ANALYSIS II EXAMPLES 1

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The Basic Questions are cover examinable material from the course. The Additional Questions are for those wishing to take things a bit further. The questions are not all equally difficult; I have tried to ensure that the hardest appear amongst the Additional Questions.
The sheet is a minor modification of last year's sheet, itself based on sheets prepared for an earlier version of this course by Gabriel Paternain.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

Basic Questions

1. Define $f_n : [0,2] \to \mathbb{R}$ by

$$\begin{aligned} f_n(x) &= 1 - n|x - n^{-1}| \qquad \text{for } |x - n^{-1}| \le n^{-1}, \\ f_n(x) &= 0 \qquad \text{otherwise.} \end{aligned}$$

Show that the f_n are continuous and sketch their graphs. Show that f_n converges pointwise on $[0,2]$ to the zero function. Is the convergence uniform? (How does this example differ from that discussed in lectures?)

2. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Show that the sequence $x^n f(x)$ is uniformly convergent on $[0,1]$ if and only if $f(1) = 0$.

3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}\,.$$

(i) Show that f_n is uniformly convergent on $(-\infty, \infty)$.
(ii) Is f_n' uniformly convergent on $[0,1]$?
(iii) What are $\lim_{n\to\infty} f_n'(x)$ and $(\lim_{n\to\infty} f_n)'(x)$?

4. Let f and g be uniformly continuous real-valued functions on a set $E \subseteq \mathbb{R}$.
(i) Show that the (pointwise) sum $f+g$ is uniformly continuous on E, as also is the scalar product λf for any real constant λ.
(ii) Is the product fg necessarily uniformly continuous on E? Give a proof or counter-example as appropriate.

5. Which of the following functions f are (a) uniformly continuous, (b) bounded on $[0,\infty)$?
(i) $f(x) = \sin x^2$.
(ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$.
(iii) $f(x) = (\sin x^3)/(x+1)$.

6. Suppose that f is continuous on $[0,\infty)$ and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Is f necessarily uniformly continuous on $[0,\infty)$? Give a proof or a counterexample as appropriate.

7. (i) Show that if (f_n) is a sequence of uniformly continuous functions on $\mathbb{R}$, and $f_n \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous.

(ii) Give an example of a sequence of uniformly continuous functions f_n on $\mathbb{R}$, such that f_n converges pointwise to a continuous function f, but f is not uniformly continuous.
[Hint for part (ii): choose the limit function f first, and take the f_n to be a sequence of 'approximations' to it.]

8. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = n^p x \exp(-n^q x)$ where p, q are positive constants.

(i) Show that f_n converges pointwise on $[0,1]$, for any p and q.
(ii) Show that if $p < q$ then f_n converges uniformly on $[0,1]$.
(iii) Now suppose that $p \geq q$. Show that f_n does not converge uniformly on $[0,1]$. Take $0 < \epsilon < 1$. Does f_n converge uniformly on $[0, 1-\epsilon]$? Does f_n converge uniformly on $[\epsilon, 1]$? Justify your answers.

9. Let $f_n(x) = n^\alpha x^n(1-x)$, where α is a real constant.

(i) For which values of α does $f_n(x) \to 0$ pointwise on $[0,1]$?
(ii) For which values of α does $f_n(x) \to 0$ uniformly on $[0,1]$?
(iii) For which values of α does $\int_0^1 f_n(x)\,dx \to 0$?
(iv) For which values of α does $f_n'(x) \to 0$ pointwise on $[0,1]$?
(v) For which values of α does $f_n'(x) \to 0$ uniformly on $[0,1]$?

10. Consider the sequence of functions $f_n\colon (\mathbb{R}\setminus\mathbb{Z}) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{m=0}^{n} (x-m)^{-2} \ .$$

(i) Show that f_n converges pointwise on $\mathbb{R}\setminus\mathbb{Z}$ to a function f.
(ii) Show that f_n does not converge uniformly on $\mathbb{R}\setminus\mathbb{Z}$.
(iii) Why can we nevertheless conclude that the limit function f is continuous, and indeed differentiable, on $\mathbb{R}\setminus\mathbb{Z}$?

11. Suppose f_n is a sequence of continuous functions from a bounded closed interval $[a,b]$ to $\mathbb{R}$, and that f_n converges pointwise to a continuous function f.

(i) If f_n converges uniformly to f, and (x_m) is a sequence of points of $[a,b]$ converging to a limit x, show that $f_n(x_n) \to f(x)$. [Careful — this is not quite as easy as it looks!]

