Michaelmas Term 2013 J. M. E. Hyland

Linear Algebra: Example Sheet 4

The first 12 questions cover the course and should give good understanding. I hope that the remaining questions will be of independent interest.

1. An endomorphism π of a vector space V is *idempotent* just when $\pi^2 = \pi$. Let $W \leq V$ with V an inner product space. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

2. Let S be a real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

3. Suppose that $\mathbf{x}_1, \cdots, \mathbf{x}_n$ is a basis for an inner product space and $\mathbf{f}_1, \cdots, \mathbf{f}_n$ the basis obtained by the Gram-Schmidt orthogonalization process (as in lectures, without normalising the vectors). Let $A = (a_{ij})$ be the matrix with $a_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$ and $B = (b_{ij})$ the matrix with $b_{ij} = \langle \mathbf{f}_i, \mathbf{f}_j \rangle$. Show that $\det A = \det B$.

4. An endomorphism α of a finite-dimensional inner product space V is *positive definite* if and only if it is self-adjoint and satisfies $\langle \mathbf{x}, \alpha(\mathbf{x}) \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
(i) Prove that a positive definite endomorphism has a unique positive definite square root.
(ii) Let α be a non-singular endomorphism of V with adjoint α^*. By considering $\alpha^*\alpha$ show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.
(iii) Can you say anything for a general endomorphism α?

5. Find a linear transformation which reduces the pair of real quadratic forms
$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$
to the forms
$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$
for some $\lambda, \mu, \nu \in \mathbb{R}$.
Does there exist a linear transformation which reduces the quadratic forms $x^2 - y^2$ and $2xy$ simultaneously to diagonal form?

6. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

7. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$;
(e) $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4\}$.

8. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

9. (i) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(ii) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* just when $f(\mathbf{v}) = 0$ for all $f \in F$ implies $\mathbf{v} = \mathbf{0}$.

10. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional complex vector space and let $\alpha^* : V^* \to V^*$ be its dual. Show that a complex number λ is an eigenvalue for α if, and only if, it is an eigenvalue for α^*. How are the algebraic and geometric multiplicities of λ for α and α^* related? How are the minimal and characteristic polynomials for α and α^* related?

11. For A an $n \times m$ and B an $m \times n$ matrix over the field F, let $\tau_A(B)$ denote $\mathrm{tr}AB$. Show that, for fixed A, τ_A is a linear map $\mathrm{Mat}_{m,n} \to F$ from the space $\mathrm{Mat}_{m,n}$ of $m \times n$ matrices to F.
Now consider the mapping $A \mapsto \tau_A$. Show that it is a linear isomorphism $\mathrm{Mat}_{n,m} \to \mathrm{Mat}^*_{m,n}$.

12. (i) Let U, V be finite dimensional vector spaces and suppose $\beta : U \times V \to F$ is a bilinear map. Show that for any $X \leq U$ we have

$$\dim X + \dim X^{\perp} \geq \dim V \ .$$

Show that equality holds if β is non-degenerate. (Can you give a necessary and sufficient condition?)
(ii) Suppose that β is a bilinear form on V. Take $U \leq V$ with $U = W^{\perp}$ for some $W \leq V$. Suppose that $\psi|_U$ is non-singular. Show that ψ is non-singular.

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13. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define

$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt \, .$$

Show that $\langle \ , \ \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$

is self-adjoint. What are the eigenvalues of α?
Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1-t^2)^k$. Prove the following.
(i) For $i \neq j$, $\langle s_i, s_j \rangle = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

14. Consider the space P of polynomials in variables x_1, ... , x_n. We have linear operators $\partial_i = \frac{\partial}{\partial x_i}$; so for any polynomial $f(x_1, \ldots, x_n) \in P$ we have a corresponding linear operator $\hat{f} = f(\partial_1, \ldots, \partial_n)$. Consider

$$\langle f, g \rangle = \hat{f}(g)(\mathbf{0}) \, ,$$

that is the result of applying $f(\partial_0, \ldots, \partial_n)$ to $g(x_1, \ldots, x_n)$ and then evaluating at $(0, \ldots, 0)$. Show that $\langle f, g \rangle$ is an inner product on P.
Fix $g \in P$. What is the adjoint of the map $P \to P; h \to gh$?
Now consider the subspaces $P(d)$ of polynomials homogeneous of degree d. Show that the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2 : P(d) \to P(d-2)$ is surjective.

15. Let A be a positive definite matrix. Show that $\det A \leq \prod_i a_{ii}$.

16. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.
In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
(a) The map D defined by $D(p)(t) = p'(t)$.
(b) The map S defined by $S(p)(t) = p(t^2)$.
(c) The map E defined by $E(p)(t) = p(t-1)$.
(d) The composite DS.
(e) The composite SD.
Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.