\mathcal{X}(P)$ satisfies the $C(n)$ condition.

Mainly to establish the notation that will be used later on, we review the construction of the Cayley complex associated to a group presentation. Let $P = \langle s_1, \ldots, s_n \mid R = \{r_1, \ldots, r_k\} \rangle$ and let B_n denote a bouquet of n loops labelled by the generators $s_1, \ldots, s_n$. Recall that the *Cayley complex* $\widetilde{\mathcal{X}}(P)$ for P is the universal cover of the complex obtained by coning-off the cycles $c_1 \to B_n, \ldots, c_k \to B_n$ corresponding to the relations $r_1, \ldots, r_k$ in B_n. The subgroup $ker(F_n \to G(P)) = \langle\langle r_1, \ldots, r_k \rangle\rangle$ is associated to a regular covering space $\hat{B}_n \to B_n$, and $\widetilde{\mathcal{X}}(P)$ is obtained from $\hat{B}_n$ by coning-off the set $\{g\tilde{c}_i\}_{i \in I, g\langle\langle R \rangle\rangle \in F_n / \langle\langle R \rangle\rangle}$. In other words, $\hat{B}_n$ is the Cayley graph of $G(P)$.

Before stating our main result, we make a standard observation, which we prove for the sake of completeness.

Lemma 3.2. *Let $P = \langle s_1, \dots, s_n \mid r_1, \dots, r_k \rangle$ be a $C(6)$ small-cancellation presentation. Then each lift $g\tilde{c}_i \to \widetilde{\mathcal{X}}(P)$ of $c_i \to \mathcal{X}(P)$ embeds in $\widetilde{\mathcal{X}}(P)$.*

Proof. Assume $g\tilde{c}_i \to \widetilde{\mathcal{X}}(P)$ is not an embedding, let σ be a non-closed subpath of c_i whose induced lift $\tilde{\sigma} \to \widetilde{\mathcal{X}}(P)$ factors through $g\tilde{c}_i \to \widetilde{\mathcal{X}}(P)$ and bounds a disc diagram D, which we can assume to be of minimal area. Let $\tilde{\beta} \to \widetilde{\mathcal{X}}(P)$ be such that $\tilde{\sigma}\tilde{\beta} = g\tilde{c}_i$. Let D' be a minimal area disc diagram with boundary path $\tilde{\sigma}\tilde{\beta}$ and finally let $D'' = D \cup D'$. Since D' consists of a single 2-cell C, then D'' is reduced, and since the endpoints of σ are assumed to be distinct, Theorem 2.7 implies that D'' is either a ladder or has at least 3 spurs. Both cases lead to a contradiction, since the only shell on $\partial D''$ is C and $g\tilde{c}_i$ cannot contain any spurs, as P is cyclically reduced. ⬡

Notation 3.3. In view of Lemma 3.2, we may drop the "$\sim$" from the notation and write gc_i to denote a translate $g\tilde{c}_i$ of a cycle $\widetilde{c}_i$ in $\hat{B}_n$.

The rest of this section is dedicated to proving that the Cohen–Lyndon property holds for $C(6)$ quotients of free groups. This is Theorem 1.1 from the introduction, which we restate below.

Recall that a subset $T \subset G$ is a *full left transversal for H in G* if and only if every left coset of H contains exactly one element of T. Let $S = \{s_1, \dots, s_n\}$ and let r_i be a word on $S \cup S^{-1}$. We use the notation $N(\langle r_i \rangle)$ to denote the normaliser of $\langle r_i \rangle$ in the free group $F(S)$.

Theorem 3.4. *Let $P = \langle s_1, \dots, s_n \mid r_1, \dots, r_k \rangle$ be a $C(6)$ small-cancellation presentation. Then $(\langle s_1, \dots, s_n \rangle, \{\langle r_i \rangle\}_i)$ has the Cohen–Lyndon property. That is, there exist full left transversals T_i of $N(\langle r_i \rangle)\langle\langle r_1, \dots, r_k \rangle\rangle$ in $F(S)$ such that*

$$\langle\langle r_1, \dots, r_k \rangle\rangle = *_{i \in I, t \in T_i} \langle r_i \rangle^t.$$

We now introduce the objects and constructions derived from the presentation $P = \langle s_1, \dots, s_n | r_1, \dots, r_k \rangle$ that will be used in the proof of this theorem.

Definition 3.5 (The structure graph)**.** Define the *structure graph* Λ of the presentation complex $\mathcal{X}(P)$ as follows. The vertex set $V(\Lambda) := V$ of Λ has two types of vertices:

(1) The first type corresponds to translates gc_i of the cycles $c_1, \dots, c_k$ in $\hat{B}_n$. The set of vertices of this type will be denoted V_T.
(2) To describe the second type, we start by defining the *untethered hull* of $\hat{B}_n$. This is the subcomplex $\mathcal{F}$ of $\hat{B}_n$ consisting of all 1-cells of $\hat{B}_n$ that do not lie in any gc_i. In general, $\mathcal{F}$ is not connected. An *untethered component* is a connected component $\mathcal{F}_\iota$ of $\mathcal{F}$. The second type of vertices of Λ corresponds to the untethered components of $\mathcal{F}$. The set of vertices of this type will be denoted V_U.

Let X_v denote the subcomplex of $\hat{B}_n$ corresponding to $v \in V$. Let

$$\mathcal{U} = \{X_v : v \in V\}.$$

Note that $\mathcal{U}$ is a topological cover of $\hat{B}_n$.

The edges of Λ are also of two types: they either correspond to non-empty intersections $gc_i \cap g'c_j$, or $gc_i \cap \mathcal{F}_\iota$ where g, g' range over the left cosets of the $\langle r_i \rangle$'s, $i, j \in I$, and ι ranges over the connected components of $\mathcal{F}$. By Lemma 3.1, each intersection $gc_i \cap g'c_j$ is either a piece p or a vertex, so an edge of Λ corresponds to a piece in $\hat{B}_n$, or to a vertex that arises from either an intersection between a pair of distinct gc_i's, or from an intersection between a gc_i and an untethered component of $\mathcal{F}$ (indeed, a non-empty intersection $gc_i \cap \mathcal{F}_\iota$ must be a single vertex because $\hat{B}_n$ is the Cayley graph of $G(P)$). Note also that Λ is a simplical graph. Indeed, Λ has no bigons by construction, and no loops by Lemma 3.2.

Remark 3.6. Note that, by construction, each $\mathcal{F}_\iota$ is a tree, and hence contractible. Lemma 3.1 shows that each intersection $gc_i \cap g'c_j$ is also contractible.

There is a Helly property for the elements of $\mathcal{U}$. It can be proven using induction and Greendlinger's Lemma, and can also be found in [OP18, 6.11]:

Lemma 3.7. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(6)$ small-cancellation presentation P, let Λ be its structure graph, and let $V' \subset V(\Lambda)$ be finite. If each pairwise intersection $X_v \cap X_v'$ with $v, v' \in V'$ is non-empty, then the total intersection $\bigcap_{v \in V'} X_v$ is non-empty.*

The following lemma, a version of which is proven in [OP18, 6.12], provides finer information about the total intersections in Lemma 3.7. We include a proof since it is illustrative of the methods of this paper.

