Michaelmas Term 2013 J. M. E. Hyland

Linear Algebra: Recapitulation from IA

This sheet contains basic definitions with which you should be familiar from IA. Though we shall go through the material again at the start of the IB course, if you do not feel at home with the ideas, the sooner you become so the better.

A *vector space* V over a field $\mathbb{F}$ is a set V equipped with the structure of an abelian group together with a compatible action *scalar multiplication* of $\mathbb{F}$: the structure and axioms are as follows.

Abelian Group V is a commutative group under *addition* $+ : V \times V \to V$: the additive identity is $\mathbf{0}$ and the additive inverse of $\mathbf{v}$ is $-\mathbf{v}$.

Field Action Scalar multiplication $(-.-) : \mathbb{F} \times V \to V$ satisfies the action laws

(i) $1.\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;

(ii) $\lambda.(\mu.\mathbf{v}) = (\lambda\mu).\mathbf{v}$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$;

and the distributive laws

(i) $(\lambda + \mu).\mathbf{v} = \lambda.\mathbf{v} + \mu.\mathbf{v}$ for all $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$;

(ii) $\lambda.(\mathbf{u} + \mathbf{v}) = \lambda.\mathbf{u} + \lambda.\mathbf{v}$ for all $\lambda \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$.

The point of the definition is that finite linear combinations of the form $\sum_1^n \lambda_i \mathbf{x}_i$ can be handled in the way with which we are familiar.
Note: all our linear combinations are finite.

A *subspace* W of a vector space V is a subset of V containing $\mathbf{0}$ and closed under addition and scalar multiplication. Then W forms a vector space under the induced operations. We write $W \leq V$. For W to be a subspace of V it is necessary and sufficient that W be non-empty and closed under $\lambda.\mathbf{u} + \mu.\mathbf{v}$.

Suppose that V and W are vector spaces. A map $\alpha : V \to W$ is *linear* just when

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha(\mathbf{u}) + \alpha(\mathbf{v}) \quad \text{and} \quad \alpha(\lambda.\mathbf{u}) = \lambda.\alpha(\mathbf{u})\,.$$

A linear map preserves linear combinations $\alpha(\sum_1^n \lambda_i \mathbf{x}_i) = \sum_1^n \lambda_i \alpha(\mathbf{x}_i)$. It is sufficient to check the equality $\alpha(\lambda.\mathbf{u} + \mu.\mathbf{v}) = \lambda.\alpha(\mathbf{u}) + \mu.\alpha(\mathbf{v})$.

If $\alpha : V \to W$ is linear, then its kernel $\ker\alpha$ is a subspace of V and its image $\mathrm{Im}\alpha$ is a subspace of W.

A subset $\{\mathbf{e}_i\}$ of a vector space V (or sequence in V according to context) is *linearly independent* just when no non-trivial linear combination is $\mathbf{0}$: that is, when $\sum \lambda_i \mathbf{e}_i = \mathbf{0}$ implies $\lambda_i = 0$ for all i.
A set (sequence) which is not linearly independent is *linearly dependent*. Note that by the definition, the empty set $\emptyset$ is always linearly independent. Also any set containing the zero vector $\mathbf{0}$ is linearly dependent (because $1.\mathbf{0} = \mathbf{0}$).

A subset $\{\mathbf{e}_i\}$ of a vector space V (or sequence in V) *spans* V (or *is a spanning set* in V) just when any $\mathbf{x}$ in V is a linear combination of the $\mathbf{e}_i$: that is when we can write any $\mathbf{x}$ as $\mathbf{x} = \sum x_i \mathbf{e}_i$. Note that for any vectors $\mathbf{e}_i$ in V, the set of linear combinations $\sum x_i \mathbf{e}_i$ forms a subspace $\langle \mathbf{e}_i \rangle$ of V; and it is trivial that the $\mathbf{e}_i$ span $\langle \mathbf{e}_i \rangle$.

A linearly independent spanning set (or sequence) in V is a *basis* for V. If $\{\mathbf{e}_i\}$ is a basis for V, then any $\mathbf{x}$ in V can be written uniquely as a linear combination of the $\mathbf{e}_i$: that is for any vector $\mathbf{x}$ there are unique *coordinates* x_i of $\mathbf{x}$ with respect to the basis $\mathbf{e}_i$ such that we have $\mathbf{x} = \sum x_i \mathbf{e}_i$.
WARNING Of course the first coordinate x_1 depends on the whole basis $\mathbf{e}_i$, and not just on the vector $\mathbf{e}_1$.

The number of elements in a basis is the *dimension*, $\dim V$, of a vector space V. A little thought is required to show that this makes good sense.

The important aspect is this. Say that a vector space V is *finite dimensional* just when it has a finite basis. One can then prove that any two bases of a finite dimensional vector spaces have the same number of elements so the notion of dimension makes sense. It is true but not completely trivial that any subspace of a finite dimensional vector space is finite dimensional.

The correctness of this kind of analysis for general vector spaces depends on the Axiom of Choice. It is not considered in the Linear Algebra course.

The *rank* $\mathrm{r}(\alpha)$ of a linear map $\alpha : V \to W$ (between finite dimensional vector spaces) is the dimension of the image; and the *nullity* $\mathrm{n}(\alpha)$ is the dimension of the kernal. So $\mathrm{r}(\alpha) = \dim(\mathrm{Im}\alpha)$ and $\mathrm{n}(\alpha) = \dim(\ker \alpha)$.

The fundamental principle for counting dimensions is the *rank-nullity theorem*:

$$\text{for } \alpha : V \to W,\ \mathrm{r}(\alpha) + \mathrm{n}(\alpha) = \dim V.$$

It is important to appreciate that this theorem has a very concrete reading. An $n \times m$ matrix A over a field $\mathbb{F}$ gives by multiplication a linear map $A : \mathbb{F}^n \to \mathbb{F}^m$ between the spaces of column vectors. Then $\ker A$ is the space of solutions of m linear equations in n unknowns and the nullity of A is therefore the dimension of this solution set. On the other hand $\mathrm{Im}A$ is the space spanned by the columns of A and it follows that the rank of A is the maximal number of linearly independent columns. In the course we discuss the crucial fact that this is equal to the maximal number of linearly independent rows. So the rank nullity theorem corresponds to the fact that when considering linear equations of the form $a_1x_1 + \cdots a_nx_n = 0$ each new independent equation reduces the dimension of the solution set by one.