Michaelmas Term 2012 J. M. E. Hyland

Number Theory: Example Sheet 3

The first 12 questions are intended for the supervisions. Further questions are designed to encourage mathematical investigation without any examination emphasis.
Throughout this sheet, ϕ denotes the Euler totient function, μ the Möbius function, $\tau(n)$ the number of positive divisors of n, and $\sigma(n)$ the sum of the positive divisors of n.

1. (i) Define Liouville's function λ by $\lambda(p_1^{a_1} \cdots p_k^{a_k}) = (-1)^{a_1+\cdots+a_k}$. Can you find a product expansion for the Dirichlet series
$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}?$$
What is its product with the Riemann Zeta-function? What combinatorial identity does that give?
(ii) Find a product expansion for the Dirichlet series
$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.$$
Find an expression for it in terms of the Riemann Zeta-function. What conbinatorial identity can you derive?

2. Suppose that for $\Re(s) > 1$, we have
$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}.$$
Identify the arithmetic function $\tau_k(n)$.

3. Suppose that for $\Re(s) > k+1$, we have
$$\zeta(s-k)\zeta(s) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}.$$
Identify the arithmetic function $\sigma_k(n)$.

4. Find all natural numbers n for which $\sigma(n) + \phi(n) = n\tau(n)$.

5. Use Legendre's formula to compute $\pi(207)$.

6. Let N be a positive integer greater than 1. Prove the inequality $N! > (\frac{N}{e})^N$. Deduce that
$$\sum_{p \leqslant N} \frac{\log p}{p-1} > \log N - 1.$$

7. Prove that every non-constant polynomial with integer coefficients assumes infinitely many composite values.

8. Prove that every integer $N > 6$ can be expressed as a sum of distinct primes.

9. Prove that for every $n \geqslant 1$, the set of numbers $\{1, \cdots, 2n\}$ can be partioned into pairs $\{a_1, b_1\}, \cdots, \{a_n, b_n\}$ so that the sum $a_i + b_i$ of each pair is prime.

10. Let a be a positive integer. Determine explicitly the real number whose continued fraction is $[a, a, a, \ldots]$.

11. Determine the continued fraction expansions of $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$, $\sqrt{46}$.

12. Calculate a_0, ..., a_4 in the continued fraction expansions of e and π.

That ends the official part of the sheet. As ever the remaining questions are intended to encourage investigation so don't bother your supervisor with them.

(A) Give direct combinatorial proofs of the identities which you discovered in Question 1.

(B) A Theorem of Mertens states that

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1)$$

Try proving this.

(C) Investigate the class of continued fractions of the form $[a, b, a, b, \ldots]$ where $b|a$. At least compute some examples and find out what generally is the real represented.

Email any comments, suggestions and queries to `m.hyland@dpmms.cam.ac.uk`.