COMPLEX ANALYSIS EXAMPLES 3

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These questions are a mix of questions from recent versions of the course together with some specific to my own take on the material. The questions are not all equally difficult.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

1. Using the residue theorem establish the following.

$$\text{(i)} \int_{-\infty}^{\infty} \frac{x^2}{x^4+10x^2+9}dx = \frac{\pi}{4}; \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}};$$

$$\text{(iii)} \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \frac{\pi}{\sqrt{2}}; \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

[How many of these integrals can you calculate by standard real variable techniques?]

2. For $a,\ b > 0$ and $a \neq b$ evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$. Also evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)^2}dx$. Can the latter be deduced from the former by letting $b \to a$?

3. Compute the residue of $(1+z^2)^{-n}$ at $z = i$, and deduce that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n} = \pi\frac{(2n-2)!}{2^{2n-2}((n-1)!)^2}$.
What is the value of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x dx}{(1+x^2)^n}$?

4. For $-1 < \alpha < 1$, and $\alpha \neq 0$, compute $\displaystyle\int_0^{\infty} \frac{x^\alpha dx}{1+x+x^2}$. Letting $\alpha \to 0$ and recalculate $\displaystyle\int_0^{\infty} \frac{dx}{1+x+x^2}$. (You should get the same answer viz $2\pi/3\sqrt{3}$ as in lectures.)

5. By integrating $\dfrac{z}{a-e^{-iz}}$ around the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\infty} \frac{x \sin x}{1-2a\cos x+a^2}dx = \frac{\pi}{a}\log(1+a)\,, \qquad \text{for } 0 < a < 1\,.$$

6. Let $a > 0$. For $\omega \in \mathbb{R}$ evaluate the following integrals.

$$\text{(a)} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\omega x}dx \qquad \text{(b)} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sin x}{x}e^{-i\omega x}dx. \qquad \text{(c)} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2+a^2}dx.$$

7. (i) For a positive integer N let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists a constant $C > 0$ such that $|\cot \pi z| < C$ on every γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2 + 1}$, show that $\displaystyle\sum_0^\infty \frac{1}{n^2 + 1} = \frac{1 + \pi \coth \pi}{2}$.

(iii) Evaluate $\displaystyle\sum_0^\infty \frac{(-1)^n}{n^2 + 1}$.

8. Let $f : D \to \mathbb{C}$ be analytic and take $a \in D$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small the formula

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} z \frac{f'(z)}{f(z) - w} dz$$

defines an analytic function in some neighbourhood of $f(a)$ which is inverse to f.

9. (a) Show that $z^4 + z + 1$ has one zero in each quadrant. Show that all roots lie inside the circle $|z| = 3/2$.

(b) How many zeros does $z^4 + 12z + 1$ have in the annulus $2 < |z| < 3$? Are they distinct? Can you determine in which quadrants they lie?

(c) Find an annulus centre 0 in which $z^4 + 26z + 4$ has exactly three roots. Can you determine in which quadrants they lie?

10. Consider the polynomials
(a) $p(z) = z^4 + z^3 + 2z^2 + 5z + 2$;
(b) $p(z) = z^4 + z^3 + 2z^2 + 5z + 3$;
(c) $p(z) = z^4 + z^3 + 2z^2 + 5z + 4$.
In each case determine whether $p(z)$ has real roots and determine in which quadrants the non-real roots lie.

11 Establish the following refinement of the Fundamental Theorem of Algebra.
Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots counting multiplicities in the disc $|z| < A + 1$.

12. Prove that $z \sin z = 1$ has only real solutions. [How many real roots are there in the interval $[-(n + 1/2)\pi, (n + 1/2)\pi]$? How many roots are there in the disc $|z| < (n + 1/2)\pi$?]

13. Show that if $|a| > e$, then $az^n = e^z$ has n distinct solutions in the unit disc. Find an upper bound r such that if $|a| < r$ then $az^n = e^z$ has no solutions in the unit disc. Can you say anything when $r < |a| < e$?

14. Prove the following strengthened form of Rouché's Theorem.
Suppose that the analytic functions f and g are such that $|g| < |f| + |f + g|$ on a simple closed curve γ. Then f and $f + g$ have the same number of zeros inside γ.

Finally an additional question to think about. Perhaps for once you really will use the Jordan Curve Theorem?

15. Suppose that γ is a simple closed curve contained (with its interior) in a domain D. Suppose that $f : D \to \mathbb{C}$ is an analytic function which takes no value more than once on γ. Show that f takes no value more than once inside γ.