Lent Term 2009 J. M. E. Hyland

Set Theory and Logic: Example Sheet 2

1. (i) Show that a totally ordered set $(A, <)$ is well-ordered if and only if there are no infinite descending sequence $a_0 > a_1 > a_2 > \cdots$. Have you used AC in your proof?
(ii) Write out a proof that a countable union of countable sets is countable. Where is the use of AC in your proof?

2. Suppose that we are given $<$ a (strict) partial order on a set X. Show that $<$ can be extended to a total order on the same set X.

3. A collection $\mathcal{X} \subseteq P(X)$ of subsets of a set X, which is such that $A \in \mathcal{X}$ if and only for all finite $a \subseteq A$, $a \in \mathcal{X}$ is said to have *finite character*.
The Teichmuller-Tukey Lemma is the statement: if $\mathcal{X}$ has finite character then it has a maximal element.
Show that AC implies the Teichmuller-Tukey Lemma. Does the Teichmuller-Tukey Lemma in its turn imply AC?

4. In this question do not assume the axiom of choice!
Suppose that we know only that every set can be totally ordered. Show that any family of finite non-empty subsets of a set has a choice function.

5. Consider the statement: for any pair of cardinals $\mathbf{m}$ and $\mathbf{n}$, either $\mathbf{m} \leq \mathbf{n}$ or $\mathbf{n} \leq \mathbf{m}$. Show that it is equivalent to AC.

6. How many different partial orders (up to isomorphism) are there on a set of 4 elements? How many of these are complete?

7. A complete poset $(X, \leq)$ is one in which the supremum $\sup A$ exists for all $A \subseteq X$. Show that in a complete poset the infimum $\inf A$ also exists for all $A \subseteq X$.
Show that the fixed point constructed in the proof of the Knaster-Tarski Theorem is the greatest fixed point. Modify the proof to produce the least fixed point instead.

8. Which of the following posets (ordered by inclusion) are complete?
(i) The set of all subsets of $\mathbb{N}$ that are finite or have finite complement.
(ii) The set of all independent subsets of a vector space V.
(iii) The set of all subspaces of a vector space V.
(iv) The set of all equivalence relations $R \subseteq X \times X$ on a set X.

9. (i) What is the cardinality of the set of open subsets in $\mathbb{R}$?
(ii) What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

10. Show that any two bases of a vector space have the same cardinality. Did you use AC?

11. Define the sum $\sum_{i \in I} L_i$ and product $\prod_{i \in I} L_i$ of an indexed family $(L_i \mid i \in I)$ of sets.
Suppose that $(L_i \mid i \in I)$ and $(M_i \mid i \in I)$ are such that there are no surjections $L_i \to M_i$ for any $i \in I$. Show that there is no surjection $\sum_{i \in I} L_i \to \prod_{i \in I} M_i$.
Deduce that there is no surjection from $\aleph_\omega$ to ${\aleph_\omega}^{\aleph_0}$. Can we have the equality $2^{\aleph_0} = \aleph_\omega$?

12. Recall from lectures that we always have $\alpha \leq \omega_\alpha$. Is there an ordinal α such that $\omega_\alpha = \alpha$?

13. (i) Show that if $\mathbf{m} + \mathbf{n} = \mathbf{m}.\mathbf{n}$, then either $\mathbf{n} \leq \mathbf{m}$ or $\mathbf{m} \leq^* \mathbf{n}$.
(ii) Take κ a well-ordered cardinal (or initial ordinal). Show that if $\kappa + \mathbf{n} = \kappa.\mathbf{n}$, then either $\mathbf{n} \leq \kappa$ or $\kappa \leq \mathbf{n}$.
(iii) Deduce that if $\mathbf{m} + \mathbf{n} = \mathbf{m}.\mathbf{n}$ for all infinite cardinals $\mathbf{m}$ and $\mathbf{n}$, then AC holds.

14. Suppose that κ is an aleph (that is an infinite well-ordered cardinal). Show that if $\kappa \leq \mathbf{m}.\mathbf{n}$ then either $\kappa \leq \mathbf{m}$ or $\kappa \leq \mathbf{n}$.

Cardinal arithmetic without the axiom of choice is a subject which I find fascinating. But it not an essential part of the course as I teach it. So the following questions are for those who are interested in playing with the ideas.

15. Let AC_n be the principle that any family of n-element sets has a choice function.
(i) Show that AC_{rs} implies AC_r for all $r, s \geq 1$.
(ii) Show that AC_2 implies AC_4.

16. (i) Prove that, even without AC, a countable union of countable sets certainly cannot have cardinality $\aleph_2$. (This should encourage quiet reflection on the usual proof that a countable union of countable sets is countable.)
(ii) Show that the cardinality of the set of well-orderings of the set $\mathbb{N}$ is $2^{\aleph_0}$.
Deduce that, even without AC, $\aleph_1 \leq^* 2^{\aleph_0}$.

Comments, corrections and queries can be sent to me at m.hyland@dpmms.cam.ac.uk.