COMPLEX ANALYSIS EXAMPLES 2

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These questions are a mix of questions from recent versions of the course together with some specific to my own take on the material. The questions are not all equally difficult.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

1. Find the radii of convergence about 0 of the functions

$$\tan z\,, \qquad z\cot z\,, \qquad \frac{z}{e^z-1}\,.$$

By definition $\dfrac{z}{e^z-1} = \displaystyle\sum_0^\infty \frac{B_n}{n!} z^n$ where B_n are the Bernoulli numbers.

(i) Show that the Bernoulli numbers are rational, and that $\dfrac{z}{e^z-1} + \dfrac{z}{2}$ is an even function so that $B_{2k+1} = 0$ for $k \geq 1$. (In an alternative convention the odd Bernoulli numbers are not mentioned.)

(ii) Explicit calculation will show that the first few Bernoulli numbers are small. Why is it evident that the Bernoulli numbers are unbounded?

2. Suppose that f is a non-constant analytic function with power series expansion $f(z) = \sum_0^\infty c_n z^n$ in $|z| < R$. For $r < R$ let γ_r be the image of the circle $|z| = r$ under f, and consider the winding number $n(\gamma_r, c_0)$ of this curve about $f(0) = c_0$. Take n least with $n > 0$ and $c_n \neq 0$. Show that for $r > 0$ sufficiently small, $n(\gamma_r, c_0) = n$.

3. What are the zeros of the function $\sin \dfrac{1}{1-z}$ in the unit disc $D(0,1)$? Is what you find consistent with the isolation of zeros of analytic functions?

4. Show that the power series $\sum_1^\infty z^{n!}$ defines an analytic function in $D(0,1)$. Show that f cannot be analytically continued to any domain properly condaining $D(0,1)$. [Hint: consider $z = \exp 2\pi i p/q$ with $p/q \in \mathbb{Q}$.]

5.Let $f : D \to \mathbb{C}$ be analytic. Suppose that the real part $\Re f$ of f has a local maximum at some $a \in D$. Show that f is constant.

6. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with $f(\frac{1}{n}) = \frac{1}{n}$ for all positive integers n. Show that $f(z) = z$.

(ii) Show that there is no analytic function $f : D(0,2) \to \mathbb{C}$ with $f(\frac{1}{n}) = \frac{1}{n+1}$ for all positive integers n.

7. (i) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Suppose that $f(n) = n^2$ for all $n \in \mathbb{Z}$. Does it follow that $f(z) = z^2$?

(ii) Let $g : \mathbb{C} \to \mathbb{C}$ be analytic except possibly at some singularities. Suppose that $g(n) = \frac{1}{n^2}$ for all $n \in \mathbb{Z}$. What more do you need to know to identify g?

8. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial function of degree $n \geq 1$. Show that $|f(z)| \to \infty$ as $|z| \to \infty$, and deduce that $|f(z)|$ takes a minimum at some $a \in \mathbb{C}$. Hence deduce that f has a zero in $\mathbb{C}$.
[This may appear to use a theorem of the course, but it should be clear that once you have the minimum you can proceed with bare hands.]

9. Suppose that f is analytic on $r < |z| < R$. Suppose that $r < \rho < \sigma < R$ and take α, β real with $\exp\alpha = \rho$ and $\exp\beta = \sigma$. What is the image γ under the transformation exp of the boundary of

$$\{z : \alpha < \Re z < \beta\,, \quad -\pi < \Im z < \pi\}\,?$$

Using the transformation exp compute $\int_\gamma f(z)dz$. Hence show that $\int_{|z|=\rho} f(z)dz = \int_{|z|=\sigma} f(z)dz$.

10. Find the Laurent expansion in powers of z of the function $1/(z^2-3z+2)$ in each of the domains

$$\{z \in C : |z| < 1\} \qquad \{z \in C : 1 < |z| < 2\} \qquad \{z \in C : |z| > 2\}\,.$$

What is its Laurent expansion in powers of $(z-1)$ in the domain $\{z : 0 < |z-1| < 1\}$?

11. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function.

(i) Show that f is a polynomial of degree $\leq n$ if and only if for some constant M we have $|f(z)| < M(1 + |z|^n)$ for all $z \in \mathbb{C}$.

(ii) Show that f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

12. Suppose that f_n are analytic, and that $f_n \to f$ locally uniformly and with f not constant. Show that for any $a \in D$ there is $N(a) \in \mathbb{N}$ and a sequence a_n for $n \geq N(a)$ with $a_n \to a$ as $n \to \infty$ and with $f_n(a_n) = f(a)$.

13. Suppose that $f : D(0,1) \to D(0,1)$ is analytic with $f(0) = 0$.

(i) Show that the function $g(z) = f(z)/z$ has a removable singularity at 0. Use the maximum modulus principle to deduce show that $|f(z)| \leq |z|$ for all $z \in D(0,1)$.

[Careful. You are not told and do not need to know anything about behaviour at the boundary.]

(ii) Suppose that $|f(a)| = |a|$ for some $a \neq 0$. Show that $f(z) = \omega z$ for some ω with $|\omega| = 1$.

14. Classify the singularities of each of the following functions.

$$\frac{z}{\sin z}\,, \quad \frac{1}{z^4+z^2}\,, \quad \cos\frac{\pi}{z^2}\,, \quad \frac{1}{z^2}\cos\frac{\pi z}{z+1}\,.$$

15 Let f be a function analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all sufficiently large z, then f is a rational function (i.e. the quotient of two polynomials).

16. Suppose that f is an analytic function on $\{z : 0 < |z-a| < R\}$. Show that if the singularity at $z = a$ is not removable then $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\Re f(z) < M$ for $0 < |z-a| < R$ then f has a removable singularity at $z = a$.

17. Can you find conformal isomorphisms between the following pairs of domains?

(i) The unit disc $D(0,1)$ and $\{z \in D(0,1) : \Re z > 0 \text{ and } \Im z > 0\}$.

(ii) $\{z : 0 < \Im z < 1\}$ and $\{z : \Re z > 0 \text{ and } \Im z > 0\}$.

(iii) $\{z : |z+1| > 0 \text{ and } |z+2| < 2\}$ and the unit disc $D(0,1)$.

18. (i) Find a Möbius transformation which sends the region between the two circles $|z| = 1$ and $|z-1| = 5/2$ into an annulus $\{w : 1 < |w| < R\}$. What is R? Did you have a choice of R?

(ii) Find a Möbius transformation which sends the unbounded region outside both of the circles $|z-5| = 4$ and $|z+5| = 4$ into an annulus $\{w : 1 < |w| < R\}$. Again what is R? And again did you have a choice?