Lemma 3.8. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(6)$ small-cancellation presentation P and let Λ be its structure graph. For all $I \subset V(\Lambda)$ with $\infty > |I| \geq 2$, the intersection $\bigcap_{v \in I} X_v$ is non-empty if and only if it is connected (and hence contractible).*

Proof. We proceed by induction on $|I|$. The base case is $|I| = 2$ and follows from Lemma 3.1. Assume that the claim holds for intersections $\bigcap_{v \in I} X_v \neq \emptyset$ where $|I| \leq k$, and consider $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v$. If this intersection is empty, then there is nothing to show. So we may assume that $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v \neq \emptyset$, and thus contains a vertex b of $\hat{B}_n$. The induction hypothesis implies that each intersection $\bigcap_{v \in J} X_{v_j}$ with $J \subsetneq I$ and $I = J \cup \{v_j\}$ is contractible, and is in fact a tree as each X_v is 1-dimensional. As $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v \subset \bigcap_{v \in I} X_v$, it suffices to show that $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v$ is connected. To this end, let x, y be vertices in $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v$.

Consider paths $\beta \to X_{v_{k+1}}$ and $\alpha \to \bigcap_{v \in I} X_v$ with endpoints x, y, so that $\beta\alpha^{-1}$ bounds a disc diagram D in $\widetilde{\mathcal{X}}(P)$. Moreover, amongst all pairs of paths with endpoints x, y as above, choose β and α to have minimal length. Consider the disc diagram $D^+ = D \cup_\beta X_{v_{k+1}} \cup_\alpha X_v$, where $v \in I$. We claim that D^+ is a degenerate disc diagram, that is, it contains no 2-cells, so $\alpha = \beta$ and x and y lie in the same connected component of $\bigcup_{v \in I \cup \{v_{k+1}\}} X_v$. To prove this assertion, note that the choices of α and β above imply that ∂D cannot contain any spurs, as such spurs could be removed to shorten α and/or β. Thus D must contain a shell S, and the outerpath of S is either a subpath of α, a subpath of β, or contains either x or y. In the first and second cases, $\partial S \cap \partial D$ is a single piece, and the innerpath of S is the concatenation of at most 3 pieces, so ∂S is the concatenation of at most 4 pieces, contradicting the $C(6)$ condition. In the third case, similarly, $\partial S \cap \partial D$ is the concatenation of at most 2 pieces, so the innerpath of S is the concatenation of at most 3 pieces, and ∂S is the concatenation of at most 5 pieces, again contradicting the $C(6)$ condition.

Thus, as asserted, D must be a degenerate disc diagram, so $\bigcap_{v \in I \cup \{v_{k+1}\}} X_v$ is connected and contractible and the induction is complete. ⬡

As noted in Definition 3.5, the set $\mathcal{U} = \{X_v : v \in V\}$ provides a topological cover of $\hat{B}_n$. The structure graph Λ, being simplicial by construction and by Lemma 3.2, is the 1-skeleton of the geometric realisation of the nerve complex of $\mathcal{U}$.

Definition 3.9 (The nerve of $\mathcal{U}$)**.** The *nerve complex* $\mathbf{N}(\mathcal{U})$ of a topological covering $\mathcal{U}$ is the abstract simplicial complex

$$\mathbf{N}(\mathcal{U}) = \{V' \subset V : \bigcap_{v \in V'} X_v \neq \emptyset, |V'| < \infty\}.$$

It is natural to use $\mathbf{N}(\mathcal{U})$ to organise the data of $\mathcal{U}$. Concretely, we do this by defining an ordering on the vertices of $\mathbf{N}(\mathcal{U})$; this ordering is key to the proof of Theorem 3.4.

Definition 3.10 (The ordering on $\mathcal{U}$)**.** Choose $v_0 \in V(\Lambda)$ corresponding to a gc_i. We define a total ordering $\leq$ on $\mathcal{U}$, that is, an injective function $\varphi : V \to \mathbb{N}$. To do this, we first define an ordering on the simplices of $\mathbf{N}(\mathcal{U})$ inductively as follows.

Start by setting $\varphi(v_0) = 0$, and define

$$A_0 = \{u \in V \ : \{u, v_0\} \in \mathbf{N}(\mathcal{U})\} \cup \{v_0\}.$$

Choose $u_1 \in A_1$, let $\varphi(u_1) = 1$, and let

$$A_{01} = \{u \in V : \{v_0, u_1, u\} \in \mathbf{N}(\mathcal{U})\} \cup \{v_0, u_1\}.$$

Inductively, assume that φ has been defined for a subset of cardinality k, so $\varphi(v_0) = 0, \dots, \varphi(v_k) = k$. For each non-empty simplex $\{v_i, \dots, v_\ell\}$ of $\mathbf{N}(\mathcal{U})$ where $\varphi(v_i), \dots, \varphi(v_\ell)$ are already defined, let

$$A_{v_i \dots v_\ell} = \{u \in V : \{v_i, \dots, v_\ell\} \cup \{u\} \in \mathbf{N}(\mathcal{U})\} \cup \{v_i \dots v_\ell\}.$$

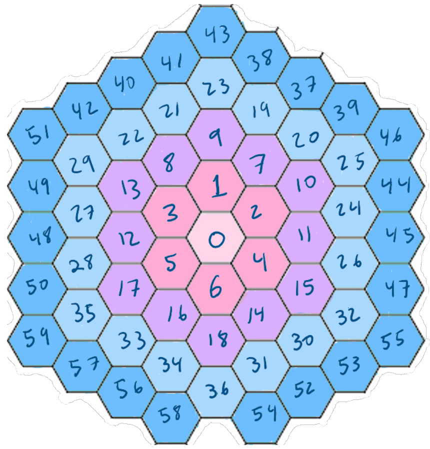


FIGURE 2. The ordering in Definition 3.10 for a portion of a hexagonal grid.

We view each simplex $\{v_i, \ldots, v_\ell\}$ as an ordered tuple $(v_i, \ldots, v_\ell)$ where $\varphi(v_i) < \varphi(v_{i+1}) < \ldots < \varphi(v_\ell)$, and order the simplices using the *Lusin–Sierpiński order*[1], which is defined as follows. For a pair of simplices, set $\{v_i, \ldots, v_\ell\} < \{w_{i'}, \ldots, w_{\ell'}\}$ if either

(1) there exists $j \leq \min\{\ell, \ell'\}$ with $v_\iota = w_\iota$ for all $\iota < j$, and $v_j < w_j$, or
(2) $\ell > \ell'$ and $j = j'$ for all $j' \leq \ell'$.

Now consider a least simplex $\{v_i \ldots v_\ell\}$ such that there exists $u \in A_{v_i \ldots v_\ell}$ whose image is not yet defined and choose such a vertex u arbitrarily. Set $\varphi(u) = k + 1$. In Figure 2 we illustrate the ordering for a simple example.

A priori, φ is only defined for a subset $V' \subset V$; we show in Lemma 3.11 that in fact $V' = V$.

Recall that, for vertices $v, v' \in V$, $\mathsf{d}(v, v')$ denotes the usual graph metric.

Lemma 3.11. *Let $v, v' \in V$. If $\varphi(v) < \varphi(v')$, then $\mathsf{d}(v, v_0) \leq \mathsf{d}(v', v_0)$. In particular, the function $\varphi : V \to \mathbb{N}$ is well-defined.*

Proof. We prove the lemma by induction on $N = \varphi(v')$. If $N = 1$ there is nothing to show, so assume that the result holds for all pairs of vertices with image $< N_0$.

Let $v, v' \in V$ and assume $\varphi(v) < \varphi(v') \leq N_0$. Note that if $\varphi(v)$ and $\varphi(v')$ are not consecutive integers, then there exists $u \in V$ with $\varphi(v) < \varphi(u) = \varphi(v') - 1$, and by the induction hypothesis $\mathsf{d}(v, v_0) \leq \mathsf{d}(u, v_0)$, so it would suffice in any case to show that $\mathsf{d}(u, v_0) \leq \mathsf{d}(v', v_0)$ to prove the lemma. Thus, we assume that $\varphi(v)$ and

[1] Also known as the *Kleene–Brouwer order*. Perhaps it would be more suitable to call it the *long-lex order*, in analogy with the *short-lex order*, which is used frequently in geometric group theory.

