Lent Term 2009 J. M. E. Hyland

Set Theory and Logic: Example Sheet 4

1. Prove informally the equivalence $\forall x. \bigcup x \subseteq x \leftrightarrow x \subseteq Px$. Write out the sentence giving the equivalence in the primitive language of set theory. Which axioms of set theory are needed to prove this sentence?

2. (i) Suppose that all the elements of a set x are transitive sets. Show that $\in$ is a transitive relation on x. Is the converse true?
(ii) Suppose that $(x_i | i \in I)$ is a non-empty family of transitive sets. Prove that the intersection $\bigcap\{x_i | i \in I\}$ is a transitive set. Why the requirement that I be non-empty?

3. Show that a set x is transitive if and only if $x = \bigcup\{S(y) | y \in x\}$. (Here $S(y) = y \cup \{y\}$.)

4. (i) Show that a set x is transitive if and only if its power set Px is transitive.
(ii) Show that a set x is transitive if and only if its successor $S(x)$ is transitive.
(It may be worth considering what you are assuming about sets in proving these.)

5. (i) Show that Separation and Infinity together imply Empty Set.
(ii) Show that Replacement, Empty Set and Power Set together imply Pairing.

6. (i) Show that (in the presence of some other axioms) Replacement implies Separation. Show further that it implies the following Axiom of Collection

$$\forall x. \exists y. \phi(x, y) \rightarrow \forall u. \exists v. \forall x \in u. \exists y \in v. \phi(x, y)\,,$$

where for simplicity we suppress parameters.
(ii) Show (again in the presence of some other axioms) that Collection and Separation together imply Replacement.

7. Show that for all $x \in V$

$$TC(x) = x \cup \bigcup\{TC(y) | y \in x\}\,.$$

Why not use this recursive definition of $TC(x)$ in the development of set theory?

8. A set x is called *hereditarily transitive* if each member of $TC(\{x\})$ is transitive. Prove that the class of hereditarily transitive sets is the class On of ordinals. For what purpose would this not be a good definition of the ordinals? (Think about the axioms used to prove the equivalence!)

9. A set x is called *hereditarily finite* if each member of $TC(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with V_ω. Which of the axioms of ZF are satisfied in the class HF of hereditarily countable sets with the standard notion of membership?

10. A set x is called *hereditarily countable* if each member of $TC(\{x\})$ is countable. Which of the axioms of ZF are satisfied in the class HC of hereditarily countable sets with the standard notion of membership?
Show that the class HC is in fact a set and determine its rank. Does HC coincide with any V_α?

11. Suppose that a set x has rank α.
(i) Find the ranks of the transitive closure $TC(x)$ of x, the singleton $\{x\}$ of x and the power set Px of x.
(ii) Can you determine the rank of the union $\bigcup x$?

12. (i) Show that $\mathrm{rk}(x) = \bigcup\{\mathrm{rk}(y) + 1 | y \in x\}$.
(ii) Show that $\mathrm{rk}(x) = \{\mathrm{rk}(y) | y \in TC(x)\}$.

13. In lectures we considered which axioms of ZF are satisfied in the structure $V_{\omega+\omega}$ with the standard notion of membership. Repeat the exercise for V_{ω_1} again with the standard notion of membership? Can you find a sentence in the language of set theory true in one model and not in the other?

14. In lectures we defined a general Mostowski collapse for any well-founded class. What is the general Mostowski collapse of the set difference $V_{\omega+\omega} - V_\omega$ with the standard notion of membership? What about $V_\omega - \omega$?

The remaining questions are for enthusiasts.

15. Show that the following are characterizations of V_ω.
(i) V_ω is the collection $\{x \,|\, TC(\{x\}) \text{ is finite}\}$.
(ii) V_ω is the least collection containing 0 and closed under power set and under taking arbitrary subsets.
(ii) V_ω is the least collection containing 0 and closed under pairs and unions.

16. A set x is *small* if and only if there is an injection $X \to V_{\omega+n}$ for some $n \in \omega$. It is *hereditarily small* if each member of $TC(\{x\})$ is small. Which of the axioms of ZF are satisfied in the class HS of hereditarily small sets with the standard notion of membership?

17. Assume that ZF is consistent. Extend the language of ZF by adding uncountably many new constants, and extend the axioms of ZF by adding the assertions that these constants are distinct and all belong to ω. Explain why this theory has a model. In this model of ZF, ω is uncountable – doesn't this contradict the fact that ω is countable?

18. Assume ZF is consistent. Extend the language of ZF by adding countably many new constants, α_n for $n \in \mathbb{N}$. Extend the axioms of ZF by adding the assertions $\alpha_n \in On$ and $\alpha_{n+1} \in \alpha_n$. Explain why this theory has a model. In this model of ZF, On is not well-founded – doesn't this contradict a thoerem of the course?

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.