(ii) If f_n does **not** converge uniformly, show that we can find a convergent sequence $x_n \to x$ in $[a,b]$ such that $f_n(x_n)$ does not converge to $f(x)$.

12. (i) Suppose f is defined and differentiable on a (bounded or unbounded) interval $E \subseteq \mathbb{R}$, and that its derivative f' is bounded on E. Use the Mean Value Theorem to show that f is uniformly continuous on E.

(ii) Give an example of a function f which is (uniformly) continuous on $[0,1]$, and differentiable at every point of $[0,1]$ (here we interpret $f'(0)$ as the 'one-sided derivative' $\lim_{h\to 0^+}((f(h)-f(0))/h)$, and similarly for $f'(1)$), but such that f' is unbounded on $[0,1]$.
[Hint: last year you probably saw an example of an everywhere differentiable function whose derivative is discontinuous; you will need to 'tweak' it slightly.]

Additional Questions

13. Let f be a bounded function defined on a set $E \subseteq \mathbb{R}$, and for each positive integer n let g_n be the function defined on E by

$$g_n(x) = \sup\{|f(y) - f(x)| : y \in E, |y - x| < 1/n\} .$$

Show that f is uniformly continuous on E if and only if $g_n \to 0$ uniformly on E as $n \to \infty$.

14. Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

converges for $s > 1$, and is uniformly convergent on $[1+\varepsilon, \infty)$ for any $\varepsilon > 0$.
Show that ζ is differentiable on $(1, \infty)$. (First think what its derivative ought to be!)

15. (Dirichlet's Test) Let a_n and b_n be real-valued functions on $E \subseteq \mathbb{R}$. Suppose that the partial sums $s_n(x) = \sum_0^n a_k(x)$ are uniformly bounded in the sense that there is a constant K with $|s_n(x)| \leq K$ for all n and all $x \in E$. Suppose further that the $b_n(x)$ are a monotonically decreasing sequence converging uniformly to 0 on E. (That is, $b_n(x) \geq b_{n+1}(x) \geq 0$ in E and $b_n \to 0$ uniformly on E.) Show that the sum $\sum_0^\infty a_n(x)b_n(x)$ is uniformly convergent on E.

16. Suppose that g_n are continuous functions with $g_n(x) \geq g_{n+1}(x) \geq 0$ for all $x \in \mathbb{R}$, and with $g_n \to 0$ uniformly in $\mathbb{R}$.
(i) Show that both $\sum_{n=0}^\infty g_n(x)\cos nx$ and $\sum_{n=0}^\infty g_n(x)\sin nx$ converge uniformly on any interval of the form $[\delta, 2\pi - \delta]$, where $\delta > 0$. (Note what this tells you about Fourier series.)
(ii) Give an example to show that we do not necessarily have convergence uniformly on $[0, 2\pi]$.

17. (i) (Abel's Test) Let a_n and a_n be real-valued functions on $E \subseteq \mathbb{R}$. Suppose that $\sum_0^\infty a_n(x)$ is uniformly convergent on E. Suppose further that the $b_n(x)$ are uniformly bounded on E, and that $b_n(x) \geq b_{n+1}(x) \geq 0$ for all $x \in E$. Show that the sum $\sum_0^\infty a_n(x)b_n(x)$ is uniformly convergent on E.
(ii) Deduce that if $\sum_0^\infty a_n$ is convergent, then $\sum_0^\infty a_n x^n$ is uniformly convergent on $[0, 1]$. (But note that $\sum_0^\infty a_n x^n$ need not be convergent at -1; you almost certainly know a counterexample!)

18. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a continuous function $f : [0, 1] \to \mathbb{R}$ on the unit interval $[0, 1]$. Suppose that $f_n(x)$ is a decreasing sequence for each $x \in [0, 1]$. Show that $f_n \to f$ uniformly on $[0, 1]$.
[If you have done Metric and Topological Spaces then you may prefer to find a topological proof.]

19. Define $\varphi(x) = |x|$ for $x \in [-1, 1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x + 2) = \varphi(x).$$

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s - t|$ for all s and t.
(ii) Define $f(x) = \sum_{n=0}^\infty \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well defined and continuous.
(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}\, 4^{-m}$ where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. Prove that

$$\left|\frac{f(x + \delta_m) - f(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.

20. *A space-filling curve (Exercise 14, Chapter 7 of Rudin's book).* Let f be a continuous real function on $\mathbb{R}$ with the following properties: $0 \le f(t) \le 1$, $f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & \text{for } t \in [0, 1/3]; \\ 1 & \text{for } t \in [2/3, 1]. \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is continuous and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.