$\varphi(v')$ are consecutive integers. By definition, there is a simplex σ which is least in the Lusin–Sierpiński order described above and such that $\{v'\} \cup \sigma$ is a simplex.

We first note that for any vertex u adjacent to v' and for any vertex w' in σ with $\varphi(w') < N_0$, if $\mathsf{d}(w', v_0) > \mathsf{d}(u, v_0)$, then by the induction hypothesis $\varphi(w') > \varphi(u)$. In particular, this holds for the least vertex w' of σ, so $\mathsf{d}(u, v_0) \geq \mathsf{d}(w, v_0)$ for every vertex u adjacent to v', or otherwise this would contradict that σ is the least simplex adjacent to v'. Thus, the distance $\mathsf{d}(v', v_0)$ is realised by a path passing through σ, and in particular, passing through its least vertex w'. There are now several cases to consider:

(1) If v is a vertex of σ, then v is adjacent to v', and by the discussion on the previous paragraph, $\mathsf{d}(v, v_0) \leq \mathsf{d}(v', v_0)$,

(2) if v is not a vertex of σ, then either

  (a) $\{v\} \cup \sigma$ is also a simplex, in which case, as claimed:

$$\mathsf{d}(v, v_0) = \mathsf{d}(v, w') + \mathsf{d}(w', v_0) = 1 + \mathsf{d}(w', v_0) = \mathsf{d}(v', w') + \mathsf{d}(w', v_0) = \mathsf{d}(v', v_0),$$

  (b) or $\varphi(w') < \varphi(v)$. Since $\varphi(v) < \varphi(v')$, the least vertex, say w, adjacent to v must satisfy that $\varphi(w) < \varphi(w')$ and in particular that $\varphi(w) < N_0$. Thus $\mathsf{d}(w, v_0) \leq \mathsf{d}(w', v_0)$ by the induction hypothesis applied to w' and w, so

$$\mathsf{d}(v, v_0) = 1 + \mathsf{d}(w, v_0) \leq 1 + \mathsf{d}(w', v_0) = \mathsf{d}(v', v_0).$$

And the induction is complete. ○

Write $V = \{v_j\}_{j \in \mathbb{N}}$, where the indexing agrees with the ordering in Definition 3.10, and recall that X_{v_j} denotes the subcomplex of $\hat{B}_n$ corresponding to $v_j \in V$. The ordering on V induces an ordering on $\mathcal{U}$, so $\varphi(v) < \varphi(v')$ if and only if $X_v < X_{v'}$. Thus, for ease of notation, in what follows we refer to the ordering of V and the ordering of $\mathcal{U}$ interchangeably.

The next lemma is our main technical result.

Lemma 3.12. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(6)$ small-cancellation presentation P, let $V = \{v_j\}_{j \in \mathbb{N}}$ be the vertices of its structure graph Λ, and for each $v_j \in V$, let X_{v_j} be the corresponding element of $\mathcal{U}$. For each $k \in \mathbb{N}$, the intersection*

$$\bigcup_{j<k} X_{v_j} \cap X_{v_k}$$

is contractible.

Proof. The proof is by induction on k. When $k = 1$ there is nothing to show, and the case of $k = 2$ holds because each intersection $X_v \cap X_{v'}$ is either a piece or a single vertex.

Assume that $\bigcup_{j<k} X_{v_j} \cap X_{v_k}$ is contractible for all $k < k_0$. We now show that $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is contractible.

Suppose towards a contradiction that $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is not contractible. Then, since this intersection is 1-dimensional, it is either disconnected, or there is a non-nullhomotopic loop $\sigma \to \bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$. We first note that the second case can be reduced to the first, and is thus precluded by the induction hypothesis.

Non-simply-connected intersection: If $\sigma \to \bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is essential, then $\sigma \to \bigcup_{j<k_0} X_{v_j}$ is also essential, since $\hat{B}_n$ is a graph. The image of σ in $\bigcup_{j<k_0} X_{v_j}$ is covered by a collection of arcs in the X_{v_j}'s. By the Seifert Van-Kampen Theorem, the image of σ cannot be covered by two contractible sets with connected intersection, so in particular, viewing σ as a concatenation of two arcs τ' and τ'' where τ' traverses a single $X_{v_{j_0}}$ with $1 \leq j_0 < k_0$ and τ'' traverses $\bigcup_{j<k_0, j\neq j_0} X_{v_j}$, it follows that the intersection $\bigcup_{j<k_0, j\neq j_0} X_{v_j} \cap X_{v_{j_0}}$ must be disconnected, contradicting the induction hypothesis since $j_0 < k_0$.

Disconnected intersection: Assume now that there exist vertices x, y lying in distinct connected components of $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$, and let $X_v, X_{v'}$ be the corresponding elements of $\mathcal{U}$ containing x and y. Note that x and y are connected by a path τ in $\bigcup_{j\leq k} X_{v_j}$, since by the induction hypothesis, this union is path-connected. Let I_τ be such that $\bigcup_{I_\tau} X_{v_j}$ are the elements of $\mathcal{U}$ traversed by τ, so $\bigcup_{I_\tau} X_{v_j} \cup X_{v_{k_0}}$ is a union of boundaries of 2-cells in $\widetilde{\mathcal{X}}(P)$ defining an annular diagram A_τ whose boundary paths are cycles in $\bigcup_{I_\tau} X_{v_j} \cup X_{v_{k_0}}$, and A_τ collars a reduced disc diagram D_τ in $\widetilde{\mathcal{X}}(P)$, as in Figure 1. Amongst all possible paths satisfying the above, choose τ so that $|I_\tau|$ is the least possible, and choose D_τ to have the least number of cells amongst all disc diagrams collared by A_τ.

If $\mathsf{Area}(D_\tau) = 0$, then D_τ is a tree. Note that if D_τ has branching, then removing an edge (or several edges corresponding to a piece) from D_τ corresponds to shortening τ by pushing it away from a pair $X_{v_j}, X_{v_{j'}}$ and towards $X_{v_{k_0}}$, as in Figure 3; such a reduction contradicts the choices in the previous paragraph, so we may assume that D_τ is a possibly degenerate subpath of $X_{v_{k_0}}$ with $\partial D_\tau = \tau\tau^{-1}$. Since $\tau \subset \bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$, this contradicts the hypothesis that x, y lie in distinct connected components of $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$.

Hence, $\mathsf{Area}(D_\tau) \geq 1$. We may assume, by performing the same reductions to τ as in the $\mathsf{Area}(D_\tau) = 0$ case, that ∂D_τ has no spurs, so by Greendlinger's Lemma D_τ must have at least one shell; we claim that $\partial S < X_{v_{k_0}}$ for every shell S in D_τ. Assuming the claim (see Claim 3.13, and proven below), we now explain how to finish the proof of Lemma 3.12. Let S be a shell in D_τ. Since S is a shell, it intersects at least 3 consecutive cells C_1, C_2, C_3 of A_τ as in the centre of Figure 5, and so the path τ' obtained from τ by pushing across C_2 traverses at most as many cells as τ, but bounds a disc diagram $D_{\tau'}$ with $\mathsf{Area}(D_{\tau'}) < \mathsf{Area}(D_\tau)$, contradicting our initial choices.

Thus, assuming the claim, we arrive at a contradiction in all cases, so $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ must be contractible and the induction is complete. ○

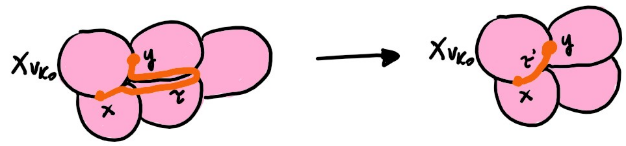


FIGURE 3. Possible reductions in the last part of the proof of Lemma 3.12 when $\mathsf{Area}(D_\tau) = 0$ and D_τ has branching.

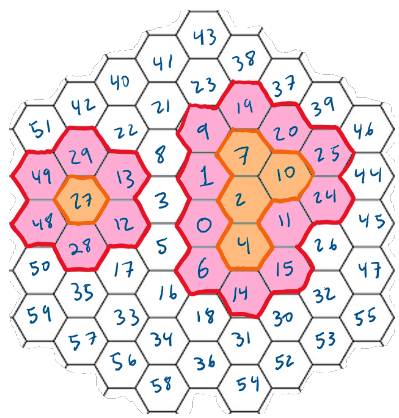


FIGURE 4. Some annuli collaring disc diagrams in the example from Figure 2, and exhibiting the behaviour explained in Claim 3.13.

Claim 3.13. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(6)$ small-cancellation presentation P, let $V = \{v_j\}_{j\in\mathbb{N}}$ be the vertices of its structure graph Λ, and for each $v_j \in V$, let X_{v_j} be the corresponding element of $\mathcal{U}$. Let A_β be an annular diagram in $\widetilde{\mathcal{X}(P)}$ collaring a reduced disc diagram D_β so that $\partial D_\beta = D_\beta \cap A_\beta = \beta$, and finally, let $X_{v_{k_0}}$ be the maximal element in the ordering of Definition 3.10 corresponding to the boundary of a 2-cell in A_β. Then $\partial S < X_{v_{k_0}}$ for every shell S in D_β.*

Proof of Claim. We prove the claim by induction on $\mathsf{Area}(D_\beta)$. For the base case, assume $\mathsf{Area}(D_\beta) = 1$, so $D_\beta = S$ is a single cell, and in fact, a shell. Consider the union $\bigcup_{j<k_0} X_{v_j}$. Since $X_{v_{k_0}}$ is the next element in the ordering, then $X_{v_{k_0}}$ lies in a simplex σ that contains a simplex σ' which is least in the Lusin–Sierpiński ordering, and there is a vertex in $\sigma - \sigma'$, namely v_{k_0}, that is not in $\bigcup_{j<k_0} X_{v_j}$. Let $X_{v_{j_1}} < \ldots < X_{v_{j_m}} < X_{v_{k_0}}$ be the boundaries of the 2-cells in A_β. If all of the vertices $v_{j_1}, \ldots, v_{j_m}$ lie in σ, then, by Lemma 3.7 and Lemma 3.8, $\bigcap_{v\in\sigma} X_v$ is contractible, so a boundary component of A_β, say $\partial_{out} A_\beta$, bounds a reduced disc diagram E in $\widetilde{\mathcal{X}(P)}$ such that the boundaries of the 2-cells in E lie in $\bigcup_{v\in\sigma} X_v$, and either $\partial S = \partial E$, implying that E is a single cell[2] and $\partial S < X_{v_{k_0}}$, or the small-cancellation condition is violated. To see that the first assertion is true, note that if $\partial S = \partial E$, then if E is not a single cell, it must contain a shell S' by Greendlinger's Lemma, but the outerpath of S' is its intersection with S, and therefore it is a single piece, contradicting that the outerpath of a shell must be the concatenation of at least 3 pieces. To see that the second assertion holds, consider the union $E \cup A_\beta$, where E is glued to A_β along its boundary path.

If $\partial S \neq \partial E$, then in particular $S \neq E$. Note that E may be assumed to be non-trivial and to have no spurs, since any spurs could be pushed out of E without changing the cellular structure of $E \cup A_\beta$. Now $\partial S \to S$ factors through $E \cup A_\beta$ by construction, and $E \cup A_\beta$ is simply-connected, so ∂S bounds a minimal area disc diagram $D_\cup$ in $E \cup A_\beta$. By Greendlinger's Lemma, $D_\cup$ has a shell S', but the outerpath of S' is the intersection $S' \cap S$, which is a single piece by Lemma 3.1, thus contradicting the $C(6)$ condition.

We may thus assume that there exists a cell C of A_β with $\partial C = X_{v_{j_i}}$ for some $1 \leq i \leq m$ such that v_{j_i} does not lie on σ. Consider the longest chain of cells of A_β in σ and let j_i be the first index not corresponding to such a $X_{v_{j_i}}$. If $\partial S > X_{v_{k_0}}$, then since S is adjacent to $X_{v_{j_i}}$ and $X_{v_{j_i}} < X_{v_{k_0}}$, then σ must contain a vertex v_n with $X_{v_n} < X_{v_{j_i}}$. But, by hypothesis, $X_{v_{j_i}} < \ldots < X_{v_{j_m}} < X_{v_{k_0}}$, so there exists a simplex θ satisfying that $\theta \cup \{v_\ell\}$ is a simplex for each $\ell \in \{n, j_i, \ldots, j_m\}$. Let $\theta^+ := \theta \cup \{v_n, v_{j_i}, \ldots, v_{j_m}\}$. As before, Lemma 3.1, Lemma 3.7, and Lemma 3.8 imply that $\bigcap_{v \in \sigma \cup \theta^+} X_v$ is contractible, so A_β bounds a disc diagram E in $\widetilde{\mathcal{X}(P)}$ such that the boundaries of the 2-cells in E lie in $\bigcup_{v \in \sigma \cup \theta^+} X_v$, and arguing as in the previous paragraph, either $\partial S = \partial E$, implying that $\partial S < X_{v_{k_0}}$ or the small-cancellation condition is again violated. Thus, the base case of the induction is complete.

Assuming the claim when $\mathsf{Area}(D_\beta) < N$, let D_β be a disc diagram as hypothesised and with $\mathsf{Area}(D_\beta) = N$. By Greendlinger's Lemma, D_β is either a single cell, a ladder, or has at least 3 shells or spurs. Since we may assume that $N > 1$, and any spurs on ∂D_β can be removed to reduce the number of cells in D_β, we may assume that D_β has at least 2 shells. Let $S_1, \ldots, S_n$ be the shells of D_β. Consider the diagram $D_{\beta'}$ obtained from D_β by removing S_1, so $D_{\beta'}$ is collared by the annular diagram $A_{\beta'}$ obtained from A_β by removing the cells in the outerpath of ∂S and adding S_1. Every shell of $D_\beta - S_1$ is still a shell of $D_{\beta'}$, so by the induction hypothesis, $\partial S_i < X_{v_{M'}}$ for each shell S_i with $1 < i \leq n$, where $X_{v_{M'}}$ is the maximal element in the ordering corresponding to the boundary of a 2-cell in $D_{\beta'}$.

[2]We warn the reader that, a priori, we cannot conclude from this observation that $S = E$. Indeed, if a relator r in the presentation P is a proper power, then E and S could be different lifts of the same disc – the disc corresponding to r – in $\mathcal{X}(P)$.

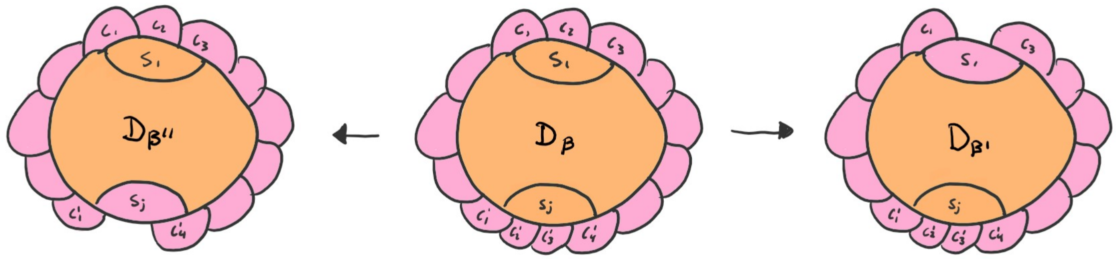


FIGURE 5. The reductions used in the final part of the proof of Lemma 3.12 (replace β with τ in the labelling) and in the inductive step of the proof of Claim 3.13.

It may be, a priori, that $X_{v_{M'}} = \partial S_1$; we explain now why this isn't the case. Repeating the same construction as above to obtain a disc diagram $D_{\beta''}$ collared by an annular diagram $A_{\beta''}$, but this time removing a shell $S_j \neq S_1$, the induction hypothesis again implies that $\partial S_i < X_{v_{M''}}$ for each shell S_i with $1 \leq i \leq n$ and $i \neq j$ and where $X_{v_{M''}}$ is defined analogously to $X_{v_{M'}}$, i.e., it is the maximal element in the ordering corresponding to the boundary of a 2-cell in $D_{\beta''}$. If $X_{v_{M''}} = S_j$, then $\partial S_j < \partial S_1$ and $\partial S_j > \partial S_1$, which is impossible. Thus, either $X_{v_{M'}} \neq \partial S_1$ or $X_{v_{M''}} \neq \partial S_j$, and since X_{v_M} is the boundary of a 2-cell of either $A_{\beta'}$ or $A_{\beta''}$ – as it can only be excluded when removing one of the shells –, then $\partial S_i < X_{v_{k_0}}$ for each shell S_i with $1 \leq i \leq n$, and the induction is complete. ⬡

We therefore conclude:

Proof of Theorem 3.4. By Lemma 3.1, for each $k \in \mathbb{N}$, the intersection $\bigcup_{j<k} X_{v_j} \cap X_{v_k}$ is contractible. Abusing notation, let e denote the basepoint of $\hat{B}_n$ (which is also the vertex representing the trivial element). We prove, by induction on k, that this implies that there exist left coset representatives $\{g_{v_1} \dots, g_{v_k}\}$ for the left cosets of $N(\langle r_i \rangle)\langle\langle r_1, \dots, r_k \rangle\rangle$ in $F(S)$ corresponding to the X_{v_j} with $j \leq k$ such that

$$\pi_1(\bigcup_{j \leq k} X_{v_j}, e) = \langle \{r_i^{g_{v_j}}\} \rangle. \tag{1}$$

For $k = 1$, this is immediate since X_{v_j} is a lift of some $r_i \in R$ based at e of $\hat{B}_n$. We may thus take $g_{v_1} = e$ and the assertion follows.

Assume that (1) holds for all $k \leq K$, and consider $\bigcup_{j \leq K} X_{v_j} \cup X_{v_{K+1}}$, where $X_{v_{K+1}}$ is a lift to $\hat{B}_n$ of the cycle c_{i_0} representing $r_{i_0} \in R$. Let p be a point in the intersection $\bigcup_{j \leq K} X_{v_j} \cap X_{v_{K+1}}$. Since this intersection is contractible, then $\pi_1(\bigcup_{j \leq K} X_{v_j} \cup X_{v_{K+1}}, p) = \pi_1(\bigcup_{j \leq K} X_{v_j}, p) * \pi_1(X_{v_{K+1}}, p)$. Let $g\langle\langle R \rangle\rangle$ denote the left coset corresponding to translating c_{i_0} to $X_{v_{K+1}}$. Since change of basepoint corresponds to conjugation, there is a $g_{v_{K+1}} \in g\langle\langle R \rangle\rangle$ such that

$$\pi_1(\bigcup_{j\leq K+1} X_{v_j}, e) = \langle \{r_i^{g_{v_j}}\}_{i\in I, j\leq K} \cup \{r_{i_0}^{g_{v_{K+1}}}\}\rangle,$$

and the induction is complete.

Now since $\hat{B}_n$ is the infinite directed union $\hat{B}_n = \bigcup_{j\leq k, k\to\infty} X_{v_k}$, then

$$\pi_1(\hat{B}_n, e) = \lim_{k\to\infty} \pi_1(\bigcup_{j\leq k} X_{v_k}, e),$$

which finishes the proof of the theorem. ⬡

4. Appendix: the C(4)-T(4) and C(3)-T(6) cases by Macarena Arenas and Karol Duda

Definition 4.1. Let q be a natural number. We say that a 2-complex X satisfies the $T(q)$ *small cancellation condition* if there does not exist a reduced disc diagram $D \to X$ containing an internal vertex v of valence n where $2 < n < q$.

We retain most of the notational conventions used in the main body of the paper. One of the differences is in the definition of a shell.

Definition 4.2 (i-shells)**.** An i-*shell* of D is a 2-cell $C \to D$ whose boundary path $\partial C \to D$ is a concatenation $qp_1 \cdots p_i$ for some $i \geq 1$ where q is a boundary arc in D and $p_1, \ldots, p_i$ are non-trivial pieces in the interior of D. The arc q is the *outerpath* of C and the concatenation $p_1 \cdots p_i$ is the *innerpath* of C.

4.1. **The C(4)-T(4) case.**

With that change both Greendlinger's Lemma and the Ladder Theorem hold also for $C(4)$–$T(4)$ complexes.

Theorem 4.3 (Greendlinger's Lemma)**.** *Let X be a $C(4)$–$T(4)$ complex and $D \to X$ be a reduced disc diagram, then either*

(1) D is a single cell,
(2) D is a ladder,
(3) D has at least three 1-shells, 2-shells and/or spurs.

Theorem 4.4 (The Ladder Theorem)**.** *Let X be a $C(4)$–$T(4)$ complex and $D \to X$ be a reduced disc diagram. If D has exactly two 1-shells or spurs, then D is a ladder.*

Moreover, for the proof of the analogue of Lemma 3.12 we need a slightly more refined version of Greendlinger's Lemma, giving us additional information about the shells.

Theorem 4.5. *Let X be a $C(4)$–$T(4)$ complex and $D \to X$ be a reduced disc diagram. If D does not contain spurs, then it is either a single cell, or contains a 1-shell, or contains a 2-shell in D containing a boundary vertex of D of degree exactly 3.*

Proof. Assume that $area(D) \geq 2$. Consider a complex D' obtained from D by removing all vertices of degree 2, with the exception of two on the outerpath of every 1-shell and a single one on the outerpath of every 2-shell. Since D' is a disc diagram, the Euler characteristic $\chi(D')$ is equal to 1. We distribute the Euler characteristic from each edge as $\frac{-1}{2}$ to both of it ends, and from each cell $\frac{1}{p}$ to each of its p boundary vertices. It is clear that the sum of the distributed Euler characteristic over all vertices is 1. Let v be an internal vertex of D'. Since D is reduced and satisfies $T(4)$, the degree of v is $n \geq 4$. The number of cells containing v is also n. Thus, the distributed Euler characteristic for v is at most $1 - \frac{n}{2} + \frac{n}{4}$ which for $n \geq 4$ is non-positive. Let u be a boundary vertex of D'. The degree of u is $n \geq 2$ and it is greater by at least 1 than number of cells containing u. Thus, the distributed Euler characteristic for u is at most $1 - \frac{n}{2} + \frac{n-1}{4}$, where n is degree of u. The only positive value it can have is $\frac{1}{4}$ which happens exactly when $n = 2$ and u is contained in a cell. This happens for boundary vertices of degree two in 1-shells and 2-shells. Since the sum is 1 and all other values are non-positive, there are multiple cells that are either a 1-shell or a 2-shell in D. Moreover observe that if all of these cells are 2-shells, then some of them have a boundary vertex of degree exactly 3. Indeed, otherwise each of these 2-shells has two boundary vertices of degree at least 4. The distributed Euler characteristic of these vertices is at most $1 - \frac{n}{2} + \frac{n-1}{4}$ which is at most $\frac{-1}{4}$. Since each such vertex can be shared by at most two 2-shells, the sum of the distributed Euler characteristic is at most 0, a contradiction. ☐

Note that Lemma 3.1 is true for the $C(4)$–$T(4)$ condition as well.

Lemma 4.6. *Let X be a simply-connected $C(4)$–$T(4)$ small-cancellation complex. Let C_1, C_2 be 2-cells of X. Then either $\partial C_1 = \partial C_2$, or $C_1 \cap C_2 = \emptyset$, or $C_1 \cap C_2$ is contractible.*

Analogues of Lemmas 3.7 and 3.8 exist for $C(4)$–$T(4)$ complexes. The Helly property for triple intersections can be proved using induction and Greendlinger's Lemma, and can also be found in [Hod20, Proposition 3.7]. Note that this result combined with Lemma 4.6 is enough to run the proof in [OP18]. While this proof is phrased for the $C(6)$ case, it only uses the fact that any counterexample to the Helly Property would be a disc diagram with boundary consisting of at most 3 pieces, which contradicts the $C(4)$ condition as well. We thus obtain:

Lemma 4.7. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(4)$–$T(4)$ small-cancellation presentation P, let Λ be its structure graph, and let $V' \subset V(\Lambda)$ with $|V'| < \infty$. If each pairwise intersection $X_v \cap X_v'$ with $v, v' \in V'$ is non-empty, then the total intersection $\bigcap_{v \in V'} X_v$ is non-empty.*

We can improve the conclusion of Lemma 4.7 to deduce contractibility of the total intersection, as in the $C(6)$ case. A version of this result follows from putting together Propositions 3.5 to 3.8 in [Hod20].

Lemma 4.8. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(4)$–$T(4)$ small-cancellation presentation P and let Λ be its structure graph. For all $I \subset V(\Lambda)$ with $\infty > |I| \geq 2$, the intersection $\bigcap_{v\in I} X_v$ is non-empty if and only if it is connected (and hence contractible).*

The proof of Lemma 4.8 follows the proof of Lemma 3.8. The difference is in the proof of the fact that D^+ is a degenerate disc diagram.

Like in the previous proof, note that the choices of α and β imply that ∂D cannot contain any spurs, as such spurs could be removed to shorten α and/or β. Thus by Greendlinger's Lemma D must contain either a 1-shell S_1, or at least three 2-shells S_2, S_3, S_4. The outerpath of either of them is either a subpath of α, a subpath of β, or contains either x or y as an internal vertex of the path. In the first and second cases, $\partial S_i \cap \partial D$ is a single piece, and the innerpath of S_i is the concatenation of at most 2 pieces, so ∂S is the concatenation of at most 3 pieces, contradicting the $C(4)$ condition. In the third case, $\partial S_i \cap \partial D$ is the concatenation of at most 2 pieces. In the case of a 1-shell S_1 the innerpath consist of exactly a single piece, so ∂S_1 is the concatenation of at most 3 pieces, contradicting the $C(4)$ condition. A 2-shell containing either x or y can exist without contradicting $C(4)$, but then there are at least three 2-shells S_1, S_2, S_3 in D, while only two of them can contain either x or y as an internal vertex. Thus there is at least one 2-shell with an outerpath being either a subpath of α, a subpath of β, yielding again a contradiction.

We use the results collected above to deduce an analogue of Claim 3.13.

Claim 4.9. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(4)$–$T(4)$ small-cancellation presentation P, let $V = \{v_j\}_{j\in\mathbb{N}}$ be the vertices of its structure graph Λ, and for each $v_j \in V$, let X_{v_j} be the corresponding element of $\mathcal{U}$. Let A_β be an annular diagram in $\widetilde{\mathcal{X}(P)}$ collaring a reduced disc diagram D_β so that $\partial D_\beta = D_\beta \cap A_\beta = \beta$, and finally, let $X_{v_{k_0}}$ be the maximal element in the ordering of Definition 3.10 corresponding to the boundary of a 2-cell in A_β. Then $\partial S < X_{v_{k_0}}$ for every shell S in D_β.*

Proof. The proof is basically identical to that of Claim 3.13, except that we substitute Lemma 3.1, Lemma 3.7, and Lemma 3.8 with their $C(4) - T(4)$ analogues – Lemma 4.6, Lemma 4.7, and Lemma 4.8. Note that the key small-cancellation point of the argument is that in a reduced disc diagram $D \to \mathcal{X}(P)$, the outerpath of a shell cannot be a single piece, and this is also true for 1-shells and 2-shells in the present case, as well for the case of D being a single cell. □

Finally, the $C(4) - T(4)$ version of Theorem 1.1 follows verbatim from the proof of Theorem 3.4 once we have the analogue of Lemma 3.12.

Lemma 4.10. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(4)$–$T(4)$ small-cancellation presentation P, let $V = \{v_j\}_{j\in\mathbb{N}}$ be the vertices of its structure* *graph* Λ, *and for each* $v_j \in V$, *let* X_{v_j} *be the corresponding element of* $\mathcal{U}$. *For each* $k \in \mathbb{N}$, *the intersection*

$$\bigcup_{j<k} X_{v_j} \cap X_{v_k}$$

is contractible.

Proof. The proof is by induction on k, and it follows closely the proof of Lemma 3.12. As in that proof, for the inductive step we assume that $\bigcup_{j<k} X_{v_j} \cap X_{v_k}$ is contractible for all $k < k_0$, and we need to show that $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is contractible. We assume towards a contradiction that it is not, so $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is either disconnected, or connected but not simply-connected.

The non-simply-connected case is precluded by the induction hypothesis by the Seifert Van-Kampen Theorem (note that the argument given in Lemma 3.12 does not use small-cancellation in any explicit way).

Now if the intersection $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ is disconnected, again we follow the strategy of the proof in the $C(6)$ case and, retaining the notation in that proof, let D_τ denote a reduced disc diagram collared by a chain of cells $\bigcup_{j<k_0} X_{v_j}$ and by $X_{v_{k_0}}$. The contradiction is now reached by considering the area of D_τ. The case of $\mathsf{Area}(D_\tau) = 0$ is straightforward and identical to the $C(6)$ case. The differences are in the case where $\mathsf{Area}(D_\tau) \geq 1$ (specifically, in the second to last paragraph in the proof Lemma 3.12). We explain this part of the argument in detail.

If $\mathsf{Area}(D_\tau) \geq 1$ we may assume, by performing the same reductions to τ as in the $\mathsf{Area}(D_\tau) = 0$ case, that ∂D_τ has no spurs, so by Theorem 4.5 D_τ contains a cell S which either is the only cell of D_τ, or is a 1-shell or a 2-shell containing a vertex of degree 3 in $\partial S \cap \partial D$. Claim 4.9 shows that $\partial S < X_{v_{k_0}}$ in D_τ. To finish the proof of Lemma 3.12, observe that if S is the only cell of D_τ then the $C(4)$ condition implies that S intersects at least 3 consecutive cells C_1, C_2, C_3, C_4 of A_τ. If S is a 1-shell, then by the $C(4)$ condition it intersects at least 3 consecutive cells C_1, C_2, C_3 of A_τ. If S is a 2-shell, then by the $C(4)$–$T(4)$ condition it intersects at least 3 consecutive cells C_1, C_2, C_3 of A_τ. In any case, the path τ' obtained from τ by pushing across C_2 traverses at most as many cells as τ, but bounds a disc diagram $D_{\tau'}$ with $\mathsf{Area}(D_{\tau'}) < \mathsf{Area}(D_\tau)$, contradicting our initial choices.

Thus we arrive at a contradiction in all cases, so $\bigcup_{j<k_0} X_{v_j} \cap X_{v_{k_0}}$ must be contractible and the induction is complete. □

4.2. **The C(3)-T(6) case.** The statement of Greendlinger's Lemma in the case of $C(3)$–$T(6)$ complexes is more complicated.

Theorem 4.11 (Greendlinger's Lemma)**.** *Let* X *be a* $C(3)$–$T(6)$ *complex and* $D \to X$ *be a reduced disc diagram, then either*

(1) D is a single cell,
(2) D is a ladder,

(3) D has at least three spurs, 1-shells and/or pairs of 2-shells sharing an edge incident to a boundary vertex.

Note that Lemma 3.1 is true for the $C(3)$–$T(6)$ condition as well.

Lemma 4.12. *Let X be a simply-connected $C(3)$–$T(6)$ small-cancellation complex. Let C_1, C_2 be 2-cells of X. Then either $\partial C_1 = \partial C_2$, or $C_1 \cap C_2 = \emptyset$, or $C_1 \cap C_2$ is contractible.*

Moreover, in the case of $C(3)$–$T(6)$ complexes, we have some additional information about pieces of a complex, a fact which was first observed by Pride (see also [GS90], [MW02]).

Lemma 4.13. *If X is a $T(5)$ complex then every piece in X is an edge.*

Consequently, the statement of Lemma 4.12 can be refined to the following: either $\partial C_1 = \partial C_2$, or $C_1 \cap C_2 = \emptyset$, or $C_1 \cap C_2$ is a single edge or a single vertex.

As in $C(4)$–$T(4)$ case, for the analogue of the Lemma 3.12, we need a refined version of Greendlinger's Lemma.

Theorem 4.14. *Let X be a $C(3)$–$T(6)$ complex and $D \to X$ be a reduced disc diagram. If D does not contain spurs then it is either a single cell, contains a 1-shell with at most one vertex of degree more than 6 in D, or contains a 2-shell with all boundary vertices of degree at most 5 in D.*

Proof. The proof is very similar to the proof of Theorem 4.5. Let $D \to X$ be a reduced disc diagram with $area(D) \geq 2$. Since the $T(6)$ condition implies that all pieces have length 1, there are no internal vertices of degree 2 in D. To simplify the calculations, let D' be the disc diagram obtained from D by removing all vertices of degree 2 at the boundary of D, with the exception of a single vertex on the outerpath of every 1-shell. Since D' is a disc diagram, the Euler characteristic $\chi(D')$ is equal to 1. We distribute the Euler characteristic from each edge as $\frac{-1}{2}$ to both of it ends, and from each cell $\frac{1}{p}$ to each of its p boundary vertices. It is clear that the sum of the distributed Euler characteristic over all vertices is equal to 1. Let v be an internal vertex of D'. Since D is reduced and satisfies the $T(6)$ condition, the degree of v is $n \geq 6$. The number of cells containing v is also n. Thus, the distributed Euler characteristic for v is at most $1 - \frac{n}{2} + \frac{n}{3}$ which for $n \geq 6$ is non-positive. Let u be a boundary vertex of D'. The degree of u is $n \geq 2$ and it is greater by at least 1 than the number of cells containing u. Thus, the distributed Euler characteristic for u is at most $1 - \frac{n}{2} + \frac{n-1}{3}$, where n is the degree of u. Consequently, the distributed Euler characteristic for u has is non-positive if $n \geq 4$. The possible positive values are:

- $\frac{1}{3}$ for $n = 2$ which can happen only for a boundary vertex of degree two in a 1-shell;

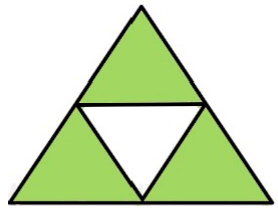


FIGURE 6. An example of a $C(3)-T(6)$ complex for which the usual Helly Property does not hold.

- $\frac{1}{6}$ for $n = 3$ if u is a vertex at the end of the piece lying in two triangles, and these triangles are either 1-shells or 2-shells;
- $\frac{1}{12}$ for $n = 3$ if u is a vertex at the end of the piece shared by a triangle and a square, such triangle is either a 1-shell or a 2-shell, a square is either a 3-shell or the intersection of its boundary with the boundary of D' has two components.

Since the sum is 1 and the distributed Euler characteristic of all other vertices is non-positive, there are multiple cells in D that are either 1-shells or 2-shells. If a vertex has degree at least 6, then its distributed Euler characteristics at most $1 - \frac{n}{2} + \frac{n-1}{3}$ which cannot exceed $\frac{-1}{3}$. Vertices of degree at least 6 can be shared by two 1-shells, or by two 2-shells or a 1-shell and a 2-shell. In either case if none of the 1-shells touch a vertex of degree less than 6 on at least one side, and none of 2-shells touch vertices of degree less than 6 on both sides, then the sum of the distributed Euler characteristic of vertices is at most 0, which is a contradiction. $\triangle$

In the $C(3)$–$T(6)$ case, the adaptations to the proof of Theorem 3.4 are a bit less straightforward – the main difficulty arises from the fact that the Helly Property does not hold in full generality, as illustrated in Figure 6. To bypass this problem, we prove a weaker form of the Helly Property and we refine the ordering of the nerve complex of $\mathcal{U}$.

Lemma 4.15. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(3) - T(6)$ small-cancellation presentation P, let Λ be its structure graph, and let $V' \subset V(\Lambda)$ with $|V'| < \infty$. If each pairwise intersection $X_v \cap X_{v'}$ with $v, v' \in V'$ is an edge, then the total intersection $\bigcap_{v \in V'} X_v$ is either a single edge or a single vertex.*

Proof. The proof is by induction on $|V'|$ where the base case is $|V'| = 3$. Let $V' = \{X_{v_1}, X_{v_2}, X_{v_3}\}$ and let $e_{ij} \subset X_{v_i} \cap X_{v_j}$ be the edges in each pairwise intersection. Observe that if $e_{ij} \cap e_{ik} \neq \emptyset$, then by Lemma 4.12, the triple intersection is also not empty and is contractible, which means that it is either a single edge or a single vertex. Thus we can assume that $e_{ij} \cap e_{ik} = \emptyset$ for any permutation of i, j, k.

For each i, let $\rho_i \to X_{v_i}$ be a spurless path joining e_{ij}, e_{ik}. Moreover, since $e_{ij} \cap e_{ik} = \emptyset$ we can choose the ρ_i so that the concatenation $\rho_1\rho_2\rho_3$ is a cycle that does not separate any two of the e_{ij}. Let $D \to \hat{B}_n$ be a reduced disc diagram with $\partial D = \rho_1\rho_2\rho_3$. Then by Lemma 4.11, D is a single cell, or is a ladder (and contains exactly two 1-shells), or it contains at least three 1-shells and/or pairs of 2-shells sharing an edge incident to a boundary vertex.

It is clear that neither of $X_{v_1}, X_{v_2}, X_{v_3}$ can be a 1- or 2-shell in D, as $e_{ij} \cap e_{ik} = \emptyset$ implies at least three pieces in the part of its boundary in the interior of D. If D is a single cell C or contains a 1-shell C, then there is a reduced disc diagram $C \cup X_{v_i} \cup X_{v_j}$ with an internal vertex of degree 3, contradicting the $T(6)$ condition. If D contains a pair of 2-shells C_1, C_2 sharing an edge incident to a boundary vertex, then there is either a reduced disc diagram $C_1 \cup C_2 \cup X_{v_i}$ with an internal vertex of degree 3 or $C_1 \cup C_2 \cup X_{v_i} \cup X_{v_j}$ with an internal vertex of degree 4, in either case contradicting $T(6)$.

Thus D must be a degenerate disc diagram, but then $e_{ij} \cap e_{ik} \neq \emptyset$.

For the inductive step, we use the fact that $\bigcap_{v \in V' - \{v_1\}} X_v$ is either a single edge or vertex, and now intersect X_{v_1} with $\bigcap_{v \in V' - \{v_1, v_2\}} X_v$. △

Let $\hat{B}^o_n = \hat{B}_n - \hat{B}^{(0)}_n$, let $\mathcal{U}^o$ be the open cover of $\hat{B}^o_n$ induced by $\mathcal{U}$, let $\mathcal{V}^o$ be the vertex set of the nerve complex $\mathbf{N}(\mathcal{U}^o)$. In general, $\hat{B}^o_n$ need not be connected, and the connected components of $\hat{B}^o_n$ partition $\mathcal{V}^o$ into subsets, which we denote V^o_ν. Let $\varphi^o_\nu : V^o_\nu \to \mathbb{N}$ denote the ordering of Definition 3.10 on $\mathbf{N}(\mathcal{U}^o_\nu)$. The vertex set V^o_ν of $\mathbf{N}(\mathcal{U}^o_\nu)$ naturally corresponds to a subset (which we also denote V^o_ν) of the vertex set V of $\mathbf{N}(\mathcal{U})$. We make the following observation:

Lemma 4.16. *If $X_v, X_w \in \mathcal{U}$ intersect only at vertices, then $X_v \cap X_w$ is a single vertex.*

Proof. Suppose $\{p, q\} \subset X_v \cap X_w$, and let $\sigma_v \to X_v, \sigma_w \to X_w$ be spurless paths connecting p, q. Since $\mathcal{X}(P)$ is simply-connected, there is a disc diagram $D \to X$ with $\partial D = \sigma_v\sigma_w^{-1}$. If $area(D) = 0$, then there is an edge $e \subset X_v \cap X_w$, contradicting our hypothesis. Suppose $area(D) > 0$. Then either X_v or X_w is an untethered component by Lemma 4.12. But then a 2-cell of D with outerpath on either $\sigma_v \to X_v$ or $\sigma_w \to X_w$ contradicts the definition of an untethered component. We have thus established that $X_v \cap X_w$ is a single vertex. △

We make the following observation:

Remark 4.17. If two elements $X_v, X_w \in \mathcal{U}$ intersect at a vertex p, then since $\hat{B}_n$ is a Cayley graph, then either p is a cut vertex of $\hat{B}_n$, or X_v and X_w lie in the same connected component of $\hat{B}^o_n$. Indeed, when viewed in $\mathcal{X}(P)$, the projections of X_v and X_w (which we denote in the same way) are either 2-cells or loops in $\mathcal{X}(P)^{(1)}$, and either $\mathcal{X}(P)$ is a wedge with X_v and X_w in different components of the wedge, or X_v and X_w are both 2-cells, and there is a non-trivial relation between a generator appearing in the attaching map of X_v and a generator appearing in the attaching map of X_w.

The previous remark shows that the orders $\{\varphi_\nu^o\}$ defined on the V_ν^o are compatible, and induce a partial order on $\mathcal{U}$ that extends to a total order $\varphi : V \to \mathbb{N}$ by the order-extension principle.

We re-index the vertex sets V_ν^o according to this order, i.e., we write $V_\nu^o = \{v_{\nu,j}^o\}_{j\in\mathbb{N}}$ (or simply $V_\nu^o = \{v_{\nu,j}^o\}_{j\in\mathbb{N}}$ when the V_ν^o is fixed) where the indexing agrees with φ_ν^o. As in previous sections, we refer to this order and the induced order on the corresponding subset of $\mathcal{U}$ interchangeably.

We prove Theorem 1.1 for $C(3) - T(6)$ presentations by first showing that, for each $V_\nu^o \subset V$, the intersection

$$\bigcup_{j<k} X_{v_j} \cap X_{v_k}$$

is contractible for each $k \in \mathbb{N}$. We then assemble the various subcomplexes corresponding to the $V_\nu^o \subset V$ together using the ordering φ obtained by the order-extension principle. This will yield the desired homotopy equivalence by Remark 4.17.

We now apply the results collected above to deduce an analogue of Claim 3.13.

Claim 4.18. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(3)$–$T(6)$ small-cancellation presentation P. Fix a ν and consider $V_\nu^o = \{v_j\}_{j\in\mathbb{N}}$. For each $v_j \in V_\nu^o$, let X_{v_j} be the corresponding element of $\mathcal{U}$. Let A_β be an annular diagram in $\widetilde{\mathcal{X}(P)}$ collaring a reduced disc diagram D_β so that $\partial D_\beta = D_\beta \cap A_\beta = \beta$, and finally, let $X_{v_{k_0}}$ be the maximal element in the ordering φ_ν^o corresponding to the boundary of a 2-cell in A_β. Then $\partial S < X_{v_{k_0}}$ for every shell S in D_β.*

Proof. The proof is basically identical to that of Claim 3.13, except that we substitute Lemma 3.1, Lemma 3.7, and Lemma 3.8 with their $C(3) - T(6)$ analogues – Lemma 4.12, Lemma 4.15, and Lemma 4.16. Note that there is a difference in a key small-cancellation point of the argument the outerpath of a shell can be a single piece in the case of a 2-shell. But by Greendlinger's Lemma, if we do not have at least two 1-shells, then there are at least two pairs of 2-shells sharing an edge incident to a boundary vertex. Such pair of 2-shells cannot be bounded by ∂S, as it would contradict the $T(6)$ condition. △

We can now conclude:

Lemma 4.19. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(3) - T(6)$ small-cancellation presentation P. Fix a ν and let $V_\nu^o = \{v_j\}_{j\in\mathbb{N}}$. For each $v_j \in V_\nu^o$, let X_{v_j} be the corresponding element of $\mathcal{U}$. For each $k \in \mathbb{N}$, the intersection*

$$\bigcup_{j<k} X_{v_j} \cap X_{v_k}$$

is contractible.

Proof. This proof follows the scheme of the proof in the $C(4)$–$T(4)$ case. Again we prove that we have a cell S that intersects at least 3 consecutive cells C_1, C_2, C_3 of A_τ. By Theorem 4.14, there is a cell S in D_τ that is either the only cell of a D_τ, or a 1-shell touching a vertex of degree at most 5, or a 2-shell touching two boundary vertices of degree at most 6. If S is the only cell, the fact that S intersects at least three consecutive shells follows from the $C(3)$ condition; in the case of a 1-shell, intersection with two of the consecutive shells follows from the $C(3)$ condition, and intersection with the third one from the $T(6)$ condition; in the case of a 2-shell, it follows from the $T(6)$ condition and the fact that such a shell has at least a single boundary edge. △

By Remark 4.17, distinct connected components of $\hat{B}_n^o$ intersect in a single vertex; we immediately deduce an analogue to Lemmas 3.12 and 4.10:

Corollary 4.20. *Let $\mathcal{X}(P)$ be the presentation complex associated to a $C(3)-T(6)$ small-cancellation presentation P. Let $V = \{v_j\}_{j\in\mathbb{N}}$ be the vertices of its structure graph Λ. For each $v_j \in V$, let X_{v_j} be the corresponding element of $\mathcal{U}$. For each $k \in \mathbb{N}$, the intersection*

$$\bigcup_{j<k} X_{v_j} \cap X_{v_k}$$

is contractible.

Reasoning in exactly the same way as in the proof of Theorem 3.4, the $C(3) - T(6)$ version of Theorem 1.1 now follows